DIFFUSION OF CURRENT INTO CONDUCTORS

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Currents are established on the surface of conductors by the propagation of electromagnetic waves in the insulating material between them. If the load is less than the characteristic impedance of the insulating material of the line, multiple reflections and retransmissions eventually build up the line current to that required by the load. The currents are initially established on the surface of the conductors before diffusing relatively slowly into the interior and gives rise to the skin effect. The diffusion velocity depends the conductivity, permeability, thickness of the conductor, and the frequency of the excitation, and such effects of the diffusion process are difficult to conceptually appreciate. Fortunately, the diffusion of heat into solids is very similar, and will be used as an analogy to aid understanding. This diffusion is the means whereby current moves into conductors and flux into of magnetic cores.

1. INTRODUCTION

Current flow in conductors is often associated with liquid flow in pipes, at least conceptually. The 'liquid' being the sea of free electrons, in the conduction band of the material, that drift along the conductor with a velocity that is proportional to the electric field and gives rise to the current, $J = \sigma E$. This liquid flow analogy is quite reasonable for steady dc currents, giving a conceptual understanding of Ohms law. However, while good conductors such as copper present little opposition to electrons moving at constant velocity, they strongly oppose electron accelerations, because the relatively large currents ($J=\sigma E$) produce magnetic fields that are much higher than those produced by displacement currents in free space. These high magnetic fields move at relatively low velocities since the back emfs induced by their motion cannot be any larger than the driving E field. These induced emfs generate eddy currents within the conductor, which also oppose the diffusion due to associated energy losses. These limited back emfs and eddy currents cause any changes in the surface electric field (current) to move very slowly into the interior of conductors and suffer significant.

The relatively slow velocity of penetration depends on the conductivity, and permeability of the material. The higher the conductivity, permeability, and size of the conductor, the slower is diffusion velocity. For a very large copper conductor the penetration velocity at a frequency 50Hz is approximately 8 m/s. This relatively low velocity is not very apparent in every day applications because the currents needed to energise electrical loads initially propagate along the cable (transmission line) to the load as displacement currents in the insulation, at velocities approaching *c*. The displacement current builds up the line current on the surface of the conducting cables by multiple reflections, and this current diffuses into the interior of the conductor [1]. Thus current changes (electron accelerations) actually move into the conductors from the outside surfaces and only have to diffuse through the thickness of the conductors (half the diameter) rather than along the whole length of the cable from the power source to the load. If it were not for the displacement current setting up the surface currents in the first instance, energy transmission, (other than via relatively steady dc currents) via copper conductors would be virtually impossible because of the long diffusion times and attenuations.

The skin effect results from the fact that at the particular frequency of operation the surface currents only have time to diffuse into the conductor to the skin depth in the ¼ period of the supply frequency. Surface currents move into the conductor on the rising part of the current waveform. Once the surface currents peak and start to fall, the interior currents move back out towards the surface of the conductor. At 50Hz the skin depth in copper is approx 10mm.

The same diffusion process also applies to surface magnetic flux moving into the interior of magnetic cores, since most cores are electrically conductive. Surface flux changes produce a driving H field ($\Delta H_s = \Delta B_s/\mu$) that drives any change in surface flux into interior of the core. The changing flux levels induce back emfs as they move into the core, with resulting eddy currents, whose H fields oppose the driving H field. These induced H fields, due to changing surface flux levels moving into the interior of the core cannot be any greater than the driving H field at any point in the conductor and this is the fundamental reason for the low diffusion velocity. Eddy current losses also attenuate the fields and further reduce the diffusion velocity. The phase velocity for transformer steels at a 50Hz ≈ 110mm/sec resulting in a skin depth of ≈ 0.5mm, and is the reason for laminating the core.

The velocity at which a wave penetrates into a conductive material also depends upon the thickness of the conductor and frequency of the excitation. These are difficult to appreciate but fortunately the diffusion of heat into solids is similar, and will be used to conceptually understand this diffusion.

