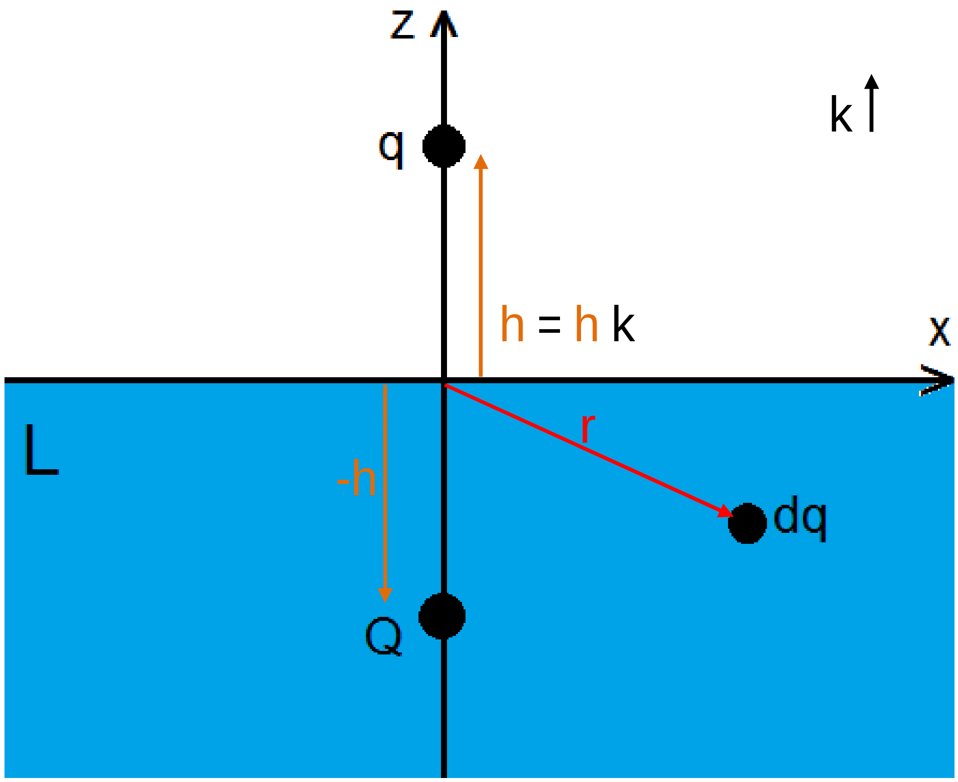
For the solution let us assume that:

- The liquid's dimensions are big enough compared to anything else in the problem, so we can assume that they are infinity.
- The dielectric is linear.

Let $\mathbf{P}$ be the dielectric polarization. At the field due to bounded charges contribute a volume charge density $\rho_b = -\nabla \cdot \mathbf{P}$ and a surface charge density $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{k}}$. (Proof: D. Griffiths "Introduction to Electrodynamics" chapter 4.2.1).



We will work as described below (all the fields are computed for the point where q is):

1. We will find the field $\mathbf{E_1}$ due to ρ_b including the field due to Q.
2. We will find σ_b field $\mathbf{E_2}$.
3. The sum $\mathbf{E_1} + \mathbf{E_2}$ is the total field that acts on q, so we can compute the force.

Dielectric is linear:

$$\mathbf{P} = (\varepsilon - 1)\varepsilon_0 \mathbf{E_f}$$

where $\mathbf{E_f}$ is the field produced by the free charges (Q and q).

We can find $\mathbf{E_f}$ using Coulomb's law:

$$\mathbf{E_f} = \frac{q}{4\pi\varepsilon_0}\frac{\mathbf{r}-\mathbf{h}}{|\mathbf{r}-\mathbf{h}|^3} + \frac{Q}{4\pi\varepsilon_0}\frac{\mathbf{r}+\mathbf{h}}{|\mathbf{r}+\mathbf{h}|^3} \quad (1)$$

Finding $\mathbf{E_1}$

As said above:

$$\rho_b = -\nabla\cdot\mathbf{P} = -(\varepsilon-1)\varepsilon_0\nabla\cdot\mathbf{E_f} \qquad (2)$$

Using Gauss's law:

$$\varepsilon_0\nabla\cdot\mathbf{E_f} = \rho_f$$

where ρ_f is the density of the free charges. Since we have point charges we will use Dirac's 3D delta function:

$$\rho_f = q\delta^3(\mathbf{r}-\mathbf{h}) + Q\delta^3(\mathbf{r}+\mathbf{h})$$

We note that $\delta^3(\mathbf{r})$ is 0 everywhere except from the point for which $\mathbf{r}=\mathbf{0}$ where it is infinity.

