The Chebyshev polynomials play an important role in the theory of approximation. The N^h-order Chebyshev polynomial can be computed by using

$$
\begin{aligned}
T_N(\Omega) &= \cos(N\cos^{-1}(\Omega)) \quad , \; |\Omega| \le 1 \\
&= \cosh(N\cosh^{-1}(\Omega)) \quad , \; |\Omega| > 1.
\end{aligned}
\tag{1.1}
$$

The first few Chebyshev polynomials are listed in Table 1, and some are plotted on Figure 1. Using $T_0(\Omega) = 1$ and $T_1(\Omega) = \Omega$, the Chebyshev polynomials may be generated recursively by using the relationship

$$
T_{N+1}(\Omega) = 2\Omega T_N(\Omega) - T_{N-1}(\Omega), \tag{1.2}
$$

$N \ge 1$. They satisfy the relationships:

1. For $|\Omega| \le 1$, the polynomial magnitudes are bounded by 1, and they oscillate between ±1.
2. For $|\Omega| > 1$, the polynomial magnitudes increase monotonically with Ω.
3. $T_N(1) = 1$ for all n.
4. $T_N(0) = \pm 1$ for n even and $T_N(0) = 0$ for N odd.
5. The zero crossing of $T_N(\Omega)$ occur in the interval $-1 \le \Omega \le 1$.

N	$T_N(\Omega)$
0	1
1	Ω
2	$2\Omega^2 - 1$
3	$4\Omega^3 - 3\Omega$
4	$8\Omega^4 - 8\Omega^2 + 1$

Table 1: Some low-order Chebyshev polynomials

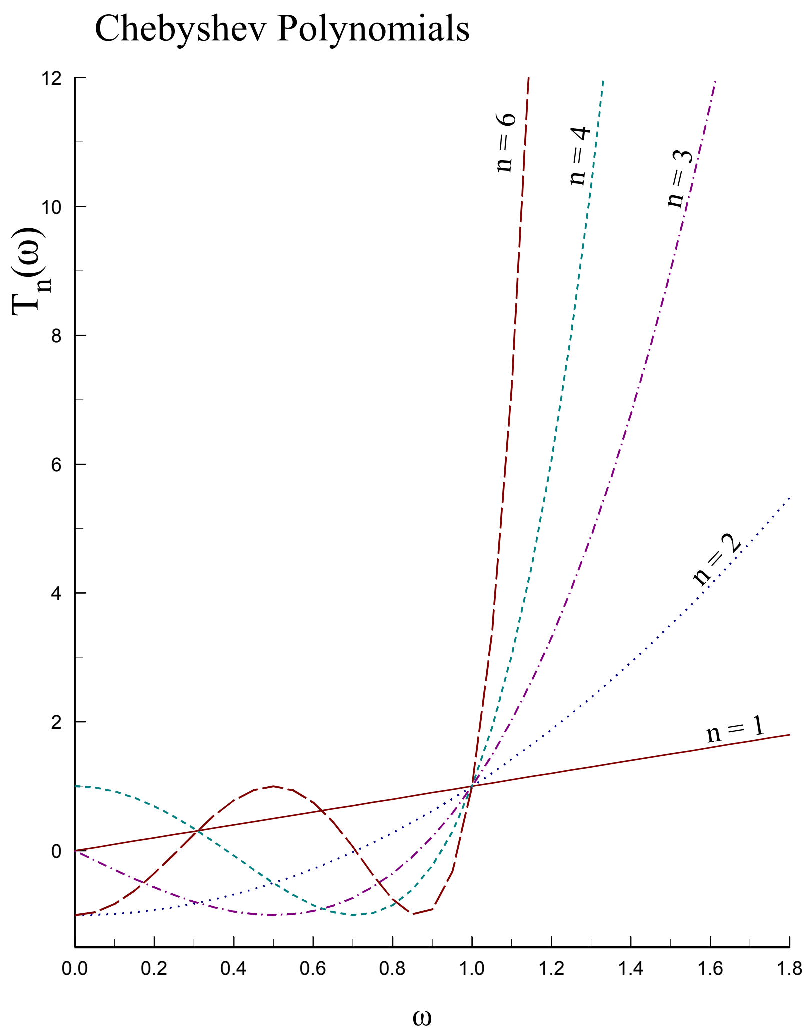

Figure 1: Some low-order Chebyshev polynomials.

Chebyshev Low-pass Filters

There are two types of Chebyshev low-pass filters, and both are based on Chebyshev polynomials. A *Type I* Chebyshev low-pass filter has an all-pole transfer function. It has an equi-ripple pass band and a monotonically decreasing stop band. A *Type II* Chebyshev low-pass filter has both poles and zeros; its pass-band is monotonically decreasing, and its has an equirriple stop band. By allowing some ripple in the pass band or stop band magnitude response, a Chebyshev filter can achieve a "steeper" pass- to stop-band transition region (*i.e.*, filter "roll-off' is faster) than can be achieved by the same order Butterworth filter.

Type I Chebyshev Low-Pass Filter

A Type I filter has the magnitude response

$$\left|H_a(j\Omega)\right|^2 = \frac{1}{1+\varepsilon^2 T_N^2(\Omega/\Omega_p)}, \tag{1.3}$$

where N is the filter order, ε is a user-supplied parameter that controls the amount of pass-band ripple, and Ω_p is the upper pass band edge. Figure 2 depicts the magnitude response of several Chebyshev Type 1 filters, all with the same normalized pass-band edge $\Omega_p = 1$. As order N increases, the number of pass-band ripples increases, and the "roll – off" rate increases. For N odd (alternatively, even), there are (N+1)/2 (alternatively, N/2) pass-bank peaks. As ripple parameter ε increases, the ripple amplitude and the "roll – off" rate increases.

