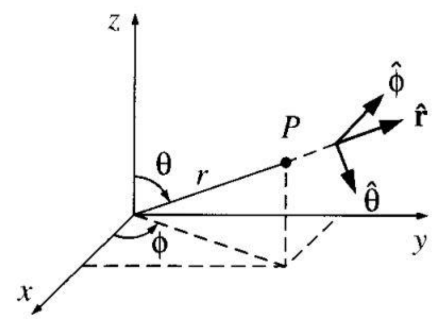
Derivation of the Laplacian in Spherical Coordinates from First Principles.

First, let me state that the inspiration to do this came from David Griffiths "Introduction to Electrodynamics" textbook Chapter 1, Section 4. I have also borrowed Figure 1 and the first two sets of equations from that textbook for reference and to give us a starting place to build off of. These are what I am calling the "First" Principles:

Figure 1



$$x = r \cdot \sin(\theta)\, cos(\phi) \qquad y = r \cdot \sin(\theta) \sin(\phi) \qquad z = r \cdot cos(\theta) \qquad Eq. 1$$

$$r = \sqrt{x^2 + y^2 + z^2} \qquad \theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right) \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right) \qquad Eq. 2$$

We will begin by deriving a very general expression before returning to Equations 1 and 2 to determine the specifics.

First, derive the Laplacian in Cartesian Coordinates:

$$\nabla \cdot \left(\nabla T_{x,y,z}\right) = \nabla^2 T_{(x,y,z)} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \qquad Eq. 3$$

Now, substitute:

$$T_{(x,y,z)} = G_{(r,\theta,\phi)} \qquad Eq. 4$$

And work out each of the Cartesian 2nd Derivatives in the new variables. We shall do this one at a time. This can get rather complicated so I will try and do things piecemeal so the process is obvious. I will begin with the "x" component first.

The spherical variables of "G", which are "r", "θ", and "ϕ", are all equations of "x," "y," and "z", so we must use the chain rule.

$$\frac{\partial}{\partial x}\left(G_{(r,\theta,\phi)}\right) = \frac{\partial G}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial G}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}+\frac{\partial G}{\partial \phi}\cdot\frac{\partial \phi}{\partial x} \qquad Eq.5$$

So:

$$\frac{\partial^2 G}{\partial x^2} = \frac{\partial}{\partial x}\cdot\left(\frac{\partial G}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial G}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}+\frac{\partial G}{\partial \phi}\cdot\frac{\partial \phi}{\partial x}\right) \qquad Eq.6$$

Now distribute:

$$\frac{\partial^2 G}{\partial x^2} = \frac{\partial}{\partial x}\cdot\left(\frac{\partial G}{\partial r}\cdot\frac{\partial r}{\partial x}\right) + \frac{\partial}{\partial x}\cdot\left(\frac{\partial G}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}\right) + \frac{\partial}{\partial x}\cdot\left(\frac{\partial G}{\partial \phi}\cdot\frac{\partial \phi}{\partial x}\right) \qquad Eq.7$$

I will tackle these three components one at a time, starting with the "r" variable:

$$\frac{\partial}{\partial x}\cdot\left(\frac{\partial G}{\partial r}\cdot\frac{\partial r}{\partial x}\right) = \frac{\partial}{\partial x}\cdot\left(G_r\cdot\frac{\partial r}{\partial x}\right) \qquad Eq.8$$

Here I use "G" with a subscript "r" to denote the partial derivative of "G" with respect to "r". It is a bit of a mixing of notation, but I find it is easier to "see". We will need to use the Product Rule to work Equation 8 out:

$$\frac{\partial}{\partial x}\cdot\left(G_r\cdot\frac{\partial r}{\partial x}\right) = G_r\cdot\frac{\partial^2 r}{\partial x^2} + \frac{\partial r}{\partial x}\cdot\frac{\partial (G_r)}{\partial x} \qquad Eq.9$$

If you note, the last part of Equation 9 contains the partial derivative of "G" with respect to "r" taken as a partial derivative with respect to "x". The original "G" equation was:

$$G_{(r,\theta,\phi)}$$

Therefore, its derivative with respect to "r" must also be an equation of the same variables, so using the chain rule once again on this last part:

$$\frac{\partial (G_r)}{\partial x} = \frac{\partial (G_r)}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial (G_r)}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} + \frac{\partial (G_r)}{\partial \phi}\cdot\frac{\partial \phi}{\partial x} \qquad Eq.10$$

So Equation 9 becomes (Whenever I make an insert substitution I will try to use a color indicator):

$$\frac{\partial}{\partial x}\cdot\left(G_r\cdot\frac{\partial r}{\partial x}\right) = G_r\cdot\frac{\partial^2 r}{\partial x^2} + \frac{\partial r}{\partial x}\cdot\left(\frac{\partial (G_r)}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial (G_r)}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} + \frac{\partial (G_r)}{\partial \phi}\cdot\frac{\partial \phi}{\partial x}\right) \qquad Eq.11$$

This whole expression can be now be substituted back into Equation 7:

$$\frac{\partial^2 G}{\partial x^2} = G_r \cdot \frac{\partial^2 r}{\partial x^2} + \frac{\partial r}{\partial x} \cdot \left(\frac{\partial (G_r)}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial (G_r)}{\partial \theta} \cdot \frac{\partial \theta}{\partial x} + \frac{\partial (G_r)}{\partial \phi} \cdot \frac{\partial \phi}{\partial x} \right)$$
$$+ \frac{\partial}{\partial x} \cdot \left(\frac{\partial G}{\partial \theta} \cdot \frac{\partial \theta}{\partial x} \right) + \frac{\partial}{\partial x} \cdot \left(\frac{\partial G}{\partial \phi} \cdot \frac{\partial \phi}{\partial x} \right) \qquad Eq. 12$$

We now need to find the expansion of the two remaining pieces in Equation 12. So far we have found 1/6 of the total general expression and the magnitude of this conversion begins to become clear. Nevertheless, I shall carry on.

