6

The Electric Field in Various Circumstances

Review. Chapter 23, Vol. I, *Resonance*

6–1 Equations of the electrostatic potential

This chapter will describe the behavior of the electric field in a number of different circumstances. It will provide some experience with the way the electric field behaves, and will describe some of the mathematical methods which are used to find this field.

We begin by pointing out that the whole mathematical problem is the solution of two equations, the Maxwell equations for electrostatics:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}, \tag{6.1}$$

$$\nabla \times \boldsymbol{E} = 0. \tag{6.2}$$

In fact, the two can be combined into a single equation. From the second equation, we know at once that we can describe the field as the gradient of a scalar (see Section 3–7):

$$\boldsymbol{E} = -\nabla\phi. \tag{6.3}$$

We may, if we wish, completely describe any particular electric field in terms of its potential ϕ. We obtain the differential equation that ϕ must obey by substituting Eq. (6.3) into (6.1), to get

$$\nabla \cdot \nabla\phi = -\frac{\rho}{\epsilon_0}. \tag{6.4}$$

The divergence of the gradient of ϕ is the same as ∇^2 operating on ϕ:

$$\nabla \cdot \nabla\phi = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}, \tag{6.5}$$

so we write Eq. (6.4) as

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}. \tag{6.6}$$

The operator ∇^2 is called the Laplacian, and Eq. (6.6) is called the Poisson equation. The entire subject of electrostatics, from a mathematical point of view, is merely a study of the solutions of the single equation (6.6). Once ϕ is obtained by solving Eq. (6.6) we can find $\boldsymbol{E}$ immediately from Eq. (6.3).

We take up first the special class of problems in which ρ is given as a function of x, y, z. In that case the problem is almost trivial, for we already know the solution of Eq. (6.6) for the general case. We have shown that if ρ is known at every point, the potential at point (1) is

$$\phi(1) = \int \frac{\rho(2)\,dV_2}{4\pi\epsilon_0 r_{12}}, \tag{6.7}$$

where $\rho(2)$ is the charge density, dV_2 is the volume element at point (2), and r_{12} is the distance between points (1) and (2). The solution of the *differential* equation (6.6) is reduced to an *integration* over space. The solution (6.7) should be especially noted, because there are many situations in physics that lead to equations like

$$\nabla^2 \text{(something)} = \text{(something else)},$$

and Eq. (6.7) is a prototype of the solution for any of these problems.

The solution of electrostatic field problems is thus completely straightforward when the positions of all the charges are known. Let's see how it works in a few examples.

6–2 The electric dipole

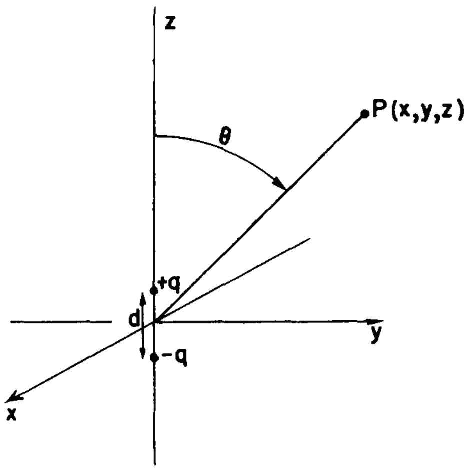

Fig. 6–1. A dipole: two charges $+q$ and $-q$ the distance d apart.

First, take two point charges, $+q$ and $-q$, separated by the distance d. Let the z-axis go through the charges, and pick the origin halfway between, as shown in Fig. 6–1. Then, using (4.24), the potential from the two charges is given by

$$\phi(x, y, z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{[z-(d/2)]^2+x^2+y^2}}+\frac{-q}{\sqrt{[z+(d/2)]^2+x^2+y^2}}\right]. \tag{6.8}$$

We are not going to write out the formula for the electric field, but we can always calculate it once we have the potential. So we have solved the problem of two charges.

There is an important special case in which the two charges are very close together—which is to say that we are interested in the fields only at distances from the charges large in comparison with their separation. We call such a close pair of charges a *dipole*. Dipoles are very common.

A "dipole" antenna can often be approximated by two charges separated by a small distance—if we don't ask about the field too close to the antenna. (We are usually interested in antennas with *moving* charges; then the equations of statics do not really apply, but for some purposes they are an adequate approximation.)

More important perhaps, are atomic dipoles. If there is an electric field in any material, the electrons and protons feel opposite forces and are displaced relative to each other. In a conductor, you remember, some of the electrons move to the surfaces, so that the field inside becomes zero. In an insulator the electrons cannot move very far; they are pulled back by the attraction of the nucleus. They do, however, shift a little bit. So although an atom, or molecule, remains neutral in an external electric field, there is a very tiny separation of its positive and negative charges and it becomes a microscopic dipole. If we are interested in the fields of these atomic dipoles in the neighborhood of ordinary-sized objects, we are normally dealing with distances large compared with the separations of the pairs of charges.