On the interval $0 < \Omega < \Omega_p$, $T_N^2(\Omega/\Omega_p)$ oscillates between 0 and 1, and this causes $|H_a(\Omega/\Omega_p)|^2$ to oscillate between 1 and $1/(1+\varepsilon^2)$, as can be seen on Figure 2. In applications, parameter ε is chosen so that the peak-to-peak value of pass-band ripple

$$\text{Peak} - \text{to} - \text{Peak Passband Ripple} = 1 - 1/\sqrt{1+\varepsilon^2}\,. \tag{1.4}$$

is an acceptable value. If the magnitude response is plotted on a dB scale, the pass-band ripple becomes

$$\text{Pass} - \text{Band Ripple in dB} \equiv 20\text{Log}\left[\frac{1}{1/\sqrt{1+\varepsilon^2}}\right] = 10\text{Log}(1+\varepsilon^2)\,. \tag{1.5}$$

In general, pass-band ripple is undesirable, but a value of 1dB or less is acceptable in most

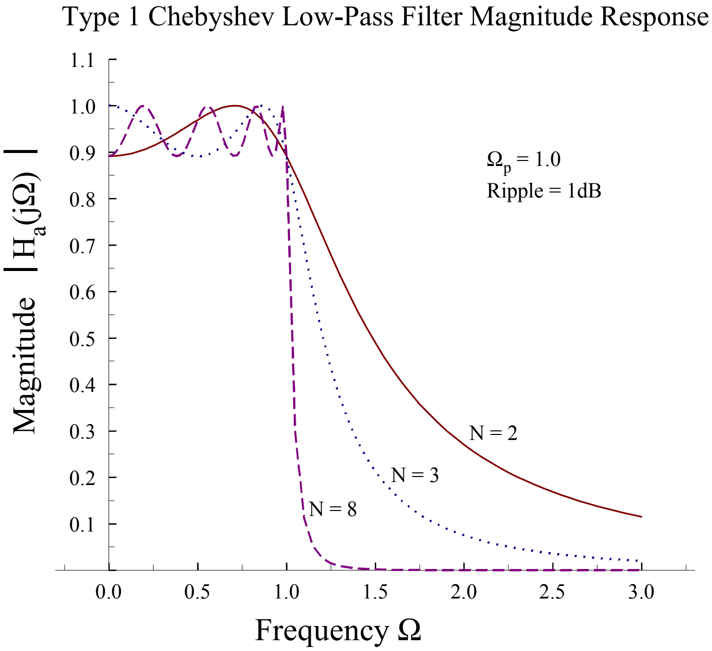

Figure 2: Type I Chebyshev magnitude response with one dB of pass-band ripple

applications.

The stop-band edge, Ω_s, can be specified in terms of a *stop-band attenuation parameter*. For $\Omega > \Omega_p$, the magnitude response decreases monotonically, and stop-band edge Ω_s can be specified as the frequency for which

$$\frac{1}{\sqrt{1+\varepsilon^2 T_N^2(\Omega/\Omega_p)}} < \frac{1}{A}, \qquad \Omega > \Omega_s > \Omega_p, \tag{1.6}$$

where A is a user-specified *Stop-Band attenuation* parameter. In Decibels, for $\Omega > \Omega_s$, the magnitude response is down 20Log(A) dB or more from the pass band peak value.

Type I Chebyshev Low-Pass Filter Design Procedure

To start, we must have Ω_p, Ω_s, pass-band ripple value and the stop-band attenuation value. These are used to compute ε, N, and the pole locations for $H_a(s)$, as outlined below.

1) Using (1.5), compute

$$\varepsilon = \sqrt{10^{\{\text{Pass-Band Ripple in dB}\}/10} - 1}\ . \tag{1.7}$$

2) Compute the necessary filter order N. At $\Omega = \Omega_s$, we have

$$\frac{1}{\sqrt{1+\varepsilon^2 T_N^2(\Omega_s/\Omega_p)}} = \frac{1}{A} \quad \Rightarrow \quad \sqrt{1+\varepsilon^2 T_N^2(\Omega_s/\Omega_p)} = A\,, \tag{1.8}$$

which can be solved for

$$T_N(\Omega_s/\Omega_p) = \cosh(N\cosh^{-1}(\Omega_s/\Omega_p)) = \sqrt{\frac{A^2-1}{\varepsilon^2}}\ . \tag{1.9}$$

Finally, N is the smallest positive integer for which

$$N \ge \frac{\cosh^{-1}\sqrt{(A^2-1)/\varepsilon^2}}{\cosh^{-1}(\Omega_s/\Omega_p)}\,. \tag{1.10}$$

3) Compute the 2N poles of $H_a(s)H_a(-s)$. The first N poles are in the left-half s-plane, and they are assigned to $H_a(s)$. Using reasoning similar to that used in the development of the Butterworth filter, we can write

$$H_a(s)H_a(-s) = \frac{1}{1+\varepsilon^2 T_N^2(s/j\Omega_p)}\,. \tag{1.11}$$

To simplify what follows, we will use $\Omega_p = 1$ and compute the pole locations for

$$H_a(s)H_a(-s) = \frac{1}{1+\varepsilon^2 T_N^2(s/j)}. \tag{1.12}$$

Once computed, the pole values can be scaled (multiplied) by any desired value of Ω_p.