Continuing my partial derivative short hand, the "θ" portion becomes:

$$\frac{\partial}{\partial x} \cdot \left(\frac{\partial G}{\partial \theta} \cdot \frac{\partial \theta}{\partial x} \right) = \frac{\partial}{\partial x} \cdot (G_\theta \cdot \theta_x) = G_\theta \cdot \frac{\partial (\theta_x)}{\partial x} + \theta_x \cdot \frac{\partial (G_\theta)}{\partial x} \qquad Eq. 13$$

And consequently, as with Equation 10, the last bit becomes:

$$\frac{\partial (G_\theta)}{\partial x} = \frac{\partial (G_\theta)}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial (G_\theta)}{\partial \theta} \cdot \frac{\partial \theta}{\partial x} + \frac{\partial (G_\theta)}{\partial \phi} \cdot \frac{\partial \phi}{\partial x} \qquad Eq. 14$$

Putting Equation 14 into 13, putting both (purple) into Equation 12, and also converting as much as possible to my short hand notation, we now have:

$$\frac{\partial^2 G}{\partial x^2} = \big(G_r \cdot r_{xx} + r_x \cdot (G_{rr} \cdot r_x + G_{r\theta} \cdot \theta_x + G_{r\phi} \cdot \phi_x) \big)$$
$$+ \big(G_\theta \cdot \theta_{xx} + \theta_x \cdot (G_{\theta r} \cdot r_x + G_{\theta\theta} \cdot \theta_x + G_{\theta\phi} \cdot \phi_x) \big) + \frac{\partial}{\partial x} \cdot \left(\frac{\partial G}{\partial \phi} \cdot \frac{\partial \phi}{\partial x} \right) \qquad Eq. 15$$

Now to finish off with the "ϕ" portion:

$$\frac{\partial}{\partial x} \cdot \left(\frac{\partial G}{\partial \phi} \cdot \frac{\partial \phi}{\partial x} \right) = \frac{\partial}{\partial x} \cdot (G_\phi \cdot \phi_x) = G_\phi \cdot \phi_{xx} + \phi_x \cdot \frac{\partial (G_\phi)}{\partial x} \qquad Eq. 16$$

As before, we need to work out that final term:

$$\frac{\partial (G_\phi)}{\partial x} = \frac{\partial (G_\phi)}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial (G_\phi)}{\partial \theta} \cdot \frac{\partial \theta}{\partial x} + \frac{\partial (G_\phi)}{\partial \phi} \cdot \frac{\partial \phi}{\partial x} \qquad Eq. 17$$

Putting Equations 16 and 17 into Equation 15, we now have:

$$\frac{\partial^2 G}{\partial x^2} = \left(G_r \cdot r_{xx} + r_x \cdot (G_{rr} \cdot r_x + G_{r\theta} \cdot \theta_x + G_{r\phi} \cdot \phi_x)\right) + \left(G_\theta \cdot \theta_{xx} + \theta_x \cdot (G_{\theta r} \cdot r_x + G_{\theta\theta} \cdot \theta_x + G_{\theta\phi} \cdot \phi_x)\right) + \left(G_\phi \cdot \phi_{xx} + \phi_x \cdot (G_{\phi r} \cdot r_x + G_{\phi\theta} \cdot \theta_x + G_{\phi\phi} \cdot \phi_x)\right) \qquad Eq.18$$

This is 1/3 of the final General Equation, our solution to converting Equation 3. Aligning the components in the fashion above allows for some interesting patterns to emerge. In fact we can use these patterns to extrapolate the other two thirds of the final equation by simply being clever and replacing variables accordingly. This is how I am going to proceed. However, the reader may choose to work the rest of the General Equation out in long hand as proof for themselves.

Dealing with the "y" component of Equation 3 we have:

$$\frac{\partial^2 G}{\partial y^2} = \left(G_r \cdot r_{yy} + r_y \cdot (G_{rr} \cdot r_y + G_{r\theta} \cdot \theta_y + G_{r\phi} \cdot \phi_y)\right) + \left(G_\theta \cdot \theta_{yy} + \theta_y \cdot (G_{\theta r} \cdot r_y + G_{\theta\theta} \cdot \theta_y + G_{\theta\phi} \cdot \phi_y)\right) + \left(G_\phi \cdot \phi_{yy} + \phi_y \cdot (G_{\phi r} \cdot r_y + G_{\phi\theta} \cdot \theta_y + G_{\phi\phi} \cdot \phi_y)\right) \qquad Eq.19$$

And the "z" component of Equation 3 becomes:

$$\frac{\partial^2 G}{\partial z^2} = \left(G_r \cdot r_{zz} + r_z \cdot (G_{rr} \cdot r_z + G_{r\theta} \cdot \theta_z + G_{r\phi} \cdot \phi_z)\right) + \left(G_\theta \cdot \theta_{zz} + \theta_z \cdot (G_{\theta r} \cdot r_z + G_{\theta\theta} \cdot \theta_z + G_{\theta\phi} \cdot \phi_z)\right) + \left(G_\phi \cdot \phi_{zz} + \phi_z \cdot (G_{\phi r} \cdot r_z + G_{\phi\theta} \cdot \theta_z + G_{\phi\phi} \cdot \phi_z)\right) \qquad Eq.20$$

