The Electromagnetic Power Flux Revisited

by Michael Bensimhoun
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ABSTRACT

After examining the common conceptions regarding the electromagnetic energy and the ambiguity of its flux density, a general form of the power flux density will be proposed, which encompasses the classical Poynting vector theory, and provides a better theoretical fundation for the energy and momentum fluxes. Of course, the general form is compatible with Maxwell equations and the energy density formula.

1. Electromagnetic Energy density and power flux

The expression of the energy density of the electromagnetic field was first derived in 1880 by J. C. Maxwell ([2]) in his monumental work. Maxwell showed essentially that the (volumic) density of E.M. energy at some point M is equal to

$$u = \frac{\varepsilon_0}{2}\mathbf{E^2} + \frac{\varepsilon_0 c^2}{2}\mathbf{B^2}, \tag{1}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields resp. at M, ε_0 is the permitivity of free space, and c is the speed of light.

This expression fitted well with the once new theory of continuous propagation of the electrical and magnetic fields, advocated by Faraday and Maxwell, in contrast to the theory of action at some distance developed by the German physicists.

Relation (1) is now considered as fundamental, and is firmly anchored into the theory of General relativity ([4], Sec. 1.2 p. 6).

Building on the work of Maxwell, J. H. Poynting introduced in 1883 the vector that bears his name ([3]). The Poynting vector is defined at any point M by

$$\mathbf{S} = \varepsilon_0 c^2 \mathbf{E} \times \mathbf{B}. \tag{2}$$

It can be shown, using Maxwell's equations, that $\mathbf{S}$ fulfills

$$-\frac{\partial u}{\partial t} = \nabla \cdot S + \mathbf{E} \cdot \mathbf{j}, \tag{3}$$

where $\mathbf{j}$ is the current density at M. Since $\mathbf{E} \cdot \mathbf{j}$ is the work done by the E.M. field, or more precisely, the power transmitted to the charge element $q = \rho dV$ centered at

M, this formula, when integrated on a closed volume $\mathcal{V}$ whose surface is oriented to the outer side, means that the rate of decrease of the E.M. energy contained inside a volume $\mathcal{V}$, or what is the same, the E.M. power wasted by $\mathcal{V}$, is equal to the sum of the flux of $\mathbf{S}$ through the surface ∂V of $\mathcal{V}$, with the power transmitted by the E.M. field to the charges inside $\mathcal{V}$.

This leads to the interpretation of $\mathbf{S}$ as the *power flux density vector* of the E.M. field, flowing out of $\mathcal{V}$ through its surface.

This interpretation has proved to work well in many circumstances, and to quote Feynman "nobody has never find anything wrong with it". Nevertheless, and that's the main point of this article, it should be observed that the Poynting vector is not the only possible expression of the power flux density susceptible of giving rise a valid theory. Indeed, the only physical way to observe the flux of power seems to be measuring the rate of decrease of the energy inside a closed volume $\mathcal{V}$; in other words, since we have

$$\nabla \cdot \mathbf{S} = -\frac{\partial u}{\partial t} - \mathbf{E} \cdot \mathbf{j}, \tag{4}$$

we obtain, integrating over $\mathcal{V}$ with the Stokes formula,

$$\int_{\partial V} \mathbf{S} \cdot \mathrm{d}\vec{s} = -\frac{\partial U}{\partial t} - P, \tag{5}$$

where U is the E.M. energy inside $\mathcal{V}$ and P is the power transmitted to the charges inside $\mathcal{V}$ by the E.M. field. But if some solenoidal field $\mathbf{R}$ were added to $\mathbf{S}$, say

$$\mathbf{S}' = \mathbf{S} + \mathbf{R},$$

then taking into account that $\nabla \cdot \mathbf{R} = \mathbf{0}$, relation (4) and (5) would hold with $\mathbf{S}'$ in place of $\mathbf{S}$ as well. So, it seems that the real form of the power flux density cannot be determined by any physical experiment: it is defined only up to a solenoidal.