From inspection of (1.12), it is clear that the poles must satisfy

$$T_N(s/j) = \pm\sqrt{-1/\varepsilon^2} = \pm j/\varepsilon \tag{1.13}$$

(two cases are required here: the first where + is used and the second where − is used). Using (1.1), we formulate

$$\cos\left[N\cos^{-1}(s/j)\right] = \pm j/\varepsilon. \tag{1.14}$$

We must solve this equation for 2N distinct roots s_k, $1 \le k \le 2N$. Define

$$\cos^{-1}(s_k/j) = \alpha_k - j\beta_k \tag{1.15}$$

so that (1.14) yields

$$\cos[N(\alpha_k - j\beta_k)] = \cos(N\alpha_k)\cosh(N\beta_k) + j\sin(N\alpha_k)\sinh(N\beta_k) = \pm\frac{j}{\varepsilon} \tag{1.16}$$

by using the identities cos(jx) = cosh(x) and sin(jx) = j{sinh(x)}. In (1.16), equate real and imaginary components to obtain

$$\begin{aligned}&\cos(N\alpha_k)\cosh(N\beta_k) = 0\\&\sin(N\alpha_k)\sinh(N\beta_k) = \pm\frac{1}{\varepsilon}.\end{aligned} \tag{1.17}$$

Since $\cosh(N\beta_k) \neq 0$, the first of (1.17) implies that $N\alpha_k$ must be an odd multiple of $\pi/2$ so that

$$N\alpha_k = (2k-1)\frac{\pi}{2} \qquad \Rightarrow \qquad \alpha_k = (2k-1)\frac{\pi}{2N}, \qquad k = 1, 2, 3, ..., 2N\,. \tag{1.18}$$

The α_k take on values that range from $\pi/2N$ to $2\pi - \pi/2N$ in steps of size $\pi/2N$. Since $\sin(N\alpha_k) = (-1)^{k-1}$, and the sign in the second (1.17) can be + or −, we can use

$$\beta_k = \frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right) \tag{1.19}$$

(all β_k are identical; the 2N distinct α_k will give us our 2N roots). Now, substitute (1.18) and (1.19) into (1.15) to obtain

$$s_k = j\cos(\alpha_k - j\beta_k) = \sigma_k + j\omega_k\,, \tag{1.20}$$

where

$$\begin{aligned}\sigma_k &= -\sin\left[(2k-1)\frac{\pi}{2N}\right]\sinh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right]\\ \omega_k &= \cos\left[(2k-1)\frac{\pi}{2N}\right]\cosh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right],\end{aligned} \tag{1.21}$$

$1 \le k \le 2N$, for the poles of $H_a(s)H_a(-s)$. The first half of the poles, s_1, s_2, … , s_n, are in the left-half of the s-plane, while the remainder are in the right-half s-plane.

For k = 1, 2, … , 2N, the poles of $H_a(s)H_a(-s)$ are located on an s-plane ellipse, as illustrated by Figure 3 for the case N = 4. The ellipse has major and minor axes of length

$$\begin{aligned} d_2 &= \cosh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right] \\ d_1 &= \sinh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right], \end{aligned} \tag{1.22}$$

respectively. To see that the poles fall on an ellipse, note that

$$\frac{\sigma_k^2}{d_1^2} + \frac{\omega_k^2}{d_2^2} = 1 \tag{1.23}$$

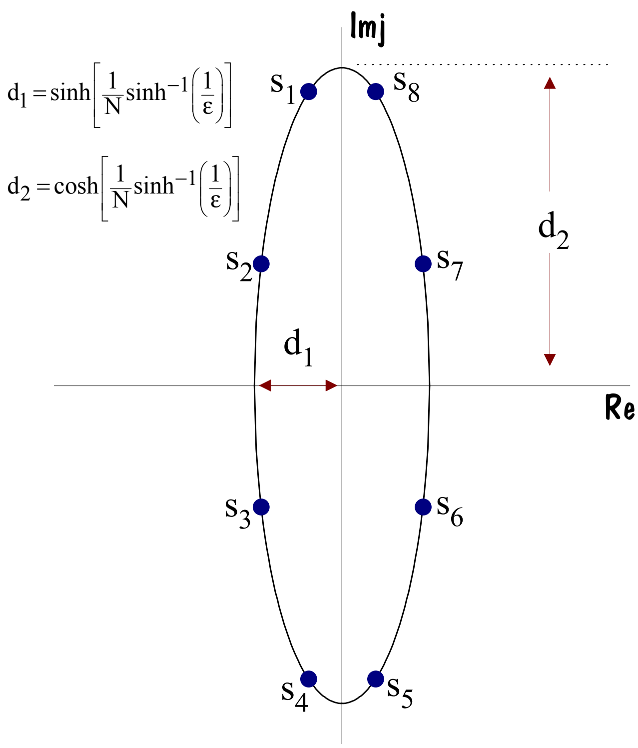

Figure 3: S-plane ellipse detailing poles of $H_a(s)H_a(-s)$ for a fourth-order (N = 4), Chebyshev filter with 1dB of passband ripple.

for $1 \le k \le 2N$.

4) Use only the left-half plane poles $s_1, s_2, \dots, s_N$, and write down the $\Omega_p = 1$ transfer function as

$$H_a(s) = K\frac{(-1)^n s_1 s_2 \cdots s_N}{(s-s_1)(s-s_2)\cdots(s-s_N)}, \tag{1.24}$$

where K is the filter DC gain. To obtain a peak pass-band gain of unity, we must use

$$\begin{aligned} K &= \frac{1}{\sqrt{1+\varepsilon^2}}, \quad N \text{ even} \\ &= 1, \quad N \text{ odd}. \end{aligned} \tag{1.25}$$

However, K can be set to any desired value, within technological constraints.

