Here is the variation of the Action using the Lagrangian (for a mechanical system)

$$\frac{dA(\varepsilon)}{d\varepsilon}=\left(\int_{t_0}^{t_1}\left(\frac{\partial L}{\partial x}\eta(t)+\frac{\partial L}{\partial \dot{x}}\dot{\eta}(t)\right)dt\right)=0$$

Now if we were to ensure that both position and velocity varied independently, the RED and BLUE eta functions MUST be different. Yes, you can probably incorporate the difference into the epsilon (below, I use two different epsilons); however, both epsilons check out of the operation (and so what if the two epsilons are the same?). THEORETICALLY, the two eta functions should NOT be related by a time derivative: they MUST be different.

As an aside, the blue one derived from taking the epsilon derivative of the velocity as follows

$$q(t;\varepsilon)=q(t)+\varepsilon_p\,\eta(t) \qquad\qquad \dot{q}(t;\varepsilon)=\dot{q}(t)+\varepsilon_v\,\dot{\eta}(t)$$

Let me shift the time derivative on the second term (recognizing ETA still satisfies end points)

$$\int_{t_0}^{t_1}\frac{\partial L}{\partial \dot{x}}\dot{\eta}(t)dt=-\int_{t_0}^{t_1}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\eta(t)dt$$

Thus, we get

$$\frac{dA(\varepsilon)}{d\varepsilon}=\int_{t_0}^{t_1}\left(\frac{\partial L}{\partial x}\eta(t)-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\eta(t)\right)dt$$

And we set this to zero

$$\int_{t_0}^{t_1}\left(\frac{\partial L}{\partial x}\eta(t)-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\eta(t)\right)dt=0$$

In most books, they now assume the two eta functions are the same

$$\int_{t_0}^{t_1}\left(\left(\frac{\partial L}{\partial x}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right)\eta(t)\right)dt=0$$

And they pull out the EL

But I cannot use the above. I must return to this when the two eta functions are different.

$$\int_{t_0}^{t_1}\left(\frac{\partial L}{\partial x}\eta(t)-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\eta(t)\right)dt=0$$

So how do you pull out the E-L now?