The Solution of the Quadratic Equation by Differentiation Method

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1. Introduction

The quadratic polynomial of the form $ax^2 + bx + c, a \neq 0$ (where a, b, c is real numbers). When we equate this polynomial to zero. Then we get a quadratic equation.[1]

2. Quadratic Equations

A quadratic equation in the variable x is an equation of the form $ax^2 + bx + c = 0$, where a, b, c are real numbers, $a \neq 0$.[1]

For Example: $2x^2 - 3x + 1 = 0, x^2 + 5x + 6 = 0$ and $12 + 7x + x^2 = 0$ are also quadratic equations.

Infact, any equation of the form P(x) =0, where P(x) is a polynomial of degree 2, is a quadratic equation but when we write the terms of p(x) in descending order of their degrees, then we get the standard form of the equation.[1]

That is, $ax^2 + bx + c = 0$, $a \neq 0$ is called the **standard form of a quadratic equation.**

3. Solution of a Quadratic Equation

In general, a real number k is called a root of the quadratic equation $ax^2 + bx + c = 0, a \neq 0$ if $ak^2 + bk + c = 0$. We also say that x=k is a solution of the quadratic equation or that k satisfies the quadratic equation.[1]

Any quadratic equation can have atmost two roots.

The methods of obtaining the roots of a quadratic equation:

a) The factorisation method
b) The method of completing the square
c) The Quadratic formula[2]

4. Nature of roots

A quadratic equation $ax^2 + bx + c = 0$ has real roots or complex roots.

Discriminant $(\mathrm{D}) = \mathrm{b}^2 - 4\mathrm{ac}$

i) $\mathrm{b}^2 - 4\mathrm{ac} > 0$, then two distinct real roots.
ii) $\mathrm{b}^2 - 4\mathrm{ac} = 0$, then two equal real roots.
iii) $\mathrm{b}^2 - 4\mathrm{ac} < 0$, then no real roots (complex roots)[3]

5. Purpose of Study

5.1. Find the roots of the quadratic equation by Differentiation Method

A quadratic equation $ax^2 + bx + c = 0, a \neq 0$

Differentiating with respect to x

$\Rightarrow$ $$2ax + b = 0$$

$\Rightarrow$ $$2ax = -\mathrm{b}$$

$\Rightarrow$ $$x = \frac{-b}{2a}$$

Adding and subtracting $\sqrt{\mathrm{D}}\ \left(\sqrt{b^2 - 4ac}\right)$ from numerator$(-\mathrm{b})$

$\therefore$ $$x = \frac{-b \pm \sqrt{D}}{2a}$$

$$x = \frac{1}{2a}\left[-b \pm \sqrt{b^2 - 4ac}\right]$$

$\therefore$ The roots of the equation are $\dfrac{-b + \sqrt{D}}{2a}$ *and* $\dfrac{-b - \sqrt{D}}{2a}$, (Here $\mathrm{D} = b^2 - 4ac$)

Example 1:

Find the roots of the equation $2x^2 - 5x + 3 = 0$ by differentiation method.

Solution:

we have $2x^2 - 5x + 3 = 0$ (1)

Here a = 2, b = -5, c = 3

D= $b^2 - 4ac = (-5)^2 - 4 \times 2 \times 3$

$= 25 - 24$

$= 1 > 0$

So given equation has two distinct real roots.

Now, Differentiate equation (1) With respect to x

$$\frac{d}{dx}[2x^2 - 5x + 3] = 0$$

$\Rightarrow \quad 4x - 5 = 0$

$\Rightarrow \quad 4x = 5$

$\Rightarrow \quad x = \frac{5}{4}$ (2)

Adding and subtracting $\sqrt{D}$ from numerator, in (2)

$\therefore \quad x = \frac{5 \pm \sqrt{D}}{4}$

$\Rightarrow \quad x = \frac{5 \pm \sqrt{1}}{4}$

$\Rightarrow \quad x = \frac{5 \pm 1}{4}$

Taking +ve Sign, $x = \frac{5+1}{4} = \frac{6}{4} = \frac{3}{2}$

Taking –ve Sign, $x = \frac{5-1}{4} = \frac{4}{4} = 1$

Hence the roots are 1 and $\frac{3}{2}$.

Example 2:

Find the roots of the equation $3x^2 - 6x + 2 = 0$ by differentiation method.

Solution:

we have $3x^2 - 6x + 2 = 0$ (1)

Here a = 3, b = -6, c = 2

$D = b^2 - 4ac = (-6)^2 - 4 \times 3 \times 2$

$= 36 - 24$

$= 12 > 0$

So given Equation has two distinct real roots.