5) For a non-unity value of Ω_p, the transfer function becomes

$$H_a(s) = K\frac{(-1)^n s_1 s_2 \cdots s_N}{\left(\frac{s}{\Omega_p} - s_1\right)\left(\frac{s}{\Omega_p} - s_2\right)\cdots\left(\frac{s}{\Omega_p} - s_N\right)}. \tag{1.26}$$

Most hand calculators do not support hyperbolic trigonometry functions. It is desirable to develop non-hyperbolic-function forms for the Chebyshev filter formulas. To accomplish this, we use $\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right)$ to implement the simplification

$$\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right) = \frac{1}{N}\ln\left(\frac{1+\sqrt{1+\varepsilon^2}}{\varepsilon}\right) = \ln(\Gamma), \tag{1.27}$$

where Γ is defined as

$$\Gamma \equiv \left[\frac{1+\sqrt{1+\varepsilon^2}}{\varepsilon}\right]^{1/N}. \tag{1.28}$$

In terms of parameter Γ, we can use (1.27) to write

$$\begin{aligned}
\cosh\left(\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right) &= \cosh\big(\ln(\Gamma)\big) = \frac{e^{\ln(\Gamma)} + e^{-\ln(\Gamma)}}{2} = \frac{\Gamma + 1/\Gamma}{2} = \frac{\Gamma^2+1}{2\Gamma} \\
\sinh\left(\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right) &= \sinh\big(\ln(\Gamma)\big) = \frac{e^{\ln(\Gamma)} - e^{-\ln(\Gamma)}}{2} = \frac{\Gamma - 1/\Gamma}{2} = \frac{\Gamma^2-1}{2\Gamma}.
\end{aligned} \tag{1.29}$$

Now, use (1.29) in the pole formulas (1.20) and (1.21) to obtain

$$s_k = -\sin\left[(2k-1)\frac{\pi}{2N}\right]\frac{\Gamma^2-1}{2\Gamma} + j\cos\left[(2k-1)\frac{\pi}{2N}\right]\frac{\Gamma^2+1}{2\Gamma}, \tag{1.30}$$

$1 \le k \le 2N$, for the $2N$ poles of $H_a(s)H_a(-s)$. Also, the poles are on an s-plane ellipse with major and minor axes

$$\begin{aligned}
d_2 &= \cosh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right] = \frac{\Gamma^2+1}{2\Gamma} \\
d_1 &= \sinh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right] = \frac{\Gamma^2-1}{2\Gamma},
\end{aligned} \tag{1.31}$$

respectively.

Example: Design a Chebyshev filter with 1dB pass band ripple and an attenuation of at least 20dB at Ω_s equal to twice the pass-band edge Ω_p, specified as $\Omega_p/2\pi = 3$kHz.

1. Use (1.7) and compute $\varepsilon = .5088$.

2. Compute the necessary filter order N. At the stop band edge $\Omega_s = 2\,\Omega_p$, the attenuation is at least 20 dB. From (1.8), we see that

$$-10\text{Log}\Big(1+\varepsilon^2 T_N(2\Omega_p/\Omega_p)\Big) = -20\text{Log}(A) = -20\,, \tag{1.32}$$

so that A = 10. Using (1.10), we compute

$$\frac{\cosh^{-1}\sqrt{(100-1)/(.5088)^2}}{\cosh^{-1}(2)} \approx 2.78\,, \tag{1.33}$$

so we select filter order N = 3.

3. Calculate the left-half plane poles. Using (1.21), we calculate

$$\begin{aligned} s_1 &= -\sin\left(\frac{\pi}{6}\right)\sinh\left[\frac{1}{3}\sinh^{-1}\left(\frac{1}{.5088}\right)\right] + j\cos\left(\frac{\pi}{6}\right)\cosh\left[\frac{1}{3}\sinh^{-1}\left(\frac{1}{.5088}\right)\right] \\ &= -.2471 + .9660j \end{aligned} \tag{1.34}$$

In a similar manner, we calculate

$$\begin{aligned} s_2 &= -.4942 \\ s_3 &= -.2471 - .9660j \end{aligned} \tag{1.35}$$

4. Calculate the frequency normalized (*i.e.,* $\Omega_p = 1$) transfer function. For the case $\Omega_p = 1$, the transfer function is

$$H_a(s) = \frac{(-s_1)(-s_2)(-s_3)}{(s-s_1)(s-s_2)(s-s_3)} = \frac{.4913}{s^3 + .9883s^2 + 1.2384s + .4913} \ . \tag{1.36}$$

The dotted line plot on Figure 2 is a plot of $|H_a(j\omega)|$ for (1.36).

5. Calculate the transfer function for the case $\Omega_p = 2\pi(3000) = 6000\pi$ as

$$H_a\left(\frac{s}{6\pi\times10^3}\right) = \frac{.4913}{\left(\frac{s}{6\pi\times10^3}\right)^3 + .9883\left(\frac{s}{6\pi\times10^3}\right)^2 + 1.2384\left(\frac{s}{6\pi\times10^3}\right) + .4913} \tag{1.37}$$

Matlab function **cheb1ord** can be used to confirm the filter order computed for this example. When supplied with the pass-band edge, stop-band edge, maximum ripple in the pass-band and minimum attenuation in the stop-band, **cheb1ord** will compute the necessary minimum filter order. Cut from a Matlab session and pasted here, the following keyboard session illustrates **cheb1ord** in action (See the Matlab documentation for **cheb1ord;** this function can do more than we have asked.).

```
>> % passband edge
>> Wp = 1;
>> % stopband edge
>> Ws = 2;
>> % passband ripple in dB
>> Rp = 1;
>> % stopband minimum attenuation in dB
>> Rs = 20;
>> N = cheb1ord(Wp,Ws,Rp,Rs,'s')

N =

     3

>>
```

We used a normalized pass-band edge `Wp = 1`, a normalized stop-band edge `Ws = 2`, a pass-band ripple `Rp = 1` dB and a minimum stop-band attenuation of `Rs = 20` dB. **Cheb1ord** returned `N = 3`, the necessary minimum filter order, in agreement with what we deduced from (1.33).

