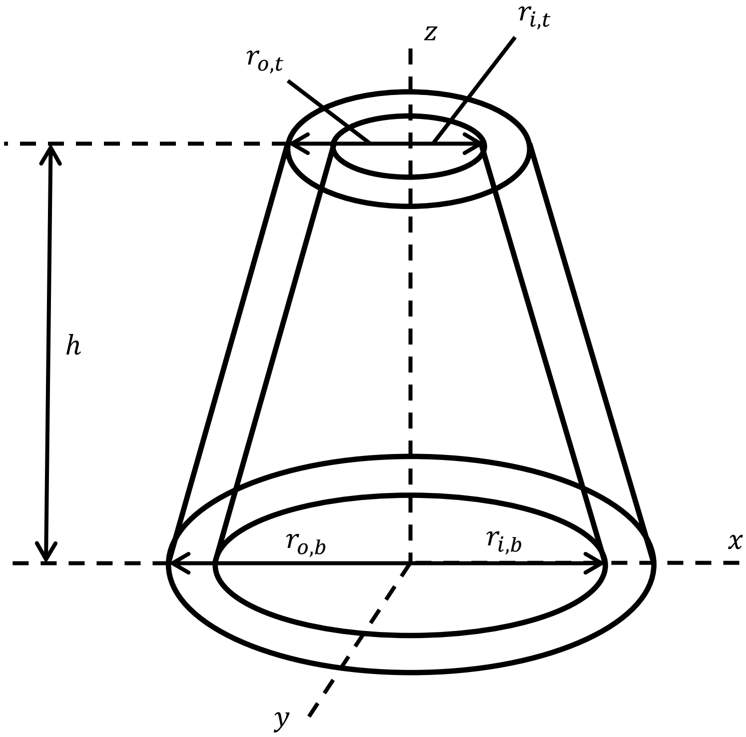
Consider a hollow conical frustum as shown below:



The outer radius of the frustum at any height z above its base is derived as follows:

$$r_o(z) = \left(\frac{h-z}{h} r_{o,b} + \frac{z}{h} r_{o,t}\right)$$

Here $r_{o,b}$ and $r_{o,t}$ denote the outer radii at the bottom and the top of the frustum, respectively. The inner radius of the frustum at any height z above its base is then derived as follows:

$$r_i(z) = \left(\frac{h-z}{h} r_{i,b} + \frac{z}{h} r_{i,t}\right)$$

Here $r_{i,b}$ and $r_{i,t}$ denote the inner radii at the bottom and the top of the frustum, respectively. The cross-sectional area of the frustum at any height z above its base is then derived as follows:

$$A(z) = \int_0^{2\pi} \int_{r_i(z)}^{r_o(z)} r\, dr d\theta = \pi[r_o(z)^2 - r_i(z)^2]$$

The volume of the frustum is determined by integrating its area over its height as follows:

$$V(z) = \int_0^h A(z)\, dz$$

Combining these equations together we obtain:

$$V = \int_0^h \pi \left[\left(\frac{h-z}{h} r_{o,b} + \frac{z}{h} r_{o,t}\right)^2 - \left(\frac{h-z}{h} r_{i,b} + \frac{z}{h} r_{i,t}\right)^2\right] dz$$

$$= \frac{\pi h}{3}\left[\left({r_{o,b}}^2 + r_{o,b} r_{o,t} + {r_{o,t}}^2\right) - \left({r_{i,b}}^2 + r_{i,b} r_{i,t} + {r_{i,t}}^2\right)\right]$$

We can now calculate the centre of gravity (COG) of the frustum as follows:

$$COG_x = \frac{1}{V}\int_0^h \int_0^{2\pi} \int_{r_i(z)}^{r_o(z)} r^2 Cos(\theta) A(z)\, dr d\theta dz = 0$$

$$COG_y = \frac{1}{V}\int_0^h \int_0^{2\pi} \int_{r_i(z)}^{r_o(z)} r^2 Sin(\theta) A(z)\, dr d\theta dz = 0$$

$$COG_z = \frac{1}{V}\int_0^h zA(z)\, dz$$

Note: Because the frustum is symmetric about the z-axis, $COG_x = COG_y = 0$. The COG is then derived as follows:

$$COG_z = \frac{1}{V}\int_0^h zA(z)\, dz = \frac{1}{V}\int_0^h z\pi \left[\left(\frac{h-z}{h} r_{o,b} + \frac{z}{h} r_{o,t}\right)^2 - \left(\frac{h-z}{h} r_{i,b} + \frac{z}{h} r_{i,t}\right)^2\right] dz$$

$$= \frac{\pi h^2}{12V}\left[\left({r_{o,b}}^2 + 2r_{o,b}r_{o,t} + 3{r_{o,t}}^2\right) - \left({r_{i,b}}^2 + 2r_{i,b}r_{i,t} + 3{r_{i,t}}^2\right)\right]$$

$$= \frac{h}{4}\left[\frac{\left({r_{o,b}}^2 + 2r_{o,b}r_{o,t} + 3{r_{o,t}}^2\right) - \left({r_{i,b}}^2 + 2r_{i,b}r_{i,t} + 3{r_{i,t}}^2\right)}{\left({r_{o,b}}^2 + r_{o,b}r_{o,t} + {r_{o,t}}^2\right) - \left({r_{i,b}}^2 + r_{i,b}r_{i,t} + {r_{i,t}}^2\right)}\right]$$

If we let:

$$A = \left({r_{o,b}}^2 + 2r_{o,b}r_{o,t} + 3{r_{o,t}}^2\right) - \left({r_{i,b}}^2 + 2r_{i,b}r_{i,t} + 3{r_{i,t}}^2\right)$$

$$B = \left({r_{o,b}}^2 + r_{o,b}r_{o,t} + {r_{o,t}}^2\right) - \left({r_{i,b}}^2 + r_{i,b}r_{i,t} + {r_{i,t}}^2\right)$$

Then the COG of a hollow conical frustum may be written as follows:

$$COG_z = \frac{hA}{4B}$$