Let us report here an entire note of Feynman in his Lectures on Physics ([1], II, 27.4):

"Before we take up some applications of the Poynting formulas [Eqs. (27.14) and (27.15)], we would like to say that we have not really "proved" them. All we did was to find a possible u and a possible $\mathbf{S}$. How do we know that by juggling the terms around some more we couldnt find another formula for u and another formula for $\mathbf{S}$? The new $\mathbf{S}$ and the new u would be different, but they would still satisfy Eq. (27.6). Its possible. It can be done, but the forms that have been found always involve various derivatives of the field (and always with second-order terms like a second derivative or the square of a first derivative). There are, in fact, an infinite number of different possibilities for u and $\mathbf{S}$, and so far no one has thought of an experimental way to tell which one is right! People have guessed that the simplest one is probably the correct one, but we must say that we do not know for certain what is the actual location in space of the electromagnetic field energy. So we too will take the easy way out and say that the field energy is given by Eq. (27.14). Then the flow vector **S** must be given by Eq. (27.15).

It is interesting that there seems to be no unique way to resolve the indefiniteness in the location of the field energy. It is sometimes claimed that this problem can be resolved by using the theory of gravitation in the following argument. In the theory of gravity, all energy is the source of gravitational attraction. Therefore the energy density of electricity must be located properly if we are to know in which direction the gravity force acts. As yet, however, no one has done such a delicate experiment that the precise location of the gravitational influence of electromagnetic fields could be determined. That electromagnetic fields alone can be the source of gravitational force is an idea it is hard to do without. It has, in fact, been observed that light is deflected as it passes near the sun–we could say that the sun pulls the light down toward it. Do you not want to allow that the light pulls equally on the sun? Anyway, everyone always accepts the simple expressions we have found for the location of electromagnetic energy and its flow. And although sometimes the results obtained from using them seem strange, nobody has ever found anything wrong with them—that is, no disagreement with experiment. So we will follow the rest of the world—besides, we believe that it is probably perfectly right".

As pointed out by Feynman and virtually all textbooks in electromagnetism, the Poynting vector leads to a representation of the power flow that does not fit well with the common natural intuition. As a example taken from the article of Poynting ([3], p. 350–351, let us examine how, according to this theory, the power flows inside an electrical wire connected to the terminals of a battery, assuming, for the sake of simplicity, that the wire is cylindrical and has a uniform resistance per unit of length, and that the current is steady.

At a point M inside the wire or at its surface, the electric field $\mathbf{E}$ is parallel to the segment of wire centered at M, since the current is flowing along this direction. Furthermore, since we assumed steady currents, $\mathbf{E}$ is constant. On the other hand, assuming a point M is located at the surface of the wire, it can be shown, with, e.g. the Biot and Savard law, that the magnetic field $\mathbf{B}$ at M is is orthogonal to the radius OM of the wire, and directed toward its axis: $\mathbf{B} = \alpha \mathbf{E} \times \overrightarrow{OM}$, with $\alpha > 0$. So, the Poynting vector $\mathbf{S} = \varepsilon_0 c^2 \mathbf{E} \times \mathbf{B}$ is radial: $\mathbf{S} = \beta \overrightarrow{OM}$, with $\beta < 0$. This means that according to this theory, the E.M. power is not flowing inside and along the wire, as one would intuitively expect, but it comes from the outside and flows perpendicularly through the surface, toward the axis of the wire.

The amount of electrical power transmitted to a length ℓ of the wire can also be easily computed: Let $\mathcal{A}$ be the area of the cylinder formed by the wire, and $\mathcal{A}'$ its area without the cross sections at the extremities: $\mathcal{A}' = 2\pi r \ell$, where r is the radius of the wire. The power transmitted to the length of wire is

$$P = -\int_{\mathcal{A}} \mathbf{S}\,\mathrm{d}\vec{s} = -\int_{\mathcal{A}'} \mathbf{S}\,\mathrm{d}\vec{s} = 2\pi r\ell\|\mathbf{S}\| = 2\pi r\ell\|\mathbf{E}\|\|\mathbf{B}\|\,.$$

But $2\pi r\|\mathbf{B}\|$ is equal to the circulation of the magnetic field around the loop of radius r, hence is equal, according to Ampere's law, to $\frac{I}{\varepsilon_0 c^2}$, where I is the current intensity in the wire. On the other hand, $V = \ell\|\mathbf{E}\|$ is equal to the positive difference of potential between the two extremities of the length of wire. Hence

$$P = VI\,,$$

which is the well known form of the electrical power transmitted to a resistor, eventually dissipated into heat.

