Take a 2 by 1 rectangle in the $y - z$ plane centered at the origin and so that the longer edge is parallel to the $z-$axis as in fig 1.

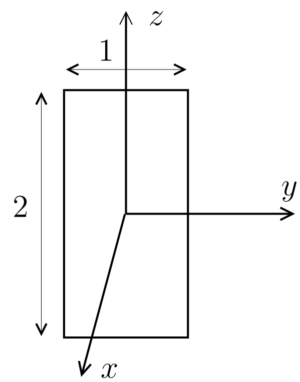

Figure 1.1: Unrotated rectangle.

We rotate the rectangle about the $z-$axis by 60 degrees and then about the $y-$axis by 60 degrees to try halve the width and length, and find the projective length (length projected onto the $y - z$ plane) of each of the diagonals. For this purpose we consider the vectors $\vec{A}, \vec{B}, \vec{C}$ and $\vec{D}$ that point from the origin to the 4 corners of the unrotated rectangle - fig 2.

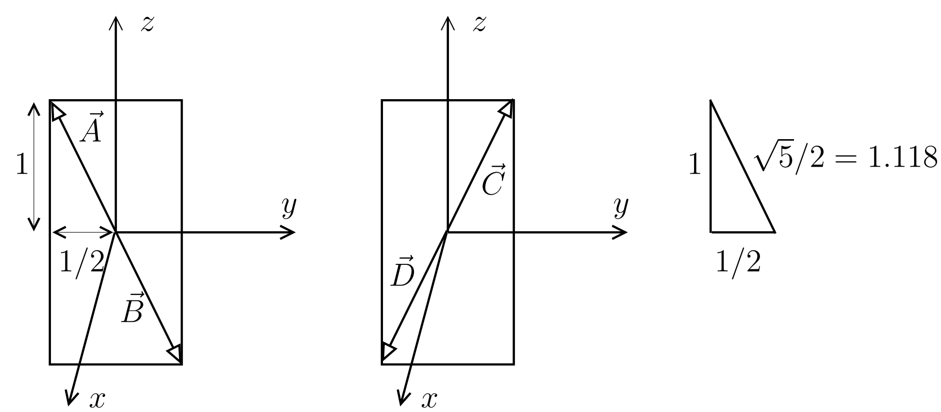

Figure 1.2: Vectors $\vec{A}, \vec{B}, \vec{C}$ and $\vec{D}$ of unrotated rectangle.

We note that the length of each of these vectors is $\sqrt{5}/2 = 1.118$.

We use a direction of rotation about the $z-$axis that moves the left hand edge of the rectangle toward you and a direction of rotation about the $y-$axis that moves the top of the rectangles away from you.

Projective length of longer diagonal

We wish to find the projective length of $\vec{A}$ after rotating it about the $y-$axis by 60 degrees and rotating it about the $z-$axis by 60 degrees (the projective length of a vector is found by disregarding the x component and using Pythagoras on the y and z components).

To start with, obviously

$$\vec{A} = -\frac{1}{2}\hat{\jmath} + \hat{k}.$$

We denote by $\vec{A}_W$ the vector we obtain after rotating $\vec{A}$ about the $z-$axis by 60 degrees. The y component will be $\frac{1}{2}\cos 60 = \frac{1}{2}\frac{1}{2}$. The x component will be $\frac{1}{2}\cos 30 = \frac{1}{2}\frac{\sqrt{3}}{2}$. So $\vec{A}_W$ will be

$$\vec{A}_W = \frac{\sqrt{3}}{4}\hat{\imath} - \frac{1}{4}\hat{\jmath} + \hat{k}.$$

We now consider the rotation about the $y-$axis. We denote by $\vec{A}'$ the resulting vector (we will only need the y and z component of this to calculate the projective length). A rotation about the $y-$axis wont change the y component, so we need only work out the z component. First note that when rotating about the $y-$axis the vector $\vec{A}_W$ will trace out a circle parallel to the $x-z$ plane and centered at $(0, -1/4, 0)$. To start with the vector $\vec{A}_W$ points to the point p on the circle (see fig 3). This is at an angle ϕ along the circle away from the horizontal.

