Corollary 5. The ring R is a finitely generated k-algebra if and only if there is some surjective k-algebra homomorphism

$$\varphi : k[x_1, x_2, \dots, x_n] \to R$$

from the polynomial ring in a finite number of variables onto R that is the identity map on k. Any finitely generated k-algebra is therefore Noetherian.

Proof: If R is generated as a k-algebra by $r_1, \dots, r_n$, then we may define the map $\varphi : k[x_1, \dots, x_n] \to R$ by $\varphi(x_i) = r_i$ for all i and $\varphi(a) = a$ for all $a \in k$. Then φ extends uniquely to a surjective ring homomorphism. Conversely, given a surjective homomorphism φ, the images of $x_1, \dots, x_n$ under φ then generate R as a k-algebra, proving that R is finitely generated. Since $k[x_1, \dots, x_n]$ is Noetherian by the previous corollary, any finitely generated k-algebra is therefore the quotient of a Noetherian ring, hence also Noetherian by Proposition 1.

Example

Suppose the k-algebra R is finite dimensional as a vector space over k, for example when $R = k[x]/(f(x))$, where f is any nonzero polynomial in $k[x]$. Then in particular R is a finitely generated k-algebra since a vector space basis also generates R as a ring. In this case since ideals are also k-subspaces any ascending or descending chain of ideals has at most $\dim_k R + 1$ distinct terms, hence R satisfies both A.C.C. and D.C.C. on ideals.

The basic idea behind "algebraic geometry" is to equate geometric questions with algebraic questions involving ideals in rings such as $k[x_1, \dots, x_n]$. The Noetherian nature of these rings reduces many questions to consideration of finitely many algebraic equations (and this was in turn one of the main original motivations for Hilbert's Basis Theorem). We first consider the principal geometric object, the notion of an "algebraic set" of points.

Affine Algebraic Sets

Recall that the set $\mathbb{A}^n$ of n-tuples of elements of the field k is called *affine n-space over k* (cf. Section 10.1). If $x_1, x_2, \dots, x_n$ are independent variables over k, then the polynomials f in $k[x_1, x_2, \dots, x_n]$ can be viewed as k-valued functions $f : \mathbb{A}^n \to k$ on $\mathbb{A}^n$ by evaluating f at the points in $\mathbb{A}^n$:

$$f : (a_1, a_2, \dots, a_n) \mapsto f(a_1, a_2, \dots, a_n) \in k.$$

This gives a ring of k-valued functions on $\mathbb{A}^n$, denoted by $k[\mathbb{A}^n]$ and called the *coordinate ring of* $\mathbb{A}^n$. For instance, when $k = \mathbb{R}$ and $n = 2$, the coordinate ring of Euclidean 2-space $\mathbb{R}^2$ is denoted by $\mathbb{R}[\mathbb{A}^2]$ and is the ring of polynomials in two variables, say x and y, acting as real valued functions on $\mathbb{R}^2$ (the usual "coordinate functions").

Each subset S of functions in the coordinate ring $k[\mathbb{A}^n]$ determines a subset $\mathcal{Z}(S)$ of affine space, namely the set of points where all functions in S are simultaneously zero:

$$\mathcal{Z}(S) = \{(a_1, a_2, \dots, a_n) \in \mathbb{A}^n \mid f(a_1, a_2, \dots, a_n) = 0 \text{ for all } f \in S\},$$

where $\mathcal{Z}(\emptyset) = \mathbb{A}^n$.

Definition. A subset V of $\mathbb{A}^n$ is called an *affine algebraic set* (or just an algebraic set) if V is the set of common zeros of some set S of polynomials, i.e., if $V = \mathcal{Z}(S)$ for some $S \subseteq k[\mathbb{A}^n]$. In this case $V = \mathcal{Z}(S)$ is called the *locus of* S in $\mathbb{A}^n$.

If $S = \{f\}$ or $\{f_1, \ldots, f_m\}$ we shall simply write $\mathcal{Z}(f)$ or $\mathcal{Z}(f_1, \ldots, f_m)$ for $\mathcal{Z}(S)$ and call it the locus of f or $f_1, \ldots, f_m$, respectively. Note that the locus of a single polynomial of the form $f - g$ is the same as the solutions in affine n-space of the equation $f = g$, so affine algebraic sets are the solution sets to systems of polynomial equations, and as a result occur frequently in mathematics.