So finally, we can now write Equation 3 in the form (This is anew new color coding scheme):

$$\nabla^2 G_{(r,\theta,\phi)} = \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + \frac{\partial^2 G}{\partial z^2} = \left(G_r \cdot r_{xx} + r_x \cdot (G_{rr} \cdot r_x + G_{r\theta} \cdot \theta_x + G_{r\phi} \cdot \phi_x)\right) + \qquad Eq.21$$
$$\left(G_\theta \cdot \theta_{xx} + \theta_x \cdot (G_{\theta r} \cdot r_x + G_{\theta\theta} \cdot \theta_x + G_{\theta\phi} \cdot \phi_x)\right) + \left(G_\phi \cdot \phi_{xx} + \phi_x \cdot (G_{\phi r} \cdot r_x + G_{\phi\theta} \cdot \theta_x + G_{\phi\phi} \cdot \phi_x)\right) +$$
$$\left(G_r \cdot r_{yy} + r_y \cdot (G_{rr} \cdot r_y + G_{r\theta} \cdot \theta_y + G_{r\phi} \cdot \phi_y)\right) + \left(G_\theta \cdot \theta_{yy} + \theta_y \cdot (G_{\theta r} \cdot r_y + G_{\theta\theta} \cdot \theta_y + G_{\theta\phi} \cdot \phi_y)\right) +$$
$$\left(G_\phi \cdot \phi_{yy} + \phi_y \cdot (G_{\phi r} \cdot r_y + G_{\phi\theta} \cdot \theta_y + G_{\phi\phi} \cdot \phi_y)\right) + \left(G_r \cdot r_{zz} + r_z \cdot (G_{rr} \cdot r_z + G_{r\theta} \cdot \theta_z + G_{r\phi} \cdot \phi_z)\right) +$$
$$\left(G_\theta \cdot \theta_{zz} + \theta_z \cdot (G_{\theta r} \cdot r_z + G_{\theta\theta} \cdot \theta_z + G_{\theta\phi} \cdot \phi_z)\right) + \left(G_\phi \cdot \phi_{zz} + \phi_z \cdot (G_{\phi r} \cdot r_z + G_{\phi\theta} \cdot \theta_z + G_{\phi\phi} \cdot \phi_z)\right)$$

Obviously, at this point some rearrangement is called for. I will take two steps to clean things up. First, I will distribute where I can (Equation 22), and then I will group together all the partial derivatives of "G" that are alike (Equation 23). But first recall that when working with partial derivatives:

$$F_{ij} = F_{ji} \qquad or \qquad \frac{\partial}{\partial j} \cdot \left(\frac{\partial F}{\partial i}\right) = \frac{\partial}{\partial i} \cdot \left(\frac{\partial F}{\partial j}\right)$$

So:

$$\nabla^2 G_{(r,\theta,\phi)} = \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + \frac{\partial^2 G}{\partial z^2} = \left(G_r \cdot r_{xx} + G_{rr} \cdot (r_x)^2 + G_{r\theta} \cdot r_x \cdot \theta_x + G_{r\phi} \cdot r_x \cdot \phi_x\right) + \quad Eq.22$$
$$\left(G_\theta \cdot \theta_{xx} + G_{\theta r} \cdot \theta_x \cdot r_x + G_{\theta\theta} \cdot (\theta_x)^2 + G_{\theta\phi} \cdot \theta_x \cdot \phi_x\right) + \left(G_\phi \cdot \phi_{xx} + G_{\phi r} \cdot \phi_x \cdot r_x + G_{\phi\theta} \cdot \phi_x \cdot \theta_x + G_{\phi\phi} \cdot (\phi_x)^2\right) +$$
$$\left(G_r \cdot r_{yy} + G_{rr} \cdot \left(r_y\right)^2 + G_{r\theta} \cdot r_y \cdot \theta_y + G_{r\phi} \cdot r_y \cdot \phi_y\right) + \left(G_\theta \cdot \theta_{yy} + G_{\theta r} \cdot \theta_y \cdot r_y + G_{\theta\theta} \cdot \left(\theta_y\right)^2 + G_{\theta\phi} \cdot \theta_y \cdot \phi_y\right) +$$
$$\left(G_\phi \cdot \phi_{yy} + G_{\phi r} \cdot \phi_y \cdot r_y + G_{\phi\theta} \cdot \phi_y \cdot \theta_y + G_{\phi\phi} \cdot \left(\phi_y\right)^2\right) + \left(G_r \cdot r_{zz} + G_{rr} \cdot (r_z)^2 + G_{r\theta} \cdot r_z \cdot \theta_z + G_{r\phi} \cdot r_z \cdot \phi_z\right) +$$
$$\left(G_\theta \cdot \theta_{zz} + G_{\theta r} \cdot \theta_z \cdot r_z + G_{\theta\theta} \cdot (\theta_z)^2 + G_{\theta\phi} \cdot \theta_z \cdot \phi_z\right) + \left(G_\phi \cdot \phi_{zz} + G_{\phi r} \cdot \phi_z \cdot r_z + G_{\phi\theta} \cdot \phi_z \cdot \theta_z + G_{\phi\phi} \cdot (\phi_z)^2\right)$$