In some molecules the charges are somewhat separated even in the absence of external fields, because of the form of the molecule. In a water molecule, for example, there is a net negative charge on the oxygen atom and a net positive charge on each of the two hydrogen atoms, which are not placed symmetrically but as in Fig. 6–2. Although the charge of the whole molecule is zero, there is a charge distribution with a little more negative charge on one side and a little more positive charge on the other. This arrangement is certainly not as simple as two point charges, but when seen from far away the system acts like a dipole. As we shall see a little later, the field at large distances is not sensitive to the fine details.

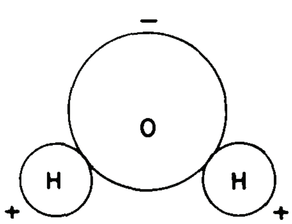

Fig. 6–2. The water molecule H_2O. The hydrogen atoms have slightly less than their share of the electron cloud; the oxygen, slightly more.

Let's look, then, at the field of two opposite charges with a small separation d. If d becomes zero, the two charges are on top of each other, the two potentials cancel, and there is no field. But if they are not exactly on top of each other, we can get a good approximation to the potential by expanding the terms of (6.8) in a power series in the small quantity d (using the binomial expansion). Keeping terms only to first order in d, we can write

$$\left(z-\frac{d}{2}\right)^2 \approx z^2 - zd.$$

It is convenient to write

$$x^2+y^2+z^2 = r^2.$$

Then

$$\left(z-\frac{d}{2}\right)^2 + x^2 + y^2 \approx r^2 - zd = r^2\left(1-\frac{zd}{r^2}\right),$$

and

$$\frac{1}{\sqrt{[z-(d/2)]^2+x^2+y^2}} \approx \frac{1}{\sqrt{r^2[1-(zd/r^2)]}} \approx \frac{1}{r}\left(1-\frac{zd}{r^2}\right)^{-1/2}.$$

Using the binomial expansion again for $[1 - (zd/r^2)]^{-1/2}$—and throwing away terms with higher powers than the square of d—we get

$$\frac{1}{r}\left(1 + \frac{1}{2}\frac{zd}{r^2}\right).$$

Similarly,

$$\frac{1}{\sqrt{[z + (d/2)]^2 + x^2 + y^2}} \approx \frac{1}{r}\left(1 - \frac{1}{2}\frac{zd}{r^2}\right).$$

The difference of these two terms gives for the potential

$$\phi(x, y, z) = \frac{1}{4\pi\epsilon_0}\frac{z}{r^3}\,qd. \tag{6.9}$$

The potential, and hence the field, which is its derivative, is proportional to qd, the product of the charge and the separation. This product is defined as the *dipole moment* of the two charges, for which we will use the symbol p (do *not* confuse with momentum!):

$$p = qd. \tag{6.10}$$

Equation (6.9) can also be written as

$$\phi(x, y, z) = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2}, \tag{6.11}$$

since $z/r = \cos\theta$, where θ is the angle between the axis of the dipole and the radius vector to the point (x, y, z)—see Fig. 6–1. The *potential* of a dipole decreases as $1/r^2$ for a given direction from the axis (whereas for a point charge it goes as $1/r$). The electric field $\boldsymbol{E}$ of the dipole will then decrease as $1/r^3$.

We can put our formula into a vector form if we define $\boldsymbol{p}$ as a vector whose magnitude is p and whose direction is along the axis of the dipole, pointing from q_- toward q_+. Then

$$\cos\theta = \boldsymbol{p}\cdot\boldsymbol{e}_r, \tag{6.12}$$

where $\boldsymbol{e}_r$ is the unit radial vector (Fig. 6–3). We can also represent the point (x, y, z) by $\boldsymbol{r}$. Then

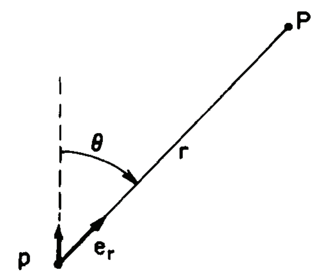

Fig. 6–3. Vector notation for a dipole.

Dipole potential:

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}\cdot\boldsymbol{e}_r}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}\cdot\boldsymbol{r}}{r^3}. \tag{6.13}$$

This formula is valid for a dipole with any orientation and position if $\boldsymbol{r}$ represents the vector from the dipole to the point of interest.

