Purpose

Figure out how Kitaev's paper (arXiv:0901.2686), which considers both unitary and antiunitary symmetries, can be understood in terms of how the basic classification of Hamiltonians, which is done with mostly antiunitary symmetries, breaks down with the introduction of other unitary symmetries (Morimoto/Furusaki PRB 88, 125129).

Here we go

We will go from least symmetric to most symmetric cases of Kitaev and consider the $d = 0$ case.

1. Classifying space R_2. Class D.

   - Kitaev considers a free-fermion non-particle-conserved Hamiltonian

     $$H = \vec{c}^T (iA) \vec{c}$$

     where A is a real skew-symmetric matrix. Since it is real and A skew-symmetric, this Hamiltonian generically has an antiunitary anticommuting symmetry (usually called particle-hole symmetry) which is simply complex conjugation C. However, Kitaev works with only A, so that anticommuting symmetry is actually a commuting symmetry when only A is considered:

     $$CH = -CH \longrightarrow CiA = -iCA \equiv -CiA \longrightarrow CA = AC$$

   - By conventional arguments, since the only symmetry is C and $C^2 = +1$, we are in the class D.
   - By Kitaev's Clifford algebra argument (which keeps everything real), C commutes with A, so it is not a negative (squares to -1) Clifford symmetry (anticommutes). Therefore, the size of the Clifford algebra of all negative Clifford symmetries of A is zero: $Cl_{0,0}$. This identifies with $Cl_{0,2}$ in Kitaev's paper, whose classifying space is R_2.
   - By Furusaki's Clifford algebra argument (which introduces a complex structure in real Clifford algebras), C and iC both anticommute with H. Note that $C^2 = 1$ and $(iC)^2 = 1$. Therefore, the size of the Clifford algebra of all negative/positive Clifford symmetries of H is two: $Cl_{0,2}$. Its classifying space is R_2.

2. Classifying space C_0. Class A.

   - Now Kitaev adds to H a real unitary particle conservation symmetry Q

     $$\begin{aligned} QH &= HQ \longrightarrow QA = AQ \\ QC &= CQ \\ Q^2 &= -1 \end{aligned}$$

   - By Kitaev's argument, A will change form into X, so we will no longer have symmetry under C: $CX \neq XC$. Kitaev shows that the classifying space of X is C_0.
   - By Furusaki's Clifford algebra argument (Sec. IV-C), note that iQ (with $(iQ)^2 = +1$) anticommutes with C:

     $$iQC = iCQ = -CiQ$$

     This means that C does not hold on eigenspaces of Q, so the original symmetry class D changes to A.

3. Classifying space R_3. Class $DIII$.

   - Now Kitaev adds to H antiunitary time reversal symmetry T

     $$\begin{aligned} TH &= HT \longrightarrow TiA = -iTA \equiv iAT \longrightarrow TA = -AT \\ TC &= CT \\ T^2 &= -1 \end{aligned}$$

   - By conventional arguments, we have time reversal and particle hole with $T^2 = -1$ and $C^2 = +1$, so we are in class $DIII$.
   - By Kitaev's argument, A has one negative Clifford symmetry, so $Cl_{1,0}$, which is associated with $Cl_{0,3}$, whose classifying space is R_3.
   - By Furusaki's argument, one can construct a Clifford algebra of three positive Clifford symmetries of H: $Cl_{0,3} = \{C, iC, iCT\}$. The classifying space associated with that is $R_{3-0} = R_3$.

4. Classifying space R_4. Class AII.

- Now Kitaev has all symmetries present

$$\begin{aligned}
QT &= -TQ \\
QTH &= QHT = HQT \longrightarrow QTA = -AQT \\
QTC &= CQT \\
QTT &= -TQT \\
QTQ &= -QQT \\
(QT)^2 &= -1
\end{aligned}$$

- By Kitaev's argument, A now has two negative Clifford symmetries: T, QT. So we have $Cl_{2,0}$, which is associated with $Cl_{0,4}$, whose classifying space is R_4.
- By Furusaki's argument [Sec. IV-C(ii)], we know that in the case of both T and C presence, we have a unitary anticommuting symmetry of H (chiral symmetry)

$$\begin{aligned}
TCH &= -THC = -HTC \\
(TC)^2 &= -1
\end{aligned}$$

  From this, we can combine with Q and construct another chiral symmetry

$$\begin{aligned}
(iTCQ)\,H &= iTCHQ = -iTHCQ = -H\,(iTCQ) \\
(iTCQ)^2 &= +1
\end{aligned}$$

  which then is an example of Sec. IV-B. In this section, we consider how the unitary anticommuting symmetry $iTCQ$ causes a shift in the original class. Since $(iTCQ)\,T = +T\,(iTCQ)$ and $(iTCQ)\,C = -C\,(iTCQ)$, we have $(+,-)$. Using Table V(a), the original index R_3 is shifted by one into R_4.