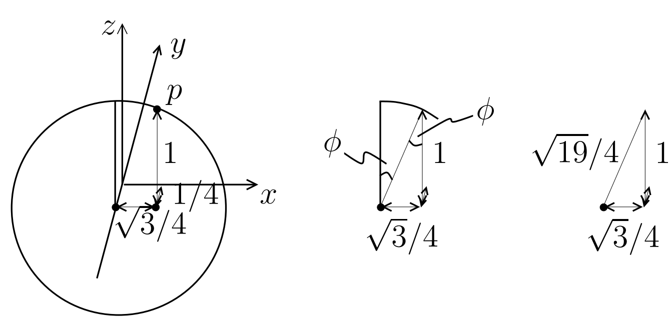

Figure 1.3: Viewing the circle from the side.

The radius of the circle is

$$R = \sqrt{1 + (\sqrt{3}/4)^2} = \frac{\sqrt{19}}{4}$$

The angle ϕ satisfies

$$\tan\phi = \frac{\sqrt{3}/4}{1}.$$

So

$$\phi = 23.4.$$

We now rotate by 60 degrees about the $y-$axis (this rotation will be in the opposite direction to ϕ) so the z component of $\vec{A'}$ is given by

$$\frac{\sqrt{19}}{4}\cos(60 - 23.4) = 0.875$$

So

$$\vec{A'} = \ldots\hat{\imath} - \frac{1}{4}\hat{\jmath} + 0.875\hat{k}.$$

The projective length is then

$$\sqrt{(1/4)^2 + 0.875^2} = 0.91.$$

By symmetry considerations the vector $\vec{B}$ rotated will have the same projective length. The ratio of projective length and actual length of rectangle diagonal is

$$\frac{2 \times 0.91}{2 \times 1.118} = 0.81.$$

So the longer projective diagonal is 81% the length of the actual diagonal length of the rectangle.

Projective length of shorter diagonal

Obviously

$$\vec{C} = \frac{1}{2}\hat{\jmath} + \hat{k}.$$

We denote by $\vec{C}_W$ the vector we obtain after rotating $\vec{C}$ about the $z-$axis by 60 degrees. The y component will be $\frac{1}{2}\cos 60 = \frac{1}{2}\frac{1}{2}$. The x component will be $-\frac{1}{2}\cos 30 = -\frac{1}{2}\frac{\sqrt{3}}{2}$. So $\vec{C}_W$ will be

$$\vec{C}_W = -\frac{\sqrt{3}}{4}\hat{\imath} + \frac{1}{4}\hat{\jmath} + \hat{k}$$

We now consider the effect of the rotation about the $y-$axis. We denote by $\vec{C}'$ the resulting vector. Again we dont need the $x-$component, and a rotation about the $y-$axis wont change the $y-$component, so we need only work out the z component. Note that when rotating about the $y-$axis the vector $\vec{C}_W$ will trace out a circle centered at $(0, 1/4, 0)$. It has the same radius as the previous circle and the angle ϕ has the same magnitude as before but now is in the opposite direction (see fig 4).

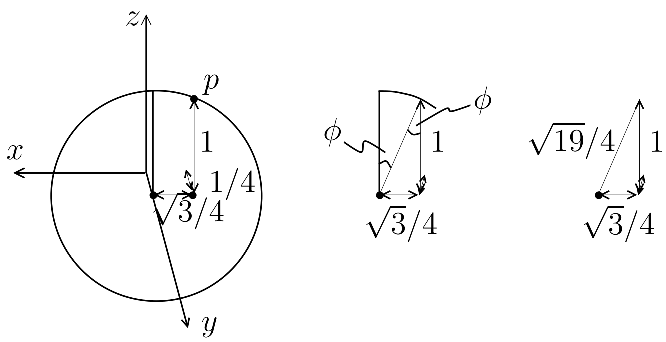

Figure 1.4: Viewing the circle from the side. The angle ϕ is same magnitude as before but in opposite direction.

We now do the rotation about the y-axis (this time it will be in the same direction as ϕ). The z component of $\vec{C}'$ is then given by

$$\frac{\sqrt{19}}{4}\cos(60 + 23.4) = 0.125$$

So

$$\vec{C}' = \ldots\hat{\imath} + \frac{1}{4}\hat{\jmath} + 0.125\hat{k}.$$

The projective length is then

$$\sqrt{(1/4)^2 + 0.125^2} = 0.28.$$

By symmetry considerations the vector $\vec{D}$ rotated will have the same projective length. The ratio of projective length and actual length of rectangle diagonal is

$$\frac{2 \times 0.28}{2 \times 1.118} = 0.25.$$

So the shorter projective diagonal is 25% the length of the actual diagonal length of the rectangle.

And the longer diagonal is 3.24 times longer than the shorter diagonal.