

Euler-Mascheroni Constant

$$\frac{1}{\Gamma(s)} = s e^{\gamma s} \prod_{k=1}^{\infty} (1 + s/k) e^{-s/k}, \quad \forall s \in \mathbb{C}$$

$$\begin{aligned} \log\left(\frac{1}{\Gamma(s)}\right) &= \gamma s + \log(s) + \sum_{k=1}^{\infty} \{\log(1 + s/k) - s/k\} \\ &= \gamma s + \log(s) + \sum_{k=1}^{\infty} \{\log(k+s) - \log(k) - s/k\} \end{aligned}$$

$$\begin{aligned} \hookrightarrow \quad 1/s \{ \log(1/\Gamma(s)) - \log(s) \} &= 1/s \log(1/s \Gamma(s)) \\ &= 1/s \log(1/\Gamma(s+1)) = \gamma + \sum_{k=1}^{\infty} \left\{ \frac{\log(k+s) - \log(k)}{s} - 1/k \right\} \end{aligned}$$

Notice that the LHS has a removable singularity at $s=0$, and is thus an entire function. Therefore, the limit as $s \rightarrow 0$ exists, and we have

$$\begin{aligned} \lim_{s \rightarrow 0} \left\{ 1/s \log(1/\Gamma(s+1)) \right\} &= \gamma + \lim_{s \rightarrow 0} \left\{ \sum_{k=1}^{\infty} \frac{\log(k+s) - \log(k)}{s} - 1/k \right\} \\ &= \gamma + \sum_{k=1}^{\infty} 1/k - 1/k \\ &= \gamma \end{aligned}$$