2.0 CURRENT DIFFUSION

The high conductivity of good conductors, such as copper, provide a good medium for electrons to move at constant velocity (steady dc currents) but opposes electron acceleration. Accelerating electrons produce changing magnetic fields with resulting back emfs that oppose the acceleration. The currents in conductors (per unit of E) are very much larger than displacement currents in free space (377 ohms), as also are the resulting changes in internal magnetic flux levels as any surface changes in E move into the interior. Changes in surface current produce the surface electric field ($\Delta Es = \Delta Js/\sigma$) that drives the current changes into the interior. As these surface current changes move into the interior, back emfs are induced that can be no bigger than the driving E field. The high magnetic fields associated with the high current levels (due to low conductivity) is the fundamental reason why electric fields move very slowly into the interior of a conductor compared with free space. The velocities of these fields in free space is also limited by back emfs but in this case the much smaller magnetic fields have to move at velocity *c*. These induced emfs, due to the accelerating electrons, also create eddy currents in the material that absorb energy and further reduce the diffusion velocity due to associated losses that reduce the effective driving E field. These eddy current losses cause any changing electro-magnetic fields to be greatly attenuated as they go into the interior of a conductors.

2.1 Propagation in Conducting Materials

At any point inside the conductor following electromagnetic equations are relevant.

Conduction Current, $J = \sigma E$ ....(1a)

Flux Density, $B = mH$ ...... (1b)

Current Density, i = J + Displacement Current

$$i = sE + \frac{\partial D}{\partial t} = sE + e_o e_r \frac{\partial E}{\partial t}$$

$$Curl\ H = i = sE + \frac{\partial D}{\partial t} = sE + e_o e_r \frac{\partial E}{\partial t} \quad .... (1c)$$

For sinusoidal excitation, $\frac{\partial E}{\partial t} = jw\,E$, so

Displacement Current = $jwe_o e_r E$

& $Curl\ H = (\sigma + j\omega\varepsilon_o\varepsilon_r)E$

The displacement current ($jwe_o e_r E$) rises with frequency and for copper equals the conduction current (sE) when $\omega = \sigma/\varepsilon \approx 10^{19}$. Thus displacement currents can be neglected in comparison to the conduction current at normal frequencies so:

$$Curl H = s\ E\ .$$

If they could occur, any net like charges inside a conductor would repel each other, and quickly move to the surface with a time constant, $\tau = \varepsilon_o/\sigma \approx 10^{-19}$, for copper. Thus at normal frequencies there is virtually no charge build up inside the conductor, and the electric field inside a conductor can be considered to be divergence free ($Div\ E = 0$).

Thus at any point inside a conductor:

$$Div E = \nabla.E = 0 \quad ...... (2a)$$

$$\&\quad Curl\ H \approx J = \sigma E \quad ...... (2b)$$

$$\text{Now,}\quad Curl\ E = -\frac{\partial B}{\partial t} = -\mu\frac{\partial H}{\partial t} \quad ..... (2c)$$

$$= -jwB = -jwmH \text{ , for sinusoids}$$

$$\&\quad \nabla.H = \nabla.B = 0 \quad ...... (2d)$$

2.2 Electromagnetic Diffusion in Conductors

Taking the Curl of both sides of (2b) we get

$$Curl\ Curl\ H = s\ Curl\ E$$

Substituting (2c) we get:

$$Curl\,Curl\ H = -sm\frac{\partial H}{\partial t} \quad ..... (3a)$$

Now, $B = m\,H$ so:

$$Curl\,Curl\,B = -sm\frac{\partial B}{\partial t} \quad ...... (3b)$$

Similarly, taking the Curl of (2c) gives

$$Curl\,Curl\ E = -\frac{\partial Curl\ B}{\partial t} = -m\frac{\partial Curl\ H}{\partial t}$$

Substituting $Curl\ H = s\ E$ gives:

$$Curl Curl\ E = -m\frac{\partial J}{\partial t} = -sm\frac{\partial E}{\partial t} \quad ..... (3c)$$

Now $J = s\ E$ so:

$$Curl\,Curl\ J = -sm\frac{\partial J}{\partial t} \quad ...... (3d)$$

These 4 equations are basically the same, so (3d) can represent them all on the understanding that J can be replaced by E, H, or B.