Differentiating equation (1) With respect to x.

$$\frac{d}{dx}[3x^2 - 6x + 2] = \frac{d}{dx}(0)$$

$\Rightarrow \quad 6x - 6 = 0$

$\Rightarrow \quad 6x = 6$

$\Rightarrow \quad x = \frac{6}{6}$

$\therefore$ Adding and subtracting $\sqrt{12}$ from numerator

$\Rightarrow \quad x = \frac{6 \pm \sqrt{12}}{6}$

$\Rightarrow \qquad x = \frac{6\pm2\sqrt{3}}{6}$

$\Rightarrow \qquad x = \frac{2(3\pm\sqrt{3})}{6}$

$\Rightarrow \qquad x = \frac{3\pm\sqrt{3}}{3}$

Taking $+ve\ sign, \quad x = \frac{3+\sqrt{3}}{3}$

Taking $-ve\ sign, \quad x = \frac{3-\sqrt{3}}{3}$

Hence the roots are $\frac{3-\sqrt{3}}{3}$ and $\frac{3+\sqrt{3}}{3}$.

Example 3:

Find the roots of the equation $2x^2 - 2\sqrt{2}x + 1 = 0$ by differentiation method.

Solution:

Given the equation

$$2x^2 - 2\sqrt{2}x + 1 = 0 \qquad (1)$$

Here $\mathrm{a} = 2, \mathrm{b} = -2\sqrt{2}, \mathrm{c} = 1$

$\therefore \quad D = b^2 - 4ac$

$= (-2\sqrt{2})^2 - 4 \times 2 \times 1$

$= 4 \times 2 - 8$

$= 8 - 8$

$= 0$

So, given quadratic equation has two equal real roots.

Differentiate (1.) With respect to x

$$\frac{d}{dx}\left[2x^2 - 2\sqrt{2} + 1\right] = \frac{d}{dx}(0)$$

$\Rightarrow \qquad 4x - 2\sqrt{2} = 0$

$\Rightarrow \qquad 4x = 2\sqrt{2}$

$\Rightarrow \qquad x = \frac{2\sqrt{2}}{4}$

$\Rightarrow \qquad x = \frac{\sqrt{2}}{2}$

$\Rightarrow \qquad x = \frac{1}{\sqrt{2}}$

Adding and subtracting $\sqrt{\mathrm{D}}$ from numerator.

$\therefore \qquad x = \frac{1 \pm \sqrt{D}}{\sqrt{2}}$

$\Rightarrow \qquad x = \frac{1 \pm \sqrt{0}}{\sqrt{2}} = \frac{1 \pm 0}{\sqrt{2}}$

Taking positive sign, $x = \frac{1+0}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

Taking negative sign, $x = \frac{1-0}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

Hence the roots are $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$.

Example 4:

Find the roots of the equation $x^2 + x + 1 = 0$ by differentiation method.

Solution:

We have $x^2 + x + 1 = 0 \qquad (1)$

Here $\mathrm{a} = 1, \mathrm{b} = 1, \mathrm{c} = 1$

$\therefore \; D = b^2 - 4ac$

$= (1)^2 - 4 \times 1 \times 1$

$= 1 - 4$

$= -3 < 0$

So given Equation has two distinct complex roots.

Now, Differentiating equation (1) With respect to x.

$$\frac{d}{dx}[x^2 + x + 1] = 0$$

$\Rightarrow \qquad 2x + 1 = 0$

$\Rightarrow \qquad 2x = -1$

$\Rightarrow \qquad x = \frac{-1}{2} \qquad (2)$

Adding and subtracting $\sqrt{\mathrm{D}}$, from numerator in (2)

$\therefore \qquad x = \frac{-1 \pm \sqrt{D}}{2}$

$\Rightarrow \qquad x = \frac{-1 \pm \sqrt{-3}}{2}$

$\Rightarrow \qquad x = \frac{-1 \pm \sqrt{3}\, i}{2} \qquad [\sqrt{-1} = i]$

Taking +ve sign, $x = \frac{-1+\sqrt{3}i}{2}$

Taking –ve sign $x = \frac{-1-\sqrt{3}i}{2}$

Hence the roots are $\frac{-1+\sqrt{3}\,i}{2}$ and $\frac{-1-\sqrt{3}i}{2}$.

References

[1] chapter 4 (class 10) NCERT BOOK, *Quadratic equation*.

[2] *CHAPTER 4(CLASS 10) NCERT BOOK*. 2022.

[3] CHAPTER 4(CLASS 10) NCERT BOOK, *Quadratic equation*. 2022.