Derivation of Faraday's law of induction

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Faraday's law of induction is formulated as $\mathcal{E} = -\frac{d\Phi_B}{dt}$ where $\mathcal{E}$ is the EMF (ElectroMotive Force) applied on the closed loop and

$$\Phi_B = \int_{\Sigma(t)} \vec{B}(t) \cdot d\vec{S} \quad (1).$$

is the magnetic flux through the surface $\Sigma(t)$ surrounded by the loop. **EMF is defined as a work done per a unit charge that travels one round trip of the loop, so the Faraday's law says that there must be some field appliable on the charge when the magnetic flux changes over time.** Φ_B changes over time due to two reasons: (a) $\vec{B}(t)$, magnetic field, changes over time, or (b) a surface of the integral, $\Sigma(t)$, changes over time. So, $\frac{d\Phi_B}{dt}$ can be expressed as a superposition of these two contributions

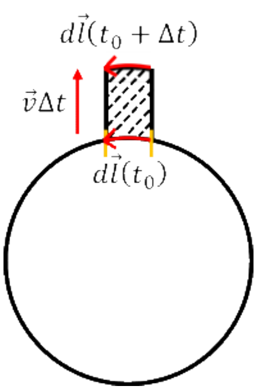
$$\left.\frac{d\Phi_B}{dt}\right|_{t=t_0} = \left(\int_{\Sigma(t_0)} \left.\frac{\partial \vec{B}}{\partial t}\right|_{t=t_0} \cdot d\vec{S}\right) + \left(\frac{d}{dt}\int_{\Sigma(t)} \vec{B}(t_0) \cdot d\vec{S}\right|_{t_0}\right) \quad (2).$$

1st term in RHS (Right Hand Side) is rewritten using Maxwell-Faraday equation as

$$\int \vec{E} \cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} \quad (3).$$

$\vec{E}$ here is a non-conservative electric field accompanied by time-varying magnetic field $\vec{B}(t)$. LHS (Left Hand Side) in this equation is a **line integral of $\vec{E}$ along the closed loop and is called a transformer EMF**.

2nd term in RHS is treated as the following. Let's look at deformation of the loop.



The deformation of an object is resulted from translation of each infinitesimally small segment of the object at its own displacement vector. Translation of whole object is just a special case of the deformation where all displacement vectors are same. The picture above shows a displacement of a line segment vector $d\vec{l}$, resulting in a local area change of the loop described by dashed-lines. We define an area vector for this new area as $\vec{v}\Delta t \times d\vec{l}(t_0)$. As a result, we have an additional magnetic flux of $\vec{B} \cdot (\vec{v}\Delta t \times d\vec{l}) = -\Delta t d\vec{l} \cdot (\vec{v} \times \vec{B})$ at $t_0 + \Delta t$. Total magnetic flux addition is obviously $-\int \Delta t d\vec{l} \cdot (\vec{v} \times \vec{B})$. So,

$$\left.\frac{d}{dt}\int_{\Sigma(t)} \vec{B}(t_0) \cdot d\vec{S}\right|_{t_0} = \lim_{\Delta t \to 0} \frac{-\int \Delta t d\vec{l} \cdot (\vec{v} \times \vec{B})}{\Delta t} = -\int d\vec{l} \cdot (\vec{v} \times \vec{B}) \quad (4).$$

$\int \boldsymbol{d\vec{l}} \cdot (\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}})$ **is called a motional EMF as it is from the motion of the loop**. $\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}$ **is just a magnetic Lorentz force per a unit charge arisen from a motion of each segment of a conducting loop with** $\vec{\boldsymbol{v}}$**.** As the segment moves with $\vec{v}$, charges in the segment also moves with $\vec{v}$, so the Lorentz force applies on the charges.

Substitution of equations (2), (3), and (4) into the equation of the Faraday's law of induction $\mathcal{E} = -\frac{d\Phi_B}{dt}$ gives

$\mathcal{E} = \int \vec{E} \cdot d\vec{l} + \int (\vec{v} \times \vec{B}) \cdot d\vec{l} = \int (\vec{E} + \vec{v} \times \vec{B}) \cdot d\vec{l}$, so the Faraday equation becomes

$$\mathcal{E} = \int (\vec{E} + \vec{v} \times \vec{B}) \cdot d\vec{l} = -\frac{d\Phi_B}{dt} \quad (5).$$

The change of the magnetic flux Φ_B through the closed loop is accompanied with EMF $\mathcal{E}$. $\vec{E}$ is from time-varying $\vec{B}(t)$ while $\vec{v} \times \vec{B}$ is from the deformation of the loop over time. $\vec{E} + \vec{v} \times \vec{B}$ is Lorentz force per a unit charge on the loop, so **EMF in the Faraday equation is the work done per a unit charge that travels one round trip of the loop by the Lorentz force, which is accompanied by time-varying magnetic flux through the loop.**

Reference

1. https://en.wikipedia.org/wiki/Faraday_paradox