$$(2) \Longrightarrow \rho_b = (1-\varepsilon)(q\delta^3(\mathbf{r}-\mathbf{h}) + Q\delta^3(\mathbf{r}+\mathbf{h}))$$

If we also add $Q\delta^3(\mathbf{r}+\mathbf{h})$ we include Q charge contribution:

$$\rho = \rho_b + Q\delta^3(\mathbf{r}+\mathbf{h}) = (1-\varepsilon)q\delta^3(\mathbf{r}-\mathbf{h}) + (2-\varepsilon)Q\delta^3(\mathbf{r}+\mathbf{h})$$

$\mathbf{E_1}$ is the field due to ρ at the point $(0,0,h)$. We can compute it using Coulomb's law:

$$d\mathbf{E_1} = \frac{dq}{4\pi\varepsilon_0}\frac{\mathbf{h}-\mathbf{r}}{|\mathbf{h}-\mathbf{r}|^3} = \frac{\rho dV}{4\pi\varepsilon_0}\frac{\mathbf{h}-\mathbf{r}}{|\mathbf{h}-\mathbf{r}|^3}$$

Integrating at liquid's region (L):

$$\mathbf{E_1} = \frac{1}{4\pi\varepsilon_0}\int_L \rho\frac{\mathbf{h}-\mathbf{r}}{|\mathbf{h}-\mathbf{r}|^3}dV =$$

$$= \frac{(1-\varepsilon)q}{4\pi\varepsilon_0}\int_L \frac{\mathbf{h}-\mathbf{r}}{|\mathbf{h}-\mathbf{r}|^3}\delta^3(\mathbf{r}-\mathbf{h})\,dV + \frac{(2-\varepsilon)Q}{4\pi\varepsilon_0}\int_L \frac{\mathbf{h}-\mathbf{r}}{|\mathbf{h}-\mathbf{r}|^3}\delta^3(\mathbf{r}+\mathbf{h})\,dV$$

Region L does not contain any point for which $\mathbf{r}=\mathbf{h}$ so:

$$\delta^3(\mathbf{r}-\mathbf{h}) = 0 \quad \forall\,\mathbf{r} \Longrightarrow \int_L \frac{\mathbf{h}-\mathbf{r}}{|\mathbf{h}-\mathbf{r}|^3}\delta^3(\mathbf{r}-\mathbf{h})\,dV = 0$$

But it contains the point for which $\mathbf{r}=-\mathbf{h}$, so:

$$\int_L \frac{\mathbf{h}-\mathbf{r}}{|\mathbf{h}-\mathbf{r}|^3}\delta^3(\mathbf{r}+\mathbf{h})\,dV = \frac{\mathbf{h}-(-\mathbf{h})}{|\mathbf{h}-(-\mathbf{h})|^3} = \frac{2\mathbf{h}}{8h^3} = \frac{\hat{\mathbf{k}}}{4h^2}$$

Finally:

$$\mathbf{E_1} = \frac{(2-\varepsilon)Q}{4\pi\varepsilon_0}\frac{\hat{\mathbf{k}}}{4h^2} = \frac{(2-\varepsilon)Q}{16\pi\varepsilon_0 h^2}\hat{\mathbf{k}}$$

Finding $\mathbf{E_2}$

As we said:

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{k}} = (\varepsilon-1)\varepsilon_0\mathbf{E_f}\cdot\hat{\mathbf{k}}$$

We know $\mathbf{E_f}$ from (1), so:

$$\sigma_b = (\varepsilon-1)\varepsilon_0\left(\frac{q}{4\pi\varepsilon_0}\frac{\mathbf{r}-\mathbf{h}}{|\mathbf{r}-\mathbf{h}|^3} + \frac{Q}{4\pi\varepsilon_0}\frac{\mathbf{r}+\mathbf{h}}{|\mathbf{r}+\mathbf{h}|^3}\right)\cdot\hat{\mathbf{k}} \Longrightarrow$$

$$\Longrightarrow \sigma_b = \frac{\varepsilon-1}{4\pi}\left(q\frac{\mathbf{r}\cdot\hat{\mathbf{k}}-h}{|\mathbf{r}-\mathbf{h}|^3} + Q\frac{\mathbf{r}\cdot\hat{\mathbf{k}}+h}{|\mathbf{r}+\mathbf{h}|^3}\right)$$