This example shows that although the power flow stemming from the Poynting vector seems counter-intuitive, the results are coherent when it comes to computing global energy transfers. This is usually the justification given by virtually all textbooks on this topic, something like "Hey, that's weird, but don't worry, just compute with the Poynting vector and everything will be OK." As we shall see, this need not be the case. It is possible to obtain a definition of the power flux density vector which gives rise a power flow that better fits with our intuition, and yet leads to the same global energy transfer results as the classical definition.

Nevertheless, if one consider the Poynting vector as a mathematical artifice, it is possible to enjoy both worlds: We could just use it to compute global energy transfers whenever convenient, since it leads to the same results as any alternative power flux density vector anyway. Nevertheless, at the local scale, those forms lead to different results and interpretations. We believe that both in practical and in theoretical situations like thermodynamics, where the E.M. energy interacts with other fluxes and with the matter, the alternative definition given in the next section is the most suitable.

2. General form of the power flux density vector

The general form proposed in this section stems from a long discussion in the Physics Forum, about the way the electrical power should be flowing through, or outside, an electrical wire traversed by a steady current. While most of the persons stuck to the classic Poynting view, according to which the electrical power is transmitted radially to the wire, the O.P., "Fluidistic", defended the idea that according to thermodynamics, there *must* exist a component of the power flow along the wire. Indeed, starting from the thermodynamic equation

$$\vec{J}_U = T\vec{J}_S + \bar{\mu}\vec{J}_e\,,$$

where $\vec{J}_U$ is the internal energy flux of the system, $\vec{J}_S$ is the entropy flux, $\vec{J}_e$ is the particle flux density and $\bar{\mu}$ is the electrochemical potential energy of the electrons, Fluidistic shown that the component $T\vec{J}_S$ of the energy flux density is radial to the wire, while $\bar{\mu}\vec{J}_e$ is parallel to the wire. After a long discussion, it was observed by the author that since $\bar{\mu}$ is the sum of the chemical energy of the electron with the potential energy of the electron (with respect to the electric potential), and since the chemical energy of an electron inside a conductor is likely to be null or negligible, $\bar{\mu}J_e$ could simply be written $\Phi\mathbf{j}$, where $\mathbf{j}$ is the current density. The author then realized that the whole situation could be formulated in a completely classical electrodynamic framework, that will now be exposed as a motivating example.

Consider again the electrical wire example of Poynting in the previous section. Along the length of wire ℓ, the electrical potential is $\Phi(z)$, z being the curvilinear coordinate oriented from the "plus" to the "minus" terminal of the length ℓ. Notice that Φ is decreasing, and it decreases linearly if the resistance per unit of length is constant. Now, consider a cross section $\mathcal{A}(z)$ of the wire, and let ρ be the volumic flowing charge density inside the wire. At a point M of $\mathcal{A}(z)$, an elementary charge ρdV has a potential electrical energy equal to $\rho\Phi(z)dV$. So, we can define an "electrical potential energy flux density" at M by

$$\mathbf{S}' = \rho\Phi\mathbf{v} = \Phi\mathbf{j}\,,$$

where $\mathbf{v}$ is the speed of the charge at M, and $\mathbf{j}$ is the current density. The amount of power "flowing" through $\mathcal{A}(z)$ is

$$\int_{\mathcal{A}(z)} \Phi\mathbf{j}\,\mathrm{d}\vec{s} = I\Phi(z)\,,$$

where I is the current intensity through $\mathcal{A}(z)$, assuming $\mathcal{A}(z)$ oriented in the same direction as the current. Now, consider as above the cylinder made by a length ℓ of wire. Orient the total surface $\mathcal{A}$ of the cylinder as usual, according to the outer normal of the surface. Then denoting by $\mathcal{A}(z_1)$ and $\mathcal{A}(z_2)$ the surfaces at the extremities of the cylinder, and assuming $z_1 < z_2$, the potential energy rate flowing out of the cylinder is

$$\int_{\mathcal{A}} \Phi\mathbf{j}\,\mathrm{d}\vec{s} = \int_{A(z_2)} \Phi\mathbf{j}\,\mathrm{d}\vec{s} - \int_{A(z_1)} \Phi\mathbf{j}\,\mathrm{d}\vec{s} = I(\Phi(z_2) - \Phi(z_1)) = -IV\,,$$

where V is the positive difference of potential between z_1 and z_2. In other words, the electrical power transmitted to the wire is IV, as should be the case for a resistor.