And then:

$$\nabla^2 G_{(r,\theta,\phi)} = G_r \cdot r_{xx} + G_r \cdot r_{yy} + G_r \cdot r_{zz} + G_{rr} \cdot (r_x)^2 + G_{rr} \cdot \left(r_y\right)^2 + G_{rr} \cdot (r_z)^2 + G_{r\theta} \cdot r_x \cdot \theta_x + G_{r\theta} \cdot r_y \cdot \theta_y +$$
$$G_{r\theta} \cdot r_z \cdot \theta_z + G_{\theta r} \cdot \theta_x \cdot r_x + G_{\theta r} \cdot \theta_y \cdot r_y + G_{\theta r} \cdot \theta_z \cdot r_z + G_{r\phi} \cdot r_x \cdot \phi_x + G_{r\phi} \cdot r_y \cdot \phi_y + G_{r\phi} \cdot r_z \cdot \phi_z + G_{\phi r} \cdot \phi_x \cdot r_x +$$
$$G_{\phi r} \cdot \phi_y \cdot r_y + G_{\phi r} \cdot \phi_z \cdot r_z + G_\theta \cdot \theta_{xx} + G_\theta \cdot \theta_{yy} + G_\theta \cdot \theta_{zz} + G_{\theta\theta} \cdot (\theta_x)^2 + G_{\theta\theta} \cdot \left(\theta_y\right)^2 + G_{\theta\theta} \cdot (\theta_z)^2 + G_{\theta\phi} \cdot \theta_x \cdot \phi_x +$$
$$G_{\theta\phi} \cdot \theta_y \cdot \phi_y + G_{\theta\phi} \cdot \theta_z \cdot \phi_z + G_{\phi\theta} \cdot \phi_x \cdot \theta_x + G_{\phi\theta} \cdot \phi_y \cdot \theta_y + G_{\phi\theta} \cdot \phi_z \cdot \theta_z + G_\phi \cdot \phi_{xx} + G_\phi \cdot \phi_{yy} + G_\phi \cdot \phi_{zz}$$
$$G_{\phi\phi} \cdot (\phi_x)^2 + G_{\phi\phi} \cdot \left(\phi_y\right)^2 + G_{\phi\phi} \cdot (\phi_z)^2 \quad Eq.23$$

A point to note from Equation 2, "ϕ" is only a function of "x" and "y". We can use this fact to eliminate a few of the terms, these being "ϕ" derivatives with respect to "z", in Eq. 23. The next logical step after that will be to factor out the partial derivatives from the remaining like terms. I have color coded the like terms in Equation 24 for easy viewing:

$$\nabla^2 G_{(r,\theta,\phi)} = G_r \cdot \left(r_{xx} + r_{yy} + r_{zz}\right) + G_{rr} \cdot \left((r_x)^2 + \left(r_y\right)^2 + (r_z)^2\right) + 2G_{r\theta} \cdot \left(r_x \cdot \theta_x + r_y \cdot \theta_y + r_z \cdot \theta_z\right)$$
$$2G_{r\phi} \cdot \left(r_x \cdot \phi_x + r_y \cdot \phi_y\right) + G_\theta \cdot \left(\theta_{xx} + \theta_{yy} + \theta_{zz}\right) + G_{\theta\theta} \cdot \left((\theta_x)^2 + \left(\theta_y\right)^2 + (\theta_z)^2\right) +$$
$$2G_{\theta\phi} \cdot \left(\theta_x \cdot \phi_x + \theta_y \cdot \phi_y\right) + G_\phi \cdot \left(\phi_{xx} + \phi_{yy}\right) + G_{\phi\phi} \cdot \left((\phi_x)^2 + \left(\phi_y\right)^2\right) \quad Eq.24$$

You will notice that we have reduced the equation to a somewhat more manageable state than that of Equation 21. Equation 24 is essentially the final form of the General Laplacian in Spherical Coordinates. However, it's still a large, ungainly beast and needs to be further simplified if we ever hope to use it. The first step in doing this will be to take the various derivates of the spherical term equations "r, θ, ϕ", substitute them in, and hope that some of this craziness cancels out.

First let me identify all of the various derivatives we will need to take and put them back into proper notation:

$$r_x = \frac{\partial r}{\partial x} \quad r_y = \frac{\partial r}{\partial y} \quad r_z = \frac{\partial r}{\partial z} \quad r_{xx} = \frac{\partial^2 r}{\partial x^2} \quad r_{yy} = \frac{\partial^2 r}{\partial y^2} \quad r_{zz} = \frac{\partial^2 r}{\partial z^2}$$
$$\theta_x = \frac{\partial \theta}{\partial x} \quad \theta_y = \frac{\partial \theta}{\partial y} \quad \theta_z = \frac{\partial \theta}{\partial z} \quad \theta_{xx} = \frac{\partial^2 \theta}{\partial x^2} \quad \theta_{yy} = \frac{\partial^2 \theta}{\partial y^2} \quad \theta_{zz} = \frac{\partial^2 \theta}{\partial z^2}$$
$$\phi_x = \frac{\partial \phi}{\partial x} \quad \phi_y = \frac{\partial \phi}{\partial y} \quad \phi_{xx} = \frac{\partial^2 \phi}{\partial x^2} \quad \phi_{yy} = \frac{\partial^2 \phi}{\partial y^2}$$

We'll take these in order one at a time. For convenience and space saving, I will make the following temporary substitution in all of the following derivatives:

$$r^2 \;=\; x^2 + y^2 + z^2 \;=\; u$$

And so:

$$r_x \;=\; \frac{\partial r}{\partial x} \;=\; \frac{\partial}{\partial x}\left(\sqrt{u}\,\right) \;=\; \frac{x}{\sqrt{u}} \;=\; \frac{r \cdot sin(\theta)\, cos(\phi)}{r} \;=\; sin(\theta)\, cos(\phi) \qquad Eq. 25$$