Matlab function **cheby1** can be used to confirm the filter transfer function computed for this example. When supplied with the filter order, maximum ripple in the pass-band and pass-band edge, **cheby1** will compute the coefficients for the numerator and denominator polynomials of transfer function $H_a(s)$. Cut from a Matlab session and pasted here, the following keyboard session illustrates **cheby1** in action (See the Matlab documentation for **cheby1;** this function can do more than we have asked.).

```
>> % filter order
>> N = 3;
>> % maximum passband ripple in dB
>> Rp = 1;
>> % cutoff frequency (where response down Rp dB)
>> Wn = 1;
>> [b,a] = cheby1(N,Rp,Wn,'s')

b =

         0         0         0    0.4913

a =

    1.0000    0.9883    1.2384    0.4913

>>
```

We used a filter order of `N = 3`, a pass-band ripple `Rp = 1` dB and a normalized cut off `Wn = 1` (where the response is down `Rp` dB). **Cheby1** returned array `b` of numerator coefficients and array `a` of denominator coefficients (ordered from highest to lowest power of s). Note the agreement with the numerator and denominator of (1.36).

Type II Chebyshev Low-Pass Filter

With a maximally flat response at $\Omega = 0$, the Type II Chebyshev low-pass filter exibits a monotonic behavior in the pass band and an equiripple response in the stop band. Chebyshev Type II filters are less common compared to the more popular Type I. They do not roll off as fast as Type I filters. Most filter theory books do not even mention this type of filter.

A Type II Chebyshev low-pass filter has a magnitude-squared transfer function given by

$$\left|\tilde{H}_a(j\Omega)\right|^2 = \frac{1}{1+\varepsilon^2\left[\dfrac{T_N^2(\Omega_s/\Omega_p)}{T_N^2(\Omega_s/\Omega)}\right]}\,. \tag{1.38}$$

Because of the dependence of (1.38) on Ω_s/Ω, the Type 2 filter is also known as the *inverse Chebyshev approximation*. Transfer function (1.38) exhibits both poles and zeros.

To design a Type 2 filter, we must know pass-band edge Ω_p, stop-band edge Ω_s, a maximum pass-band attenuation factor δ_1, and a minimum stop-band attenuation factor δ_2. We use this information to calculate ε, filter order N, and the poles and zeros of $\tilde{H}_a(s)$. The general procedure to accomplish this is outlined below.

First, the parameter ε must be determined. At $\Omega = \Omega_p$, the filter response must be equal to the known δ_1 so that

$$\delta_1^2 = \left|\tilde{H}_a(j\Omega_p)\right|^2 = \frac{1}{1+\varepsilon^2}\,. \tag{1.39}$$

This leads immediately to the value

$$\varepsilon = \frac{\sqrt{1-\delta_1^2}}{\delta_1}\,. \tag{1.40}$$

In the stop band (*i.e.*, for $\Omega \geq \Omega_s$), the square of the maximum response is the square of the stop-band ripple peak value, and it is given by

$$\delta_2^{\,2} \equiv \max_{\Omega>\Omega_s} \left|\tilde{H}_a(\Omega)\right|^2 = \frac{1}{1+\varepsilon^2 T_N^2(\Omega_s/\Omega_p)}. \tag{1.41}$$

Parameter δ_2 must be known in order to determine the required filter order N during the design phase.

Often, parameter δ_2 is given in dB. For example, as part of the filter specification, we might state "the stop-band response is at least 50dB below the maximum pass-band response". By this, we mean $-50 = 20\mathrm{Log}_{10}(\delta_2)$.

We have enough information to compute the necessary filter order N. From (1.40) and (1.41), we compute

$$T_N^2(\Omega_s/\Omega_p) = \cosh^2\left(N\cosh^{-1}(\Omega_s/\Omega_p)\right) = \frac{1-\delta_2^{\,2}}{(\varepsilon\delta_2)^2}, \tag{1.42}$$

and this yields

$$N = \frac{\cosh^{-1}\left(\sqrt{1-\delta_2^{\,2}}\Big/\varepsilon\delta_2\right)}{\cosh^{-1}(\Omega_s/\Omega_p)}. \tag{1.43}$$

Of course, fractional values of N must be rounded up to the next highest integer value.

With $\Omega_s/\Omega_p > 1$, the function $T_N(\Omega_s/\Omega_p)$ increases with increasing N (see Figure 1). So, rounding N upwards increases both sides of (1.42). Since δ_2 is fixed in the pole/zero finding algorithm used below to find the filter transfer function, rounding N upward causes a *decrease* in the effective value of ε. In the final filter design, this decrease in ε forces the original pass-band specification to be exceeded. However, the original stop-band specification is retained without change.

Example: We want a Chebyshev Type II filter with a normalized pass-band frequency of 1, a normalized stop-band frequency of 1.5, a maximum of 1 dB of pass-band attenuation, and a minimum of 40 dB of stop-band attenuation. Determine the required filter order N. First, we compute

$$20\text{Log}(\delta_1) = -1 \qquad \Rightarrow \qquad \delta_1 = 10^{-1/20} = .8913 \tag{1.44}$$

Next, we use (1.40) to compute

$$\varepsilon = \frac{\sqrt{1-\delta_1^2}}{\delta_1} = .5088 \tag{1.45}$$

Also, we compute

$$20\text{Log}(\delta_2) = -40 \qquad \Rightarrow \qquad \delta_2 = 10^{-2} = .01 \tag{1.46}$$