We see that we have obtained here the same result as Poynting, but in a completely, almost opposite, route. The power is now flowing inside and along the wire, carried by the charges, and has no component orthogonal to the wire, which certainly better corresponds to our intuition of the electrical power.

We now state the proposed form of the power flux density vector, valid for all electrodynamic systems, in full generality. It is likely to be the simplest form that reduces to $\Phi\mathbf{j}$ in the case of steady currents.

In term of the potential Φ and the vector potential $\mathbf{A}$ alone, it is defined by

$$\mathbf{S}' = \varepsilon_0 c^2 \left[\left(\Phi \nabla - \frac{\partial \mathbf{A}}{\partial t} \right) \times (\nabla \times \mathbf{A}) \right] . \tag{1}$$

In a somewhat more digest form, using Maxwell's equation $\mathbf{B} = \nabla \times \mathbf{A}$,

$$\mathbf{S}' = \varepsilon_0 c^2 \left[\left(\Phi \nabla - \frac{\partial \mathbf{A}}{\partial t} \right) \times \mathbf{B} \right] \tag{2}$$

Observe that if a gauge is chosen is such a way that $\Phi = 0$ identically (which is always possible), then formula (2) reduces to the Poynting vector, because of Maxwell's equation

$$\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t} .$$

Such a gauge is usually advantageously chosen in the case of plane wave propagating in charge free space. Then the Poynting vector is directed along the direction of the wave propagation, an intuitively desirable property, especially because the EM power flux density must be, according to general physical concepts, equal to c^2 times the EM power flux momentum.

Beside, with the help of Maxwell's equation

$$\varepsilon_0 c^2 \nabla \times \mathbf{B} = \mathbf{j} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} ,$$

it is possible to transform eq. (2) in order to exhibit the current density:

$$\mathbf{S}' = \Phi \mathbf{j} + \varepsilon_0 \Phi \frac{\partial \mathbf{E}}{\partial t} - \varepsilon_0 c^2 \frac{\partial \mathbf{A}}{\partial t} \times \mathbf{B} . \tag{3}$$

Observe that for steady currents, eq. (3) reduces to

$$\mathbf{S}' = \Phi \mathbf{j} ,$$

assuming a gauge has been chosen is such a way that $\frac{\partial \mathbf{A}}{\partial \mathbf{t}} = 0$. The meaning of this equation is that for steady currents, there is no flow of power wherever the space is free of charges: the power is carried by the charges only. This form justifies the electrical power flux used in thermodynamics, as pointed out in the previous section.

It is now well acknowledged that the electric and magnetic fields are not real, or at least less real than the electric and vector potentials. This fact was pointed out by

Feynman ([1]), and can be found in many books like [4], Sec. 1. Now, if we consider Φ and $\mathbf{A}$ as the real notions, then the Poynting vector becomes

$$\mathbf{S} = \varepsilon_0 c^2 \mathbf{E} \times \mathbf{B} = -\varepsilon_0 c^2 \left[(\nabla\Phi + \frac{\partial \mathbf{A}}{\partial t}) \times \mathbf{B} \right] \tag{4}$$

$$= -\varepsilon_0 c^2 \left[\left(\nabla\Phi + \frac{\partial \mathbf{A}}{\partial t} \right) \times (\nabla \times \mathbf{A}) \right] \tag{5}$$

Observe that

$$\mathbf{S}' - \mathbf{S} = (\Phi\nabla + \nabla\Phi) \times \mathbf{B} = \nabla \times \Phi\mathbf{B}.$$

So, $\mathbf{S}'$ and $\mathbf{S}$ differ by a solenoidal field. This implies that $\nabla \cdot \mathbf{S}' = \nabla \cdot \mathbf{S}$, and $\mathbf{S}'$ represent a valid formula for the power flux density, as explained in the previous section.

Bibliography

[1] R.P. Feynman, R.B. Leighton, and M. Sands. *The Feynman Lectures on Physics, Vol. II: The New Millennium Edition: Mainly Electromagnetism and Matter.* The Feynman Lectures on Physics. Basic Books, 2011.

[2] J. C. Maxwell. *Treatise on Electricity and Magnetism.* Oxford: Clarendon Press, 1873.

[3] J. H. Poynting. On the Transfer of Energy in the Electromagnetic Field. *Phil. Trans.of the Royal Society of London*, 175:343–361, 1884.

[4] R. M Wald. *Advanced Classical Electromagnetism.* Princeton University Press, 2022.