Examples

(1) If $n = 1$ then the locus of a single polynomial $f \in k[x]$ is the set of roots of f in k. The algebraic sets in $\mathbb{A}^1$ are $\emptyset$, any finite set, and k (cf. the exercises).

(2) The one point subsets of $\mathbb{A}^n$ for any n are affine algebraic since $\{(a_1, a_2, \ldots, a_n)\}$ is $\mathcal{Z}(x_1 - a_1,\ x_2 - a_2,\ \ldots,\ x_n - a_n)$. More generally, any finite subset of $\mathbb{A}^n$ is an affine algebraic set.

(3) One may define lines, planes, etc. in $\mathbb{A}^n$ — these are *linear algebraic sets*, the loci of sets of linear (degree 1) polynomials of $k[x_1, \ldots, x_n]$. For example, a line in $\mathbb{A}^2$ is defined by an equation $ax + by = c$ (which is the locus of the polynomial $f(x, y) = ax + by - c \in k[x, y]$). A line in $\mathbb{A}^3$ is the locus of two linear polynomials of $k[x, y, z]$ that are not multiples of each other. In particular, the coordinate axes, coordinate planes, etc. in $\mathbb{A}^n$ are all affine algebraic sets. For instance, the x-axis in $\mathbb{A}^3$ is the zero set $\mathcal{Z}(y, z)$ and the x,y plane is the zero set $\mathcal{Z}(z)$.

(4) In general the algebraic set $\mathcal{Z}(f)$ of a nonconstant polynomial f is called a *hypersurface* in $\mathbb{A}^n$. Conic sections are familiar algebraic sets in the Euclidean plane $\mathbb{R}^2$. For example, the locus of $y - x^2$ is the parabola $y = x^2$, the locus of $x^2 + y^2 - 1$ is the unit circle, and $\mathcal{Z}(xy - 1)$ is the hyperbola $y = 1/x$. The x- and y-axes are the algebraic sets $\mathcal{Z}(y)$ and $\mathcal{Z}(x)$ respectively. Likewise, quadric surfaces such as the ellipsoid defined by the equation $x^2 + \dfrac{y^2}{4} + \dfrac{z^2}{9} = 1$ are affine algebraic sets in $\mathbb{R}^3$.

We leave as exercises the straightforward verification of the following properties of affine algebraic sets. Let S and T be subsets of $k[\mathbb{A}^n]$.

(1) If $S \subseteq T$ then $\mathcal{Z}(T) \subseteq \mathcal{Z}(S)$ (i.e., $\mathcal{Z}$ is inclusion reversing or *contravariant*).

(2) $\mathcal{Z}(S) = \mathcal{Z}(I)$, where $I = (S)$ is the ideal in $k[\mathbb{A}^n]$ generated by the subset S.

(3) The intersection of two affine algebraic sets is again an affine algebraic set, in fact $\mathcal{Z}(S) \cap \mathcal{Z}(T) = \mathcal{Z}(S \cup T)$. More generally an arbitrary intersection of affine algebraic sets is an algebraic set: if $\{S_j\}$ is any collection of subsets of $k[\mathbb{A}^n]$, then

$$\cap \mathcal{Z}(S_j) = \mathcal{Z}(\cup S_j).$$

(4) The union of two affine algebraic sets is again an affine algebraic set, in fact $\mathcal{Z}(I) \cup \mathcal{Z}(J) = \mathcal{Z}(IJ)$, where I and J are ideals and IJ is their product.

/**(5)** $\mathcal{Z}(0) = \mathbb{A}^n$ and $\mathcal{Z}(1) = \emptyset$ (here 0 and 1 denote constant functions).

By (2), every affine algebraic set is the algebraic set corresponding to an *ideal* of the coordinate ring. Thus we may consider

$$\mathcal{Z} : \{\text{ ideals of } k[\mathbb{A}^n]\,\} \to \{\text{ affine algebraic sets in } \mathbb{A}^n\,\}.$$

Since every ideal I in the Noetherian ring $k[x_1, x_2, \dots, x_n]$ is finitely generated, say $I = (f_1, f_2, \dots, f_q)$, it follows from (3) that $\mathcal{Z}(I) = \mathcal{Z}(f_1) \cap \mathcal{Z}(f_2) \cap \cdots \cap \mathcal{Z}(f_q)$, i.e., *each affine algebraic set is the intersection of a finite number of hypersurfaces in* $\mathbb{A}^n$. Note that this "geometric" property in affine n-space is a consequence of an "algebraic" property of the corresponding coordinate ring (namely, Hilbert's Basis Theorem).

