I want to determine the distribution that a random variable is below an estimate quantile when estimated with a limited number of observations. The quantile of interest is the α-quantile. It is assumed that $X_t \sim N(\mu, \sigma^2)$ and $X_1, ..X_T$ are drawn from a random sample. This α-quantile is estimated by in following way:

1. draw T observations from $N(\mu, \sigma^2)$
2. determine k = roundup($\alpha \cdot T$)
3. determine the estimated quantile by taking out the k^{th} lowest value out of the T observations, this value is the estimated quantile and is denoted by $X_{k:T}$.

For example, for 525 observations and $\alpha = 0.01$, the α-quantile is estimated by taking out the 6^{th} lowest loss. But since $6/525 = 0.0114 \neq 0.01 = \alpha$, there is an estimation error.

The goal is to find following probability $Pr(X_t < X_{k:T})$. According to Bain and Engelhardt (1991, p. 244) if $\lim_{T\to\infty} \frac{k}{T} = \alpha$ (and for a continuous distribution with pdf f(x) that is continuous and nonzero at the α-quantile) then the following holds:

$$\sqrt{T}(X_{k:T} - X_\alpha) \sim N(0, c^2/T), \tag{1}$$

where $c^2 = \frac{\alpha(1-\alpha)}{[f(x_\alpha)]^2}$ and $f(x)$ is the normal probability density function. Then, the probability of a VaR exceedance equals:

$$Pr(X_t < X_{k:T}) = Pr(X_t - X_{k:T} < 0) = Pr(Q < 0), \tag{2}$$

To solve above equation, the distribution of Q has to be derived:

$$Q \sim N(\mu, \sigma^2) - N(X_\alpha, c^2/T^2) = N(\mu - X_\alpha, \sigma^2 + c^2/T^2) \tag{3}$$

So, the probability is computed by the following (by using the Central Limit Theorem):

$$Pr(Q < 0) = Pr\Big(\frac{X_t - X_{k:T} - \mu + X_\alpha}{\sqrt{\sigma^2 + c^2/T^2}} < \frac{-\mu + X_\alpha}{\sqrt{\sigma^2 + c^2/T^2}}\Big) = \Phi_{0,1}\Big(\frac{-\mu + X_\alpha}{\sqrt{\sigma^2 + c^2/T^2}}\Big) \tag{4}$$

But in really, μ and σ^2 are unknown. But the distributions of these parameters are known:

$$\sqrt{T}(\hat{\mu} - \mu) \sim N(0, \sigma^2) \tag{5}$$
$$\sqrt{T}(\hat{\sigma} - \sigma) \sim N(0, 2\sigma^4) \tag{6}$$

The questions:
- Is above derivation correct?
- How to determine probability $Pr(X_t < X_{k:T})$ with the distributions $\hat{\mu}$ and $\hat{\sigma}$?.