Finally, we use (1.43) to compute

$$N = \frac{\cosh^{-1}\left(\sqrt{1-\delta_2{}^2}\Big/\varepsilon\delta_2\right)}{\cosh^{-1}(\Omega_s/\Omega_p)} = \frac{\cosh^{-1}\left(\sqrt{1-(.01)^2}\Big/\{(.5088)(.01)\}\right)}{\cosh^{-1}(1.5)} = 6.2$$

and round up to N = 7. As shown by the keyboard session listed below, the value N = 7 is confirmed by the Matlab function **cheb2ord.**

```
% Normalized passband frequency Wp = 1
```

```
>> Wp = 1;
% Normalized stopband frequency Ws = 1.5
>> Ws = 1.5;
% Passband Ripple Rp = 1 dB
>> Rp = 1;
% Stopband Attenuation = 40 dB
>> Rs = 40;
% Call cheb2ord to compute the filter order N
>> N = cheb2ord(Wp,Ws,Rp,Rs,'s')

N =

     7
```

As discussed previously, rounding N upward decreases ε. Using (1.42) with N = 7 and δ_2 = .01, we compute the smaller value of ε as

$$\varepsilon = \sqrt{\frac{1-\delta_2^2}{\delta_2^2\cosh^2(N\cosh^{-1}(\Omega_s/\Omega_p)}} = \sqrt{\frac{1-(.01)^2}{(.01)^2\cosh^2(7\cosh^{-1}(1.5)}} = .2372 \tag{1.47}$$

For the 7th-order filter, the response at the actual pass-band edge is down

$$20\mathrm{Log}\left|\hat{H}_a(j\Omega_p)\right| = 20\mathrm{Log}\left[\frac{1}{\sqrt{1+\varepsilon^2}}\right] = 20\mathrm{Log}\left[\frac{1}{\sqrt{1+(.2372)^2}}\right] = -.2378\mathrm{dB}. \tag{1.48}$$

relative to the maximum filter response at DC. So, we have exceeded the 1dB maximum pass-band attenuation specification.

Figure 4 depicts the amplitude response of our seventh-order Type II Chebyshev low-pass filter (a dB plot is given in addition to a "straight" amplitude plot).

Computing Filter Poles and Zeros

For a Type II filter, the 2N poles of $\tilde{H}_a(s)\tilde{H}_a(-s)$ are denoted here as $\tilde{s}_k$, $1 \le k \le 2N$. They are the 2N distinct roots of

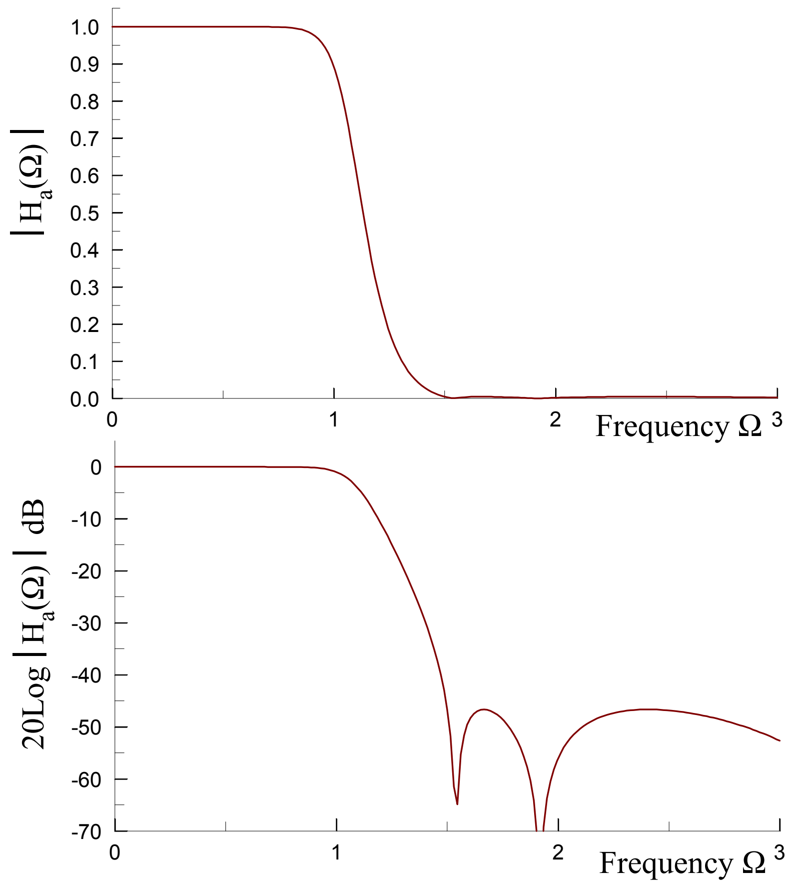

Figure 4: Response of a N = 7 order, Type 2 Chebyshev filter with $\Omega_p = 1$, $\Omega_s = 1.5$, maximum pass-band attenuation = 1dB (actual is .237dB) and minimum stop-band attenuation = 40 dB. Plots given in normalized amplitude and

$$1+\varepsilon^2\left[\frac{T_N^2(\Omega_s/\Omega_p)}{T_N^2(j\Omega_s/s)}\right]=0\,. \tag{1.49}$$

Equivalently, we are looking for the roots of

$$1+\left(\frac{1}{\varepsilon^2 T_N^2(\Omega_s/\Omega_p)}\right)T_N^2(j\Omega_s/s)=0\,. \tag{1.50}$$

With the aid of (1.42), this equation can be written as

$$1+\left(\frac{\delta_2^2}{1-\delta_2^2}\right)T_N^2(j\Omega_s/s)=0. \tag{1.51}$$

Compare this equation with the denominator of (1.12); conclude that, for a Type II filter, the 2N poles of $\tilde{H}_a(s)\tilde{H}_a(-s)$ must satisfy $j\Omega_s/\tilde{s}_k = s_k/j$, a result that can be expressed as

$$\tilde{s}_k = (j\Omega_s)\frac{j}{s_k}. \tag{1.52}$$

In (1.52), the s_k are computed by using the Type I filter pole formula (1.30) with Γ computed with $\delta_2^2/(1-\delta_2^2)$ substituted for ε^2.