If V is an algebraic set in affine n-space, then there may be many ideals I such that $V = \mathcal{Z}(I)$. For example, in affine 2-space over $\mathbb{R}$ the y-axis is the locus of the ideal (x) of $\mathbb{R}[x, y]$, and also the locus of (x^2), (x^3), etc. More generally, the zeros of any polynomial are the same as the zeros of all its positive powers, and it follows that $\mathcal{Z}(I) = \mathcal{Z}(I^k)$ for all $k \geq 1$. We shall study the relationship between ideals that determine the same affine algebraic set in the next section when we discuss radicals of ideals.

While the ideal whose locus determines a particular algebraic set V is not unique, there is a unique largest ideal that determines V, given by the set of *all* polynomials that vanish on V. In general, for any subset A of $\mathbb{A}^n$ define

$$\mathcal{I}(A) = \{f \in k[x_1, \dots, x_n] \mid f(a_1, a_2, \dots, a_n) = 0 \text{ for all } (a_1, a_2, \dots, a_n) \in A\}.$$

It is immediate that $\mathcal{I}(A)$ is an *ideal*, and is the unique largest ideal of functions that are identically zero on A. This defines a correspondence

$$\mathcal{I} : \{\text{ subsets in } \mathbb{A}^n \} \to \{\text{ ideals of } k[\mathbb{A}^n] \}.$$

Examples

(1) In the Euclidean plane, $\mathcal{I}$(the x-axis) is the ideal generated by y in the coordinate ring $\mathbb{R}[x, y]$.

(2) Over any field k, the ideal of functions vanishing at $(a_1, a_2, \dots, a_n) \in \mathbb{A}^n$ is a maximal ideal since it is the kernel of the surjective ring homomorphism from $k[x_1, \dots, x_n]$ to the field k given by evaluation at $(a_1, a_2, \dots, a_n)$. It follows that

$$\mathcal{I}((a_1, a_2, \dots, a_n)) = (x_1 - a_1,\ x_2 - a_2,\ \dots,\ x_n - a_n).$$

(3) Let $V = \mathcal{Z}(x^3 - y^2)$ in $\mathbb{A}^2$. If $(a, b) \in \mathbb{A}^2$ is an element of V then $a^3 = b^2$. If $a \neq 0$, then also $b \neq 0$ and we can write $a = (b/a)^2$, $b = (b/a)^3$. It follows that V is the set $\{(a^2, a^3) \mid a \in k\}$. For any polynomial $f(x, y) \in k[x, y]$ we can write $f(x, y) = f_0(x) + f_1(x)y + (x^3 - y^2)g(x, y)$. For $f(x, y) \in \mathcal{I}(V)$, i.e., $f(a^2, a^3) = 0$ for all $a \in k$, it follows that $f_0(a^2) + f_1(a^2)a^3 = 0$ for all $a \in k$. If $f_0(x) = a_r x^r + \cdots + a_0$ and $f_1(x) = b_s x^s + \cdots + b_0$ then

$$f_0(x^2) + x^3 f_1(x^2) = (a_r x^{2r} + \cdots + a_0) + (b_s x^{2s+3} + \cdots + b_0 x^3)$$

and this polynomial is 0 for every $a \in k$. If k is infinite, this polynomial has infinitely many zeros, which can happen only if all of the coefficients are zero. The coefficients of the terms of even degree are the coefficients of $f_0(x)$ and the coefficients of the terms of odd degree are the coefficients of $f_1(x)$, so it follows that $f_0(x)$ and $f_1(x)$ are both 0. It follows that $f(x, y) = (x^3 - y^2)g(x, y)$, and so

$$\mathcal{I}(V) = (x^3 - y^2) \subset k[x, y].$$

If k is finite, however, there may be elements in $\mathcal{I}(V)$ not lying in the ideal $(x^3 - y^2)$. For example, if $k = \mathbb{F}_2$, then V is simply the set $\{(0, 0), (1, 1)\}$ and so $\mathcal{I}(V)$ contains the polynomial $x(x - 1)$ (cf. Exercise 15).