It follows then that:

$$r_y \;=\; \frac{\partial r}{\partial y} \;=\; sin(\theta)\, sin(\phi) \qquad Eq. 25$$

$$r_z \;=\; \frac{\partial r}{\partial z} \;=\; cos(\theta) \qquad Eq. 26$$

And now to differentiate these three equations again:

$$r_{xx} \;=\; \frac{\partial^2 r}{\partial x^2} \;=\; \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{u}}\right) \;=\; \frac{\sqrt{u} - \;\; x^2 (u)^{\left(-1/2\right)}}{u} \;=\; \frac{\frac{u}{\sqrt{u}} - \frac{x^2}{\sqrt{u}}}{u} \;=\; \frac{u - \;\; x^2}{u\sqrt{u}}$$

$$\frac{r^2 - r^2 \cdot sin^2(\theta)\, cos^2(\phi)}{r^3} \;=\; \frac{1 - sin^2(\theta)\, cos^2(\phi)}{r} \qquad Eq. 27$$

And:

$$r_{yy} \;=\; \frac{\partial^2 r}{\partial y^2} \;=\; \frac{1 - sin^2(\theta)\, sin^2(\phi)}{r} \qquad Eq. 28$$

$$r_{zz} \;=\; \frac{\partial^2 r}{\partial z^2} \;=\; \frac{1 - cos^2(\theta)}{r} \qquad Eq. 29$$

Luckily, the "r" equation has a symmetry that we can use to quickly compute the various derivatives once we have taken a single or double derivative with respect to any variable. In this case, I choose "x" and so "y" and "z" followed easily. Unfortunately, the "θ" equation is not so elegant, particularly the second partial derivatives.

As a refresher, let me give you the generic form of the inverse cosine derivative:

$$\partial(cos^{-1}(w)) = \; \frac{-dw}{\sqrt{1 - w^2}}$$

Also note the following as it will come in handy soon:

$$\sqrt{u - z^2} \;=\; \sqrt{x^2 + y^2} \;=\; \sqrt{r^2 \cdot sin^2(\theta)\,(cos^2(\phi) + sin^2(\phi))} \;=\; r \cdot sin(\theta)$$

And so:

$$\theta_x \;=\; \frac{\partial \theta}{\partial x} \;=\; \frac{\partial}{\partial x}\left(cos^{-1}\left(\frac{z}{\sqrt{u}}\right)\right) \;=\; \frac{z \cdot x}{u\sqrt{u}\sqrt{1 - \frac{z^2}{u}}} \;=\; \frac{z \cdot x}{u\sqrt{u - z^2}} \;=$$

$$\frac{r^2 \cdot cos(\theta)\, sin(\theta)\, cos(\phi)}{r^2 \cdot r \cdot sin(\theta)} \;=\; \frac{cos(\theta)\, cos(\phi)}{r} \qquad Eq.30$$

The "y" partial derivative follows easily:

$$\theta_y \;=\; \frac{\partial \theta}{\partial y} \;=\; \frac{\partial}{\partial y}\left(\cos^{-1}\left(\frac{z}{\sqrt{u}}\right)\right) \;=\; \frac{z \cdot y}{u\sqrt{u}\sqrt{1 - \frac{z^2}{u}}} \;=\; \frac{z \cdot y}{u\sqrt{u - z^2}} \;=$$

$$\frac{r^2 \cdot cos(\theta)\, sin(\theta)\, sin(\phi)}{r^2 \cdot r \cdot sin(\theta)} \;=\; \frac{cos(\theta)\, sin(\phi)}{r} \qquad Eq.31$$

And for "z":

$$\theta_z \;=\; \frac{\partial \theta}{\partial z} \;=\; \frac{\partial}{\partial z}\left(cos^{-1}\left(\frac{z}{\sqrt{u}}\right)\right) \;=\; \frac{-\left(\frac{\left(\sqrt{u} - \frac{z^2}{\sqrt{u}}\right)}{u}\right)}{\sqrt{1 - \frac{z^2}{u}}} \;=\; \frac{-\left(\frac{\left(\frac{u}{\sqrt{u}} - \frac{z^2}{\sqrt{u}}\right)}{u}\right)}{\sqrt{1 - \frac{z^2}{u}}} \;=\; \frac{-\left(\frac{(u - z^2)}{u\sqrt{u}}\right)}{\sqrt{1 - \frac{z^2}{u}}} \;=$$

$$\frac{-(u - z^2)}{u\sqrt{u}\sqrt{1 - \frac{z^2}{u}}} \;=\; \frac{-(u - z^2)}{u\sqrt{u - z^2}} \;=\; \frac{-\sqrt{u - z^2}}{u} \;=\; \frac{-r \cdot sin(\theta)}{r^2} \;=\; \frac{-\,sin(\theta)}{r} \qquad Eq.32$$

Now for the fun part. In order to take the second partial derivatives, we must use the dreaded quotient rule:

$$\theta_{xx} \;=\; \frac{\partial^2 \theta}{\partial x^2} \;=\; \frac{\partial}{\partial x}\left(\frac{z \cdot x}{u\sqrt{u - z^2}}\right) \;=\; \frac{\left(u\sqrt{u - z^2}\right)z \;- z \cdot x\left(\left(\frac{u \cdot x}{\sqrt{u - z^2}}\right) + 2x\sqrt{u - z^2}\right)}{u^2\left(\sqrt{u - z^2}\right)^2} \;=$$

$$\frac{(r^2 \cdot r \cdot sin(\theta))z \;- z \cdot x\left(\left(\frac{r^2 \cdot x}{r \cdot sin(\theta)}\right) + 2x \cdot r \cdot sin(\theta)\right)}{r^4(r^2 \cdot sin^2(\theta))} \;=$$

$$\frac{r^4 \cdot sin(\theta)\,cos(\theta)\ -\left(\dfrac{r^4 \cdot sin^2(\theta)\,cos^2(\phi)\,cos(\theta)}{sin(\theta)}\right) - 2r^4 \cdot sin^3(\theta)\,cos^2(\phi)\,cos(\theta)}{r^6\,sin^2(\theta)} =$$

$$\frac{cos(\theta)}{r^2\,sin(\theta)} - \frac{cos^2(\phi)\,cos(\theta)}{r^2\,sin(\theta)} - \frac{2 \cdot sin^2(\theta)\,cos^2(\phi)\,cos(\theta)}{r^2\,sin(\theta)} =$$

$$\frac{cos(\theta)(1 - cos^2(\phi) - 2 \cdot sin^2(\theta)\,cos^2(\phi))}{r^2\,sin(\theta)} = \frac{cos(\theta)(sin^2(\phi) - 2 \cdot sin^2(\theta)\,cos^2(\phi))}{r^2\,sin(\theta)} \qquad Eq.33$$