Magnetic Diffusion Equation

Applying the vector identity

$$Curl\ Curl\ J = Grad\ Div\ J - \nabla^2 J$$

$$= -\nabla^2 J \qquad \text{, since } Div\ J = 0$$

to (4d) gives the Magnetic Diffusion Equation

$$\nabla^2 J = \sigma\mu \frac{\partial J}{\partial t} \quad \text{, or} \quad \frac{\partial J}{\partial t} = \frac{1}{sm}\nabla^2 J \qquad \text{...... (4a)}$$

where, J can be replaced by E,H, or B.

$$\textbf{\textit{Magnetic Diffusivity}},\ a = \frac{1}{sm} \qquad \text{..... (4b)}$$

The diffusivity α, and hence the penetration speed, decreases with increase in the conductivity and permeability of the material.

For copper,

$\sigma = 5.8\text{x}10^7$, $\mu = 4\pi\text{x}10^{-7}$, $\alpha = 13.7\text{x}10^{-3}\ m^2/sec$

For transformer steels,

$\sigma = 2\text{x}10^6$, $\mu = 10^4 \text{ x } 4\pi\text{x}10^{-7}$, $\alpha = 0.04\text{x}10^{-3}\ m^2/sec$

2. HEAT CONDUCTIVITY

Heat flow in solids will now be considered to gain a conceptual understanding of this diffusion.

2.1 Thermal Conductivity

Heat flows in a material from regions of high temperature to ones at lower temperature, the driving force being the temperature gradient.

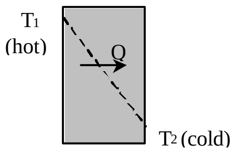


Fig 1: Steady-State Heat Flow

Fourier's Law

Heat Flux, $Q = -k\nabla T$ $\quad$ (W/m^2)

where, k = Thermal conductivity, ($W/m\ ^oC$)

$\approx 388\ W/m\ ^oC$, for copper

$\approx 62\ W/m\ ^oC$, for iron

The thermal conductance of the material = kA/L, similar to G = σA/L for electrical conductance.

In the steady state, if there are no heat sources or sinks inside the material, the temperature gradient throughout the material is constant. The heat leaving the cooler surface is equal to that flowing through the material from the hot surface and $\nabla . Q = 0$.

2.2 Thermal Diffusion

Considering a small internal elemental volume of the material, the heat flow rate equation is given by [2]:

Heat Flow In – Heat Flow Out

= Change in Internal Energy – Heat Generated

$$\therefore \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) = c\rho\frac{\partial T}{\partial t} - \frac{dq}{dt}$$

where, c = specific Heat ($J/kg\ ^oC$)

ρ = density (kg/m^3)

$$\text{Thus} \qquad k\nabla^2 T = c\rho\frac{\partial T}{\partial t} - \frac{dq}{dt}$$

$$\nabla^2 T = \frac{c\rho}{k}\frac{\partial T}{\partial t} - \frac{1}{k}\frac{dq}{dt}$$

2.2.1 Steady State Conditions

In the steady state the rate of change of internal energy is zero.

If internal energy sources exist then

$$\nabla^2 T = -\frac{1}{k}\frac{dq}{dt} \qquad \text{..... Poison's Equation}$$

If there are no internal sources then

$$\nabla^2 T = 0 \qquad \text{..... Laplace's Equation}$$

2.2.2 Unsteady (Changing) Conditions

If there are no internal sources and constant, steady state conditions have not yet been achieved then:

Net Rate of Heat Flow into an element

= Rate of increase in internal energy

Thermal Diffusion (Fourier Equation)

$$k\nabla^2 T = c\rho\frac{\partial T}{\partial t}$$

$$\nabla^2 T = \frac{c\rho}{k}\frac{\partial T}{\partial t} \text{ , or } \frac{\partial T}{\partial t} = \alpha\nabla^2 T \qquad \text{....(5)}$$

$$\textbf{\textit{Therm Diffusivity}},\ a = \frac{k}{cr} = \frac{\textit{Thermal Conductivity}}{\textit{Thermal Capacitance}}$$

$\approx 114 \text{ x } 10^{-6}\ m^2/sec$, for copper

$\approx 18 \text{ x } 10^{-6}\ m^2/sec$, for iron

The material absorbs energy as its temperature rises and so acts like a capacitance, $C = cr$. Any heat stored in the material has to flow via the conductance k, so the thermal conductivity aids diffusion while c and ρ oppose it. Thermal diffusivity is a measure of the ability of a material to propagate energy compared with its energy storage requirements. Heat flow can be modelled with a distributed RC electrical circuit.