To calculate the poles of $\tilde{H}_a(s)\tilde{H}_a(-s)$, modify Γ, given by (1.28), by using $\delta_2^2/(1-\delta_2^2)$ in place of ε^2. This leads to

$$\tilde{\Gamma} \equiv \left[\frac{1+\sqrt{1+\delta_2^2/(1-\delta_2^2)}}{\delta_2/\sqrt{1-\delta_2^2}}\right]^{1/N} = \left[\frac{1+\sqrt{1-\delta_2^2}}{\delta_2}\right]^{1/N}. \tag{1.53}$$

Use this in (1.30) to calculate the poles of a Type I filter; then, use the computed s_k in (1.52) to compute

$$\tilde{s}_k = (j\Omega_s)\frac{j}{-\sin\left[(2k-1)\frac{\pi}{2N}\right]\frac{\tilde{\Gamma}^2-1}{2\tilde{\Gamma}} + j\cos\left[(2k-1)\frac{\pi}{2N}\right]\frac{\tilde{\Gamma}^2+1}{2\tilde{\Gamma}}}, \tag{1.54}$$

$1 \le k \le 2N$, the 2N poles of $\tilde{H}_a(s)\tilde{H}_a(-s)$. Of course, N of these poles are in the left-half s-plane, and they are assigned to $\tilde{H}_a(s)$.

For $1 \le \ell \le N$, the zeros are given by $\tilde{z}_\ell = j\Omega_\ell$, where the values Ω_ℓ are the roots of

$$T_N\left(\frac{\Omega_s}{\Omega_\ell}\right) = \cos\left(N\cos^{-1}\left\{\frac{\Omega_s}{\Omega_\ell}\right\}\right) = 0. \tag{1.55}$$

Clearly, we must have

$$N\cos^{-1}\left\{\frac{\Omega_s}{\Omega_\ell}\right\} = (2\ell-1)\frac{\pi}{2} \quad \Rightarrow \quad \frac{\Omega_s}{\Omega_\ell} = \cos\left((2\ell-1)\frac{\pi}{2N}\right), \tag{1.56}$$

and this leads to

$$\tilde{z}_\ell = j\Omega_\ell = j\frac{\Omega_s}{\cos\left((2\ell-1)\dfrac{\pi}{2N}\right)}, \quad \begin{array}{l} 1 \le \ell \le N,\ \text{N an even integer} \\ \\ 1 \le \ell \le N,\ \ell \ne (N+1)/2,\ \text{N an odd integer.} \end{array}, \tag{1.57}$$

So, we have N finite zeros if N is even, and (N-1) finite zeros if N is odd. If N is odd, we have a zero (for the integer $\ell = (N+1)/2$ that is omitted from (1.57)) at infinity, and we write $H_a(s)$ with only N-1 finite zero (and one zero at ∞).

Example: Design a normalized low-pass Chebyshev Type 2 filter that satisfies the following specifications:

1) Pass-band edge $\Omega p = .6$ radian/second.

2) Stop-band edge $\Omega s = 1$ radians/second.

3) Maximum pass-band attenuation = 1dB.

4) Minimum stop-band attenuation = 35 dB.

a) Find the minimum filter order.

b) Obtain the required transfer function.

c) Calculate the actual maximum pass-band attenuation.

The pass-band specification is computed to be

$$20\mathrm{Log}\,\delta_1 = 1 \quad \Rightarrow \qquad \delta_1 = 10^{-1/20} = .89125 \tag{1.58}$$

Equation (1.40) is used to compute

$$\varepsilon = \frac{\sqrt{1-\delta_1^2}}{\delta_1} = \frac{\sqrt{1-(.89125)^2}}{.89125} = .5088 \tag{1.59}$$

The stop-band specification is computed to be

$$20\mathrm{Log}\,\delta_2 = -35 \qquad \Rightarrow \qquad \delta_2 = 10^{-35/20} = .01778 \tag{1.60}$$

Now, filter order n can be computed as

$$N = \frac{\cosh^{-1}\left(\sqrt{1-\delta_2^2}\,\Big/\varepsilon\delta_2\right)}{\cosh^{-1}(\Omega_s/\Omega_p)} = \frac{\cosh^{-1}\left(\sqrt{1-(.01778)^2}\,\Big/\{(.5088)(.01778)\}\right)}{\cosh^{-1}(1/.6)} = 4.9135\,,$$

so we round upward to N = 5. Rounding N upward will cause a decrease in the effective value of ε and the pass-band specification to be exceeded.

The Matlab *Cheb2ord* function can be used to confirm these results.

```
>> Wp =.6;
>>Ws = 1;
>>Rp = 1;
>>Rs = 35;
```

```
>>N = cheb2ord(Wp,Ws,Rp,Rs,'s')
N =
     5
```

Equation (1.53) can be used to compute

$$\tilde{\Gamma} = \left[\frac{1+\sqrt{1-\delta_2^{\,2}}}{\delta_2}\right]^{1/N} = \left[\frac{1+\sqrt{1-(.01778)^2}}{.01778}\right]^{1/5} = 2.57157 \tag{1.61}$$