As you can see, the second partial derivatives of "θ" do not simplify nearly as nicely as the others we have taken so far. It is just something we will have to deal with for now.

$$\theta_{yy} = \frac{\partial^2\theta}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{z \cdot y}{u\sqrt{u - z^2}}\right) = \frac{(u\sqrt{u - z^2})z\ - z \cdot y\left(\left(\dfrac{u \cdot y}{\sqrt{u - z^2}}\right) + 2y\sqrt{u - z^2}\right)}{u^2(u - z^2)} =$$

$$\frac{(r^2 \cdot r \cdot sin(\theta))z\ - z \cdot y\left(\left(\dfrac{r^2 \cdot y}{r \cdot sin(\theta)}\right) + 2y \cdot r \cdot sin(\theta)\right)}{r^4(r^2 \cdot sin^2(\theta))} =$$

$$\frac{r^4 \cdot sin(\theta)\,cos(\theta)\ -\left(\dfrac{r^4 \cdot sin^2(\theta)\,sin^2(\phi)\,cos(\theta)}{sin(\theta)}\right) - 2r^4 \cdot sin^3(\theta)\,sin^2(\phi)\,cos(\theta)}{r^6\,sin^2(\theta)} =$$

$$\frac{cos(\theta)}{r^2\,sin(\theta)} - \frac{sin^2(\phi)\,cos(\theta)}{r^2\,sin(\theta)} - \frac{2 \cdot sin^2(\theta)\,sin^2(\phi)\,cos(\theta)}{r^2\,sin(\theta)}$$

$$\frac{cos(\theta)(1 - sin^2(\phi) - 2 \cdot sin^2(\theta)\,sin^2(\phi))}{r^2\,sin(\theta)} = \frac{cos(\theta)(cos^2(\phi) - 2 \cdot sin^2(\theta)\,sin^2(\phi))}{r^2\,sin(\theta)} \qquad Eq.34$$

You might be able to see it already, but when we put "θ_{xx}" and "θ_{yy}" together later we will be able to cancel A LOT of the terms out. But for now, let's finish "θ_{zz}". At first glance, it appears to be another quotient rule, but in fact it is not and therefore much easier to calculate:

$$\theta_{zz} = \frac{\partial^2\theta}{\partial z^2} = -\frac{\partial}{\partial z}\left(\frac{\sqrt{u - z^2}}{u}\right) = -\frac{\partial}{\partial z}\left(\frac{\sqrt{x^2 + y^2}}{u}\right) = -\frac{\partial}{\partial z}\left(u^{-1}\sqrt{x^2 + y^2}\right) =$$

$$2z \cdot u^{-2}\sqrt{x^2 + y^2} = \frac{2z\sqrt{x^2 + y^2}}{u^2} = \frac{2\,r \cdot cos(\theta)\,(r \cdot sin(\theta))}{r^4} = \frac{2 \cdot cos(\theta)\,sin(\theta)}{r^2} \qquad Eq.35$$

And finally, we get to the "ϕ" derivatives. By the way, the generic form of the arctan derivate is:

$$\partial(tan^{-1}(w)) = \frac{dw}{1+w^2}$$

$$\phi_x = \frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}\left(tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{\left(\frac{-y}{x^2}\right)}{1+\left({y^2}/{x^2}\right)} = \frac{-y}{x^2+y^2} =$$

$$\frac{-r\cdot sin(\theta)\,sin(\phi)}{r^2\cdot sin^2(\theta)} = \frac{-\,sin(\phi)}{r\cdot sin(\theta)} \qquad Eq.36$$

And:

$$\phi_y = \frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}\left(tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{\left(\frac{1}{x}\right)}{1+\left({y^2}/{x^2}\right)} = \frac{1}{x+\frac{y^2}{x}} = \frac{1}{\left(\frac{x^2+y^2}{x}\right)} = \frac{x}{x^2+y^2} =$$

$$\frac{r\cdot sin(\theta)\,cos(\phi)}{r^2\cdot sin^2(\theta)} = \frac{cos(\phi)}{r\cdot sin(\theta)} \qquad Eq.37$$

And now the second partial derivatives:

$$\phi_{xx} = \frac{\partial^2\phi}{\partial x^2} = -\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) = -\frac{\partial}{\partial x}\left(y(x^2+y^2)^{-1}\right) = \frac{2xy}{(x^2+y^2)^2} \qquad Eq.38$$

I'm not going to simplify this one any further, and you will see why in just a moment:

$$\phi_{yy} = \frac{\partial^2\phi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = \frac{\partial}{\partial y}\left(x(x^2+y^2)^{-1}\right) = \frac{-2xy}{(x^2+y^2)^2} \qquad Eq.39$$

That will cancel out nicely!