The rate of heat flow is directly proportional to the temperature gradient, and this gradient is the driving force for the diffusive flow of heat.

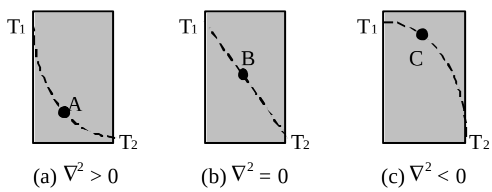


Fig2: Unsteady & Steady Heat Flows

In Fig2(a) the heat flowing into point A exceeds that flowing out so the temperature of point A rises. In

Fig2(b) the heat flowing into point B is equal to that flowing out so the temperature of B is constant. In Fig2(c) the heat flowing out of point C exceeds that flowing in, so the temperature of point C will fall.

3. COMPARISON OF DIFFUSIONS

Thermal Diffusion

At any point in the material

$$\frac{\partial T}{\partial t} = a\,\nabla^2 T \qquad \text{, where } a = \frac{k}{cr}$$

In this case T is a scalar so:

$$\nabla^2 T = \text{div of grad T} = \nabla.\nabla T$$
$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) T$$

For one-dimensional diffusion in the x direction:

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2} \qquad \text{.... (6a)}$$

Magnetic Diffusion (F = J, E, H, B, or T)

$$\frac{\partial F}{\partial t} = a\nabla^2 F \qquad \text{, where } a = \frac{1}{sm}$$

Since **F** is a vector

$$\nabla^2 F = \text{grad of div F - Curl of Curl F}$$

In rectangular co-ordinates this is equal to the vector sum of the Laplacian operation on the 3 scalar components of F.

$$\nabla^2 F = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(F_x a_x + F_y a_y + F_z a_z)$$

For one dimensional diffusion in the x direction let:

$$F = F_y a_y, \quad \text{and} \quad \frac{\partial F_y}{\partial y} = \frac{\partial F_y}{\partial z} = 0$$

$$\therefore\ \nabla^2 F = \frac{\partial^2 F_y}{\partial x^2} a_y, \quad \text{so} \quad \frac{\partial F_y}{\partial t} a_y = a\frac{\partial^2 F_y}{\partial x^2} a_y$$

$$\therefore\ \frac{\partial F_y}{\partial t} = a\frac{\partial^2 F_y}{\partial x^2} \qquad \text{.... (6b)}$$

Although the thermal (6a) and magnetic diffusion equations (6b), operate on scalar and vector fields respectively they are essentially the same. In each case the diffusive flow of the entity is due to the spatial concentration gradient of the entity itself.

Heat diffusion is opposed by the heat capacitance ($c\rho$) of the material, and aided by thermal conductivity. However, in the case of magnetic diffusion both the electrical conductivity and permeability oppose the diffusion.

Material	**Diffusivity, a (m^2/sec)**	
	Thermal	**Magnetic**
Copper	1.14×10^{-4}	137×10^{-4}
Transformer Steel	0.18×10^{-4}	0.4×10^{-4}

The thermal and electrical diffusivities of transformer steels are similar to each other and give an appreciation of the slow diffusion rates of surface flux into magnetic cores laminations. The electrical diffusivity of copper is approx 100 times its thermal. Since diffusion velocities are proportional to the $\sqrt{\alpha}$, electric fields (and hence surface currents) will diffuse into copper approx 10 times faster than heat.

These comparisons with heat give some appreciation of the impossible situation that would exist if currents had to diffuse longitudinally through the length of copper conductors instead of transversely across half their thickness. It is very fortunate that surface currents are initially established along the length of conductors at velocities around c, by means of the displacement currents that flow in the insulating medium.