Equation (1.54) yields the 2N poles of $\tilde{H}_a(s)\tilde{H}_a(-s)$; the left-half-plane poles are $\tilde{s}_6$ through $\tilde{s}_{10}$, and they are computed as

$$\tilde{s}_k = \frac{-1}{-\sin\left[(2k-1)\frac{\pi}{10}\right]\frac{(2.57157)^2-1}{2.57157} + j\cos\left[(2k-1)\frac{\pi}{2N}\right]\frac{(2.57157)^2+1}{2.57157}}, \quad 6 \le k \le 10$$

$$\begin{aligned} &\tilde{s}_6 = -.1609 + .6718j && \tilde{s}_9 = (\tilde{s}_7)^* \\ &\tilde{s}_7 = -.5746 + .5662j && \tilde{s}_{10} = (\tilde{s}_6)^* \\ &\tilde{s}_8 = -.9163 \end{aligned} \tag{1.62}$$

Matlab was used to compute these six poles; for further processing, they are left in the Matlab environment as the variables s6 through s10. The characteristic equation can be computed by using the Matlab code

```
>> sym s
>> expand((s^2-2*real(s6)*s+abs(s6)^2)*(s^2-2*real(s7)*s+abs(s7)^2)*(s-real(s8)))
```

Matlab will return a polynomial with rational coefficients. Use Matlab to evaluate these rational coefficients and obtain the characteristic polynomial

$$s^5 + 2.3874s^4 + 2.8459s^3 + 2.1304s^2 + 1.0050s + .2846 \tag{1.63}$$

The zeros of $\tilde{H}_a(s)$ are

$$z_\ell = j\frac{\Omega_s}{\cos\left[(2\ell-1)\frac{\pi}{2N}\right]}, \qquad 1 \le \ell \le 5. \tag{1.64}$$

They are computed to be

$$\begin{aligned} z_1 &= 1.0515j \\ z_2 &= 1.7013j \\ z_3 &= \infty \\ z_4 &= z_2\,* \\ z_5 &= z_1\,* \end{aligned} \tag{1.65}$$

The numerator polynomial of $\tilde{H}_a(s)$ is s^4+4s^2+3.2, a result found by using (1.65). Finally, (1.63) and (1.65) are used to write

$$\tilde{H}_a(s) = \frac{.088928(s^4+4s^2+3.2)}{s^5 + 2.3874s^4 + 2.8459s^3 + 2.1304s^2 + 1.0050s + .2846} \tag{1.66}$$

as the filter transfer function. Note that the numerator gain constant was set to obtain $\tilde{H}_a(0)$ = 1. This result can be confirmed using the Matlab *cheby2* function as

```
>> Rs =35;
>> n = 5;
>> Ws = 1;
>> [b,a]=cheby2(n,Rs,Ws,'s')

b =
     0    0.0889    0.0000    0.3557    0.0000    0.2846
a =
     1.0000    2.3874    2.8459    2.1304    1.0050    .2846
```

Finally, Matlab can be used to find the actual pass-band and stop-band attenuation values. This can be accomplished by using the code

```
>> 20*log10(abs(freqs(b,a,[.6 1])))

ans = -.8427   -35.0000
```

Note that we met exactly the stop-band specification of 35 dB attenuation, and we exceeded the passband spec. of 1dB. The Matlab-generated magnitude and phase response is given by Fig. 5.

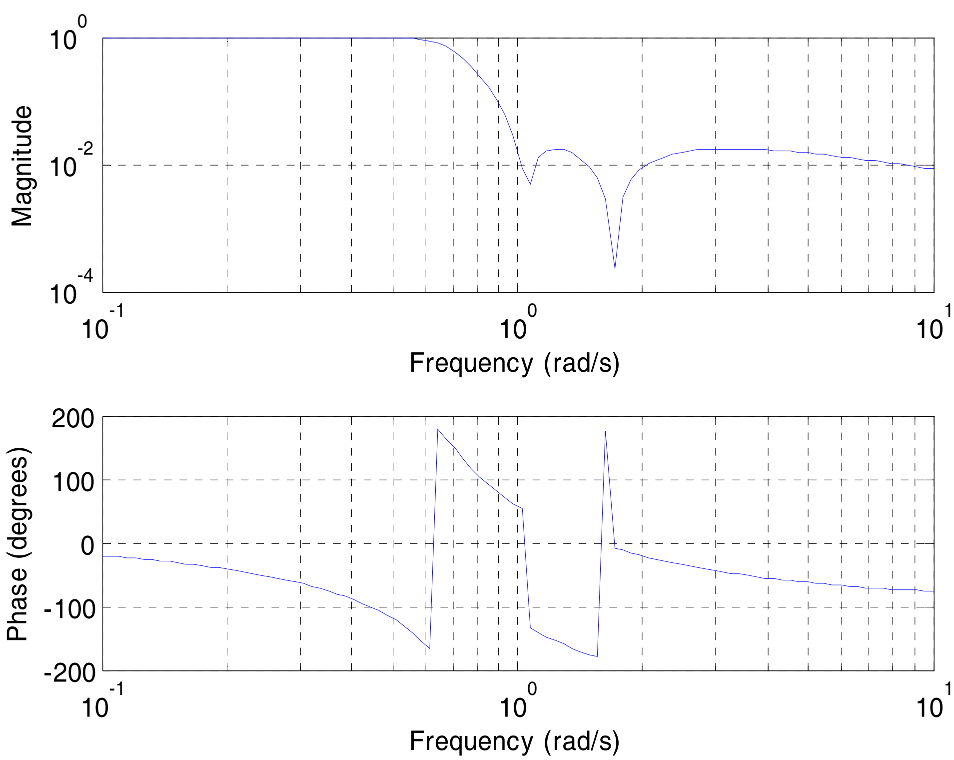

Fig. 5: Magnitude and phase response of a 5th - order, Type 2 Chebyshev filter with pass-band edge Ω_p = .6 radian/second, stop-band edge Ωs = 1 radians/second, maximum pass-band attenuation = 1dB and minimum stop-band attenuation = 35 dB.