Now we are ready to begin our massive substitution mission of the derivatives back into Equation 24. I will use the Equation 24 color coding scheme to substitute the equation out piece meal:

$$G_r\cdot\left(r_{xx}+r_{yy}+r_{zz}\right) = \frac{\partial G}{\partial r}\cdot\left(\frac{1-sin^2(\theta)\,cos^2(\phi)}{r}+\frac{1-sin^2(\theta)\,sin^2(\phi)}{r}+\frac{1-cos^2(\theta)}{r}\right) =$$

$$\frac{\partial G}{\partial r}\cdot\left(\frac{-(sin^2(\theta)\,cos^2(\phi)+sin^2(\theta)\,sin^2(\phi)+cos^2(\theta)-3)}{r}\right) = \frac{\partial G}{\partial r}\cdot\left(\frac{2}{r}\right) \qquad Eq.40$$

$$G_{rr}\cdot\left((r_x)^2+\left(r_y\right)^2+(r_z)^2\right) \;=\; \frac{\partial^2 G}{\partial r^2}\cdot\left(sin^2(\theta)\,cos^2(\phi)+sin^2(\theta)\,sin^2(\phi)+cos^2(\theta)\right) \;=\; \frac{\partial^2 G}{\partial r^2} \qquad Eq.41$$

$$2G_{r\theta}\cdot\left(r_x\cdot\theta_x+r_y\cdot\theta_y+r_z\cdot\theta_z\right) \;=$$

$$2\frac{\partial}{\partial\theta}\left(\frac{\partial G}{\partial r}\right)\cdot\left(sin(\theta)\,cos(\phi)\cdot\frac{cos(\theta)\,cos(\phi)}{r}+sin(\theta)\,sin(\phi)\cdot\frac{cos(\theta)\,sin(\phi)}{r}-cos(\theta)\frac{sin(\theta)}{r}\right) \;=$$

$$2\frac{\partial}{\partial\theta}\left(\frac{\partial G}{\partial r}\right)\cdot\left(\frac{sin(\theta)\,cos(\theta)\,cos^2(\phi)}{r}+\frac{sin(\theta)\,cos(\theta)\,sin^2(\phi)}{r}-\frac{sin(\theta)\,cos(\theta)}{r}\right) \;=$$

$$2\frac{\partial}{\partial\theta}\left(\frac{\partial G}{\partial r}\right)\cdot\left(\frac{sin(\theta)\,cos(\theta)}{r}-\frac{sin(\theta)\,cos(\theta)}{r}\right) \;=\; 0 \qquad Eq.42$$

Equation 42 is the best result I seen so far! Let's keep on going:

$$2G_{r\phi}\cdot\left(r_x\cdot\phi_x+r_y\cdot\phi_y\right) \;=$$

$$2\frac{\partial}{\partial r}\left(\frac{\partial G}{\partial\phi}\right)\cdot\left((sin(\theta)\,cos(\phi))\cdot\left(\frac{-\,sin(\phi)}{r\cdot sin(\theta)}\right)+(sin(\theta)\,sin(\phi))\cdot\left(\frac{cos(\phi)}{r\cdot sin(\theta)}\right)\right) \;=$$

$$2\frac{\partial}{\partial r}\left(\frac{\partial G}{\partial\phi}\right)\cdot\left(\left(\frac{sin(\theta)\,sin(\phi)\,cos(\phi)}{r\cdot sin(\theta)}\right)-\left(\frac{sin(\theta)\,sin(\phi)\,cos(\phi)}{r\cdot sin(\theta)}\right)\right) \;=\; 0 \qquad Eq.43$$

$$G_{\theta}\cdot\left(\theta_{xx}+\theta_{yy}+\theta_{zz}\right) \;=$$

$$\frac{\partial G}{\partial\theta}\cdot\left(\frac{cos(\theta)(sin^2(\phi)-2\cdot sin^2(\theta)\,cos^2(\phi))}{r^2\,sin(\theta)}+\frac{cos(\theta)(cos^2(\phi)-2\cdot sin^2(\theta)\,sin^2(\phi))}{r^2\,sin(\theta)}\right.$$
$$\left.+\frac{2\cdot cos(\theta)\,sin(\theta)}{r^2}\right) \;=$$

$$\frac{\partial G}{\partial\theta}\cdot\left(\frac{cos(\theta)}{r^2}\left(\frac{sin^2(\phi)+cos^2(\phi)-2\cdot sin^2(\theta)\,cos^2(\phi)-2\cdot sin^2(\theta)\,sin^2(\phi)+2\cdot sin^2(\theta)}{sin(\theta)}\right)\right) \;=$$

$$\frac{\partial G}{\partial\theta}\cdot\left(\frac{cos(\theta)}{r^2}\left(\frac{1-2\cdot sin^2(\theta)\,cos^2(\phi)-2\cdot sin^2(\theta)\,sin^2(\phi)+2\cdot sin^2(\theta)}{sin(\theta)}\right)\right) \;=$$

$$\frac{\partial G}{\partial\theta}\cdot\left(\frac{cos(\theta)}{r^2}\left(\frac{1-2\cdot sin^2(\theta)(cos^2(\phi)+sin^2(\phi))+2\cdot sin^2(\theta)}{sin(\theta)}\right)\right) \;=$$

$$\frac{\partial G}{\partial \theta}\cdot\left(\frac{cos(\theta)}{r^2}\left(\frac{1-2\cdot sin^2(\theta)+2\cdot sin^2(\theta)}{sin(\theta)}\right)\right) = \frac{\partial G}{\partial \theta}\cdot\left(\frac{cos(\theta)}{r^2\cdot sin(\theta)}\right) \qquad Eq.44$$

$$G_{\theta\theta}\cdot\left((\theta_x)^2+(\theta_y)^2+(\theta_z)^2\right) = \frac{\partial^2 G}{\partial \theta^2}\cdot\left(\left(\frac{cos(\theta)\,cos(\phi)}{r}\right)^2+\left(\frac{cos(\theta)\,sin(\phi)}{r}\right)^2+\left(\frac{-sin(\theta)}{r}\right)^2\right) =$$