4. EXAMPLES OF DIFFUSION

Irrespective of whether we consider electromagnetic or thermal diffusion the situation is described by applying the relevant boundary conditions to the generalised diffusion equation:

$$\frac{\partial F}{\partial t} = a\frac{\partial^2 F}{\partial x^2} \quad \text{, where } F = J, E, B, H \text{ or } T.$$

4.1 Semi-Infinite Slab

This is the simplest case for examining diffusion, since there is only one boundary. Although we could use J, E, B, H or T, we will use temperature (T) since it gives a conceptual understanding of this diffusion.

Consider a semi-infinite block, initially at 0^oC, whose exposed surface temperature is suddenly raised to a constant T_s of 100^oC, as indicated in Fig 3.

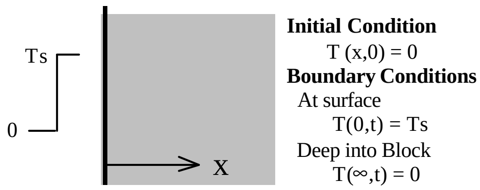


Fig 3: Step Temp at Surface of Semi-Infinite Block

As the temperature of the exposed surface rises, a temperature gradient is produced which will drive heat into the material. At any point in the block the rate of temperature rise must satisfy the diffusion equation.

$$\frac{\partial T(x,t)}{\partial t} = \alpha\nabla^2 T(x,t) = \alpha\frac{\partial^2 T(x,t)}{\partial x^2}, \quad \text{for } 0 < x < \infty$$

Taking the Laplace Transform and inserting the boundary conditions gives.

$$T(x,s) = \frac{T_s}{s} e^{-x\sqrt{s/a}} \Rightarrow T(x,t) = T_s\, erfc\left(\frac{x}{\sqrt{4at}}\right)$$

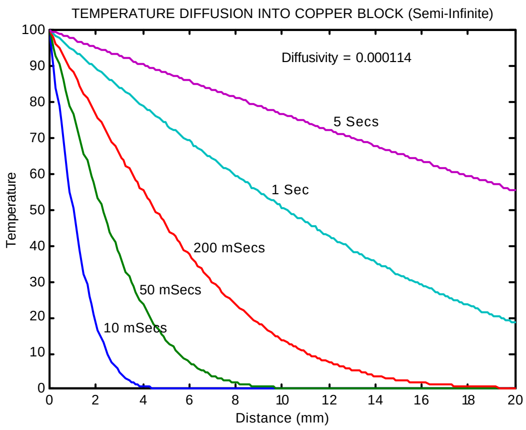
$$\text{where, } erfc(x) = 1 - erf(x) = 1 - \frac{2}{\sqrt{p}} \int_x^{\infty} e^{-y^2} .dy$$

The temperature at an internal surface, distance x into the block at time t, is given by.

$$T(x,t) = T_s\, erfc\left(\frac{x}{\sqrt{4at}}\right) \qquad \text{.... (8a)}$$

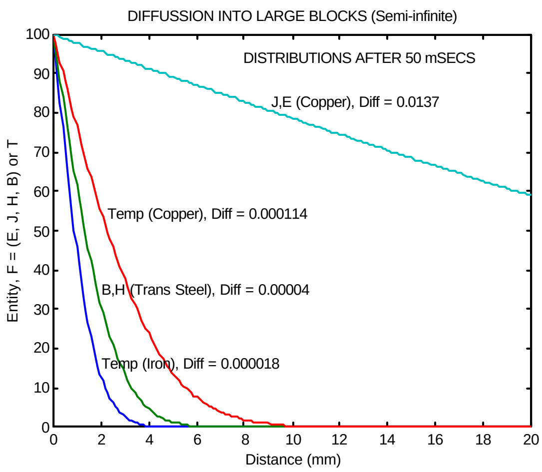
where, Ts = 100°C, $\alpha = 1.14\text{x}10^{-4}$

Temperatures in the copper block are as follows.



It can be seen that the temperature diffuses fairly slowly into the block. After 5 secs the temperature at 20mm is only 55% because material beyond 20mm still requires heat, and acts like a heat sink.