If we want the electric field of the dipole we can get it by taking the gradient of ϕ. For example, the z-component of the field is $-\partial\phi/\partial z$. For a dipole oriented along the z-axis we can use (6.9):

$$-\frac{\partial\phi}{\partial z} = -\frac{p}{4\pi\epsilon_0}\frac{\partial}{\partial z}\left(\frac{z}{r^3}\right) = -\frac{p}{4\pi\epsilon_0}\left(\frac{1}{r^3} - \frac{3z^2}{r^5}\right),$$

or

$$E_z = \frac{p}{4\pi\epsilon_0}\frac{3\cos^2\theta - 1}{r^3}. \tag{6.14}$$

The x- and y-components are

$$E_x = \frac{p}{4\pi\epsilon_0}\frac{3zx}{r^5}, \qquad E_y = \frac{p}{4\pi\epsilon_0}\frac{3zy}{r^5}.$$

These two can be combined to give one component directed *perpendicular* to the z-axis, which we will call the transverse component $E_\perp$:

$$E_\perp = \sqrt{E_x^2 + E_y^2} = \frac{p}{4\pi\epsilon_0}\frac{3z}{r^5}\sqrt{x^2 + y^2}$$

or

$$E_\perp = \frac{p}{4\pi\epsilon_0}\frac{3\cos\theta\sin\theta}{r^3}. \tag{6.15}$$

The transverse component $E_\perp$ is in the x-y plane and points directly away from the *axis* of the dipole. The total field, of course, is

$$E = \sqrt{E_z^2 + E_\perp^2}\,.$$

The dipole field varies inversely as the cube of the distance from the dipole. On the axis, at $\theta = 0$, it is twice as strong as at $\theta = 90°$. At both of these special angles the electric field has only a z-component, but of opposite sign at the two places (Fig. 6–4).

6–3 Remarks on vector equations

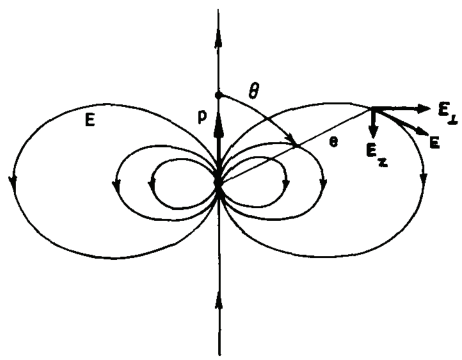

Fig. 6–4. The electric field of a dipole.

This is a good place to make a general remark about vector analysis. The fundamental proofs can be expressed by elegant equations in a general form, but in making various calculations and analyses it is always a good idea to choose the axes in some convenient way. Notice that when we were finding the potential of a dipole we chose the z-axis along the direction of the dipole, rather than at some arbitrary angle. This made the work much easier. But then we wrote the equations in vector form so that they would no longer depend on any particular coordinate system. After that, we are allowed to choose any coordinate system we wish, knowing that the relation is, in general, true. It clearly doesn't make any sense to bother with an arbitrary coordinate system at some complicated angle when you can choose a neat system for the particular problem—provided that the result can finally be expressed as a vector equation. So by all means take advantage of the fact that vector equations are independent of any coordinate system.

On the other hand, if you are trying to calculate the divergence of a vector, instead of just looking at $\nabla \cdot \boldsymbol{E}$ and wondering what it is, don't forget that it can always be spread out as

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\,.$$

If you can then work out the x-, y-, and z-components of the electric field and differentiate them, you will have the divergence. There often seems to be a feeling that there is something inelegant—some kind of defeat involved—in writing out the components; that somehow there ought always to be a way to do everything with the vector operators. There is often no advantage to it. The first time we encounter a particular kind of problem, it usually helps to write out the components to be sure we understand what is going on. There is nothing inelegant about putting numbers into equations, and nothing inelegant about substituting the derivatives for the fancy symbols. In fact, there is often a certain cleverness in doing just that. Of course when you publish a paper in a professional journal it will look better—and be more easily understood—if you can write everything in vector form. Besides, it saves print.

6–4 The dipole potential as a gradient

We would like to point out a rather amusing thing about the dipole formula, Eq. (6.13). The potential can also be written as

$$\phi = -\frac{1}{4\pi\epsilon_0}\,\boldsymbol{p} \cdot \nabla\left(\frac{1}{r}\right). \tag{6.16}$$

If you calculate the gradient of $1/r$, you get

$$\nabla\left(\frac{1}{r}\right) = -\frac{\boldsymbol{r}}{r^3} = -\frac{\boldsymbol{e}_r}{r^2}\,,$$

and Eq. (6.16) is the same as Eq. (6.13).

How did we think of that? We just remembered that $\boldsymbol{e}_r/r^2$ appeared in the formula for the *field* of a point charge, and that the field was the gradient of a *potential* which has a $1/r$ dependence.