1 Energy current

The Hamiltonian is

$$\begin{aligned} H_0 &= -\frac{\hbar^2}{2m}\int dr \psi^\dagger(r)\nabla^2\psi(r) \\ &= \frac{\hbar^2}{2m}\int dr \nabla\psi^\dagger(r)\nabla\psi(r) \end{aligned}$$

The Hamiltonian density is

$$h_0(r) = \frac{\hbar^2}{2m}\nabla\psi^\dagger(r)\nabla\psi(r)$$

The continuity equation is

$$\frac{\partial h_0(r)}{\partial t} + \nabla \cdot j_E = 0$$

To calculate the derivative of $h(r)$ I evaluate the commutator

$$\begin{aligned} [h_0, H_0] &= \frac{\hbar^2}{2m}\left[\nabla\psi^\dagger(r)\nabla\psi(r), H_0\right] \\ &= \left(\frac{\hbar^2}{2m}\right)\left(\nabla\left[\psi^\dagger(r), H_0\right]\nabla\psi(r) + \nabla\psi^\dagger(r)\nabla\left[\psi(r), H_0\right]\right) \end{aligned}$$

I have the two commutators

$$[\psi(r), H_0] = -\frac{\hbar^2}{2m}\nabla^2\psi(r)$$

$$\left[\psi^\dagger(r), H_0\right] = \frac{\hbar^2}{2m}\nabla^2\psi^\dagger(r)$$

Thus

$$\begin{aligned} [h_0, H_0] &= \left(\frac{\hbar^2}{2m}\right)^2\left(\nabla\nabla^2\psi^\dagger(r)\nabla\psi(r) - \nabla\psi^\dagger(r)\nabla\nabla^2\psi(r)\right) \\ &= \left(\frac{\hbar^2}{2m}\right)^2\nabla\cdot\left(\nabla^2\psi^\dagger(r)\nabla\psi(r) - \nabla\psi^\dagger(r)\nabla^2\psi(r)\right) \end{aligned}$$

Now because

$$\frac{\partial h_0}{\partial t} = -\frac{i}{\hbar}[h_0, H_0] = -\nabla\cdot j_E(r)$$

I have

$$j_E(r) = -\frac{i}{\hbar}\left(\frac{\hbar^2}{2m}\right)^2\left(\nabla\psi^\dagger(r)\nabla^2\psi(r) - \nabla^2\psi^\dagger(r)\nabla\psi(r)\right)$$

I now reintroduce the commutators

$$[\psi(r), H_0] = -\frac{\hbar^2}{2m}\nabla^2\psi(r)$$

$$\left[\psi^\dagger(r), H_0\right] = \frac{\hbar^2}{2m}\nabla^2\psi^\dagger(r)$$

Giving the following current

$$j_E(r) = -\frac{i}{\hbar}\left(\frac{\hbar^2}{2m}\right)\left(-\nabla\psi^\dagger(r)\left[\psi(r), H_0\right] - \left[\psi^\dagger(r), H_0\right]\nabla\psi(r)\right)$$

Which can also be expressed in terms of the time derivative

$$i\hbar\frac{\partial\psi}{\partial t} = \left[\psi(r), H_0\right]$$

So the final result is

$$j_E(r) = -\frac{\hbar^2}{2m}\left(\nabla\psi^\dagger(r)\frac{\partial\psi}{\partial t} + \frac{\partial\psi^\dagger}{\partial t}\nabla\psi(r)\right)$$