Temperature and electromagnetic diffusions into copper & steel blocks after 50msecs are shown below.



It can be seen that that electrical diffusion rates for copper are approx 10 times the thermal ones.

4.2 Semi-Infinite Plate (insulated at far surface)

Consider a plate of thickness L, whose exposed surface temperature is raised as shown below.

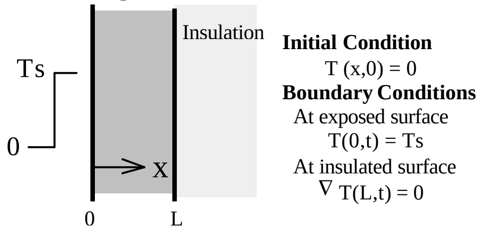


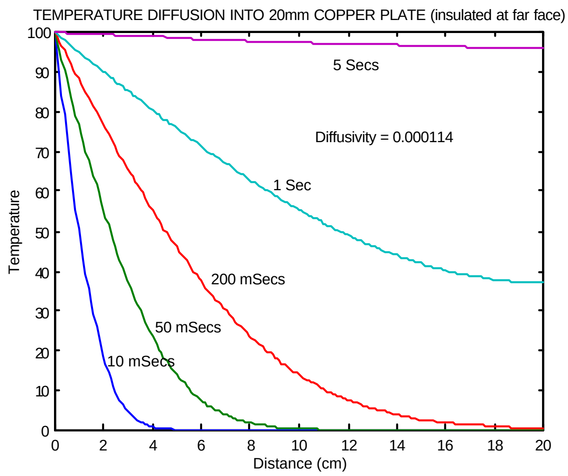
Fig 4: Step Temp at Surface of Insulated Plate

In this case the temperature gradient at the insulated face is always zero since there is no heat flow out of this surface. Using (8a), and a series of images to satisfy the boundary conditions gives:

$$T(x,t) = T_s\, erfc\left(\frac{x}{\sqrt{4at}}\right) + T_s \sum_{n=1}^{\infty} (-1)^n erfc\left(\frac{2nL + x}{\sqrt{4at}}\right)$$

$$- \sum_{n=1}^{\infty} (-1)^n erfc\left(\frac{2nL - x}{\sqrt{4at}}\right) \qquad \text{....(8b)}$$

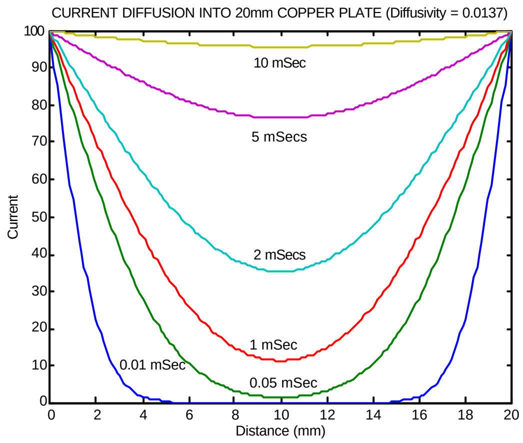
Heat diffusion into a copper plate 20mm thick is as shown below.



The temperature diffuses faster into plates since there is no heat sinking beyond the plate thickness. The end surface of the plate at 20mm rises to 96% after 5 secs. This helps to appreciate how current & flux diffusion velocities reduce with material thickness, although in these cases it is due to reductions in eddy currents.

4.3 Surface Current Diffusion into Copper Plate

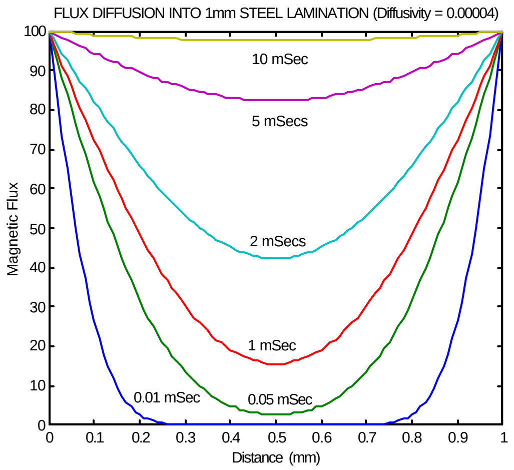
In this case surface current changes diffuse into the copper from both sides of the plate. No diffusion flows cross the centre line, that in terms of the analogous heat flow can be considered a perfectly insulating surface. With this refinement Eq (8b) can be used to graph current diffusions into copper plate copper plate 20mm thick as follows.