$$\frac{\partial^2 G}{\partial \theta^2}\cdot\left(\frac{cos^2(\theta)\,cos^2(\phi)+cos^2(\theta)\,sin^2(\phi)+sin^2(\theta)}{r^2}\right) = \frac{\partial^2 G}{\partial \theta^2}\cdot\left(\frac{1}{r^2}\right) \qquad Eq.45$$

$$2G_{\theta\phi}\cdot\left(\theta_x\cdot\phi_x+\theta_y\cdot\phi_y\right) = 2\frac{\partial}{\partial \phi}\frac{\partial G}{\partial \theta}\cdot\left(\left(\frac{cos(\theta)\,cos(\phi)}{r}\right)\cdot\left(\frac{-sin(\phi)}{r\cdot sin(\theta)}\right)+\left(\frac{cos(\theta)\,sin(\phi)}{r}\right)\cdot\left(\frac{cos(\phi)}{r\cdot sin(\theta)}\right)\right) =$$

$$2\frac{\partial}{\partial \phi}\frac{\partial G}{\partial \theta}\cdot\left(\left(\frac{cos(\theta)\,cos(\phi)}{r}\right)\cdot\left(\frac{-sin(\phi)}{r\cdot sin(\theta)}\right)+\left(\frac{cos(\theta)\,sin(\phi)}{r}\right)\cdot\left(\frac{cos(\phi)}{r\cdot sin(\theta)}\right)\right) =$$

$$2\frac{\partial}{\partial \phi}\frac{\partial G}{\partial \theta}\cdot\left(\left(\frac{cos(\theta)\,cos(\phi)\,sin(\phi)}{r^2\cdot sin^2(\theta)}\right)-\left(\frac{cos(\theta)\,cos(\phi)\,sin(\phi)}{r^2\cdot sin^2(\theta)}\right)\right) = 0 \qquad Eq.45$$

$$G_{\phi}\cdot\left(\phi_{xx}+\phi_{yy}\right) = \frac{\partial G}{\partial \phi}\cdot\left(\left(\frac{2xy}{(x^2+y^2)^2}\right)+\left(\frac{-2xy}{(x^2+y^2)^2}\right)\right) = 0 \qquad Eq.47$$

$$G_{\phi\phi}\cdot\left((\phi_x)^2+(\phi_y)^2\right) = \frac{\partial^2 G}{\partial \phi^2}\cdot\left(\left(\frac{-sin(\phi)}{r\cdot sin(\theta)}\right)^2+\left(\frac{cos(\phi)}{r\cdot sin(\theta)}\right)^2\right) =$$

$$\frac{\partial^2 G}{\partial \phi^2}\cdot\left(\frac{sin^2(\phi)+cos^2(\phi)}{r^2\cdot sin^2(\theta)}\right) = \frac{\partial^2 G}{\partial \phi^2}\cdot\left(\frac{1}{r^2\cdot sin^2(\theta)}\right) \qquad Eq.48$$

Substituting all of these Equations, 40 - 48, back into Equation 24, you get the ALMOST completed results of:

$$\nabla^2 G_{(r,\theta,\phi)} = \frac{\partial G}{\partial r}\cdot\left(\frac{2}{r}\right)+\frac{\partial^2 G}{\partial r^2}+\frac{\partial G}{\partial \theta}\cdot\left(\frac{cos(\theta)}{r^2\cdot \sin(\theta)}\right)+\frac{\partial^2 G}{\partial \theta^2}\cdot\left(\frac{1}{r^2}\right)+\frac{\partial^2 G}{\partial \phi^2}\cdot\left(\frac{1}{r^2\cdot \sin^2(\theta)}\right) \qquad Eq.49$$

In fact, this IS the final result. Some clever person later came along and condensed the expression a little more. I would guess if for nothing else then to make it all fit nicely in a text book!

We have come this far, so no reason to stop now. The first two terms, red and blue, are the result of a product rule expansions, as you can see:

$$\frac{\partial G}{\partial r}\cdot\left(\frac{2}{r}\right)+\frac{\partial^2 G}{\partial r^2} = \frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\cdot\frac{\partial G}{\partial r}\right)$$

And:

$$\frac{\partial G}{\partial \theta} \cdot \left(\frac{cos(\theta)}{r^2 \cdot \sin(\theta)}\right) + \frac{\partial^2 G}{\partial \theta^2} \cdot \left(\frac{1}{r^2}\right) = \left(\frac{1}{r^2 \cdot \sin(\theta)}\right) \cdot \frac{\partial}{\partial \theta}\left(\sin(\theta) \cdot \frac{\partial G}{\partial \theta}\right)$$

A word of caution, occasionally the "θ" and "ϕ" terms are swapped in the initial stage when deriving Equations 1. The end result is the same, but the variables are switched. The Laplacian we have derived here is typically how it is found in Physics textbooks.

So finally, after 11 pages of work, we have arrived at our answer:

$$\nabla^2 G_{(r,\theta,\phi)} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial G}{\partial r}\right) + \left(\frac{1}{r^2 \cdot \sin(\theta)}\right)\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial G}{\partial \theta}\right) + \left(\frac{1}{r^2 \cdot \sin^2(\theta)}\right)\frac{\partial^2 G}{\partial \phi^2}$$

It's pretty clear why the derivation of this formula is left up to the student. You would need an entire subsection in a text book to clearly show the steps. Having seen it magically appear out of the author's bag of tricks in a few of my classes with little comment on why it takes this form, I felt compelled to find out for myself.

If you have taken the time to come this far, thank you for reading.

-Lyle Arnett