Centre line currents (at 10mm depth) reach approx 75% of the surface currents after 5msecs.

4.3 Surface Flux Diffusion into Laminations

The case for surface flux diffusion into a transformer steel lamination 1mm thick is indicated below.



In this case centre line flux levels (at 0.5mm depth) reach approx 85% of the surface flux after 5msecs.

4.5 Sinusoidal Excitations at Surface of Block

Consider the flow of sinusoid surface temperatures into a semi-infinite block as indicated below.

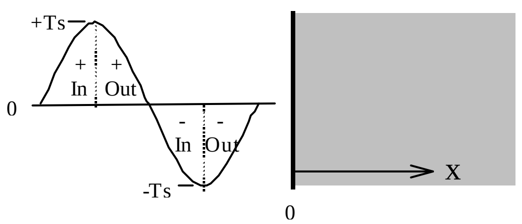


Fig 5: Sinusoidal Temp at Surface of Large Block

The diffusive flows depend upon the spatial gradient of the temperature. Heat flows into the block as the surface temperature rises and out of the block when the surface temperature falls. Positive heat flows in through the surface of the block for the first 90^o of the waveform and back out for the next 90^o. Similarly negative heat flows in and out of the block during the negative half cycle of the surface temperature. The lower the frequency the greater is the time for the heat to penetrate into the material before it moves back out. Thus the penetration depth reduces with increase in frequency. The rate of change of surface temperature increases with frequency and so do the spatial temperature gradients in the material. As gradients are the driving force for this diffusive flow, the diffusion velocity in the material increases with excitation frequency.

Magnetic Diffusion with Sinusoids.

Whether it is surface current or flux that diffuses into a conductive material, both E & H fields are produced whose amplitudes decrease as they go further in [3].

$$E = E_s e^{-\frac{x}{\delta}} Sin(\omega t - x/\delta) \qquad \text{.... (9a)}$$

$$H = H_s e^{-\frac{x}{\delta}} Sin(\omega t - x/\delta - \pi/4) \qquad \text{.... (9b)}$$

where, $\textbf{\textit{Skin Depth}}, \delta = \sqrt{\frac{2\alpha}{\omega}}$ .... (9c)

& $\textbf{\textit{Phase Velocity}}, u = wd$ .... (9d)

The amplitude of these wave is attenuated by 1/e = 0.37 in one radian length, δ, of the propagating wave. The magnetic field H in conductors lags the electric field E by 45^0 due to the magnetic fields resulting from conduction rather than displacement currents losses. E & H fields in non-conductive materials are in phase .

Diffusion Type	**Diffus** a	**SkDepth** d	**Ph Vel *u***
Copper –Elect	137×10^{-4}	9.34 mm	2.93m/s
Copper -Therm	1.14×10^{-4}	0.85mm	0.27m/s
TraSteel-Mag	0.40×10^{-4}	0.51mm	0.16m/s
TraSteel -Therm	0.18×10^{-4}	0.34mm	0.11m/s

5. CONCLUSIONS

Liquid flow in pipes gives a good realistic analogy only for constant dc current flows in conductors since good conductors are a hostile medium for electrons to accelerate in. All changes have to diffuse in and out of conductors through their longitudinal surfaces, and a good analogy for this is heat flow in solids.

References

[1] J.Edwards,T.K.Saha,"Establishment of Current in Electrical Cables via Electromagnetic Energies & the Poyting Vector", AUPEC'98 ,Vol2 385-388, Sept 98.
[2] F.Kreith,"Principles of Heat Transfer", 1973.
[3] P.Lorrain, D.Corson,"Electromagnetic Fields and Waves",2nd ed, pp 471-481, Freeman, 1970.