We are at liquid's surface, so $\mathbf{r}$ is perpendicular to $\hat{\mathbf{k}}$, which means $\mathbf{r}\cdot\hat{\mathbf{k}} = 0$. Also, using Pythagorean theorem:

$$|\mathbf{r}-\mathbf{h}| = |\mathbf{r}+\mathbf{h}| = \sqrt{r^2+h^2}$$

Hence:

$$\sigma_b = \frac{(\varepsilon-1)(Q-q)h}{4\pi(r^2+h^2)^{3/2}}$$

We will use Coulomb's law again to compute field $\mathbf{E_2}$:

$$d\mathbf{E_2} = \frac{dq}{4\pi\varepsilon_0}\frac{\mathbf{h}-\mathbf{r}}{|\mathbf{h}-\mathbf{r}|^3} = \frac{\mathbf{h}-\mathbf{r}}{4\pi\varepsilon_0(r^2+h^2)^{3/2}}dq$$

Let's find the $z'z$ axis component of this vector:

$$dE_{2z} = d\mathbf{E_2}\cdot\hat{\mathbf{k}} = \frac{\mathbf{h}\cdot\hat{\mathbf{k}} - \mathbf{r}\cdot\hat{\mathbf{k}}}{4\pi\varepsilon_0(r^2+h^2)^{3/2}}dq = \frac{h\,dq}{4\pi\varepsilon_0(r^2+h^2)^{3/2}} = \frac{\sigma_b h\,dS}{4\pi\varepsilon_0(r^2+h^2)^{3/2}}$$

$$\Longrightarrow dE_{2z} = \frac{(\varepsilon-1)(Q-q)h^2}{16\pi^2\varepsilon_0(r^2+h^2)^3}dS$$

Integrating on the surface S of the liquid using polar coordinates:

$$E_{2z} = \frac{(\varepsilon-1)(Q-q)h^2}{16\pi^2\varepsilon_0}\int_S\frac{dS}{(r^2+h^2)^3} = \frac{(\varepsilon-1)(Q-q)h^2}{16\pi^2\varepsilon_0}\int_0^{2\pi}\int_0^{\infty}\frac{rdrd\varphi}{(r^2+h^2)^3} =$$

$$= \frac{(\varepsilon - 1)(Q - q)h^2}{16\pi^2\varepsilon_0} 2\pi \left[\frac{1}{4(r^2 + h^2)^2}\right]_\infty^0 = \frac{(\varepsilon - 1)(Q - q)h^2}{8\pi\varepsilon_0} \frac{1}{4h^4} \Longrightarrow$$

$$\Longrightarrow E_{2z} = \frac{(\varepsilon - 1)(Q - q)}{32\pi\varepsilon_0 h^2}$$

Taking into account the symmetry of the problem, we can see that $\mathbf{E_2}$ cannot have components in any other axis except $z'z$. So:

$$\mathbf{E_2} = E_{2z}\hat{\mathbf{k}} = \frac{(\varepsilon - 1)(Q - q)}{32\pi\varepsilon_0 h^2}\hat{\mathbf{k}}$$

Force on q

The total field at the point where q is:

$$\mathbf{E} = \mathbf{E_1} + \mathbf{E_2} = \frac{(2 - \varepsilon)Q}{16\pi\varepsilon_0 h^2}\hat{\mathbf{k}} + \frac{(\varepsilon - 1)(Q - q)}{32\pi\varepsilon_0 h^2}\hat{\mathbf{k}} \Longrightarrow$$

$$\Longrightarrow \mathbf{E} = \frac{(4 - 2\varepsilon)Q + (\varepsilon - 1)Q + (1 - \varepsilon)q}{32\pi\varepsilon_0 h^2}\hat{\mathbf{k}} = \frac{(3 - \varepsilon)Q + (1 - \varepsilon)q}{32\pi\varepsilon_0 h^2}\hat{\mathbf{k}}$$

Hence, the total force is:

$$\mathbf{F} = \mathbf{E}q = \frac{(3 - \varepsilon)Qq + (1 - \varepsilon)q^2}{32\pi\varepsilon_0 h^2}\hat{\mathbf{k}}$$

This equation also gives the direction of $\mathbf{F}$. If $\mathbf{F} \uparrow\uparrow \hat{\mathbf{k}}$ the force is repulsive, while if $\mathbf{F} \uparrow\downarrow \hat{\mathbf{k}}$ the force is attractive. Now let's see something strange. Consider that the liquid is water, for which $\varepsilon \cong 80$.

The force is repulsive if:

$$79q^2 + 77Qq < 0 \Longrightarrow \begin{cases} -\frac{77}{79}Q < q < 0, & if\ Q > 0 \\ 0 < q < -\frac{77}{79}Q, & if\ Q < 0 \end{cases}$$

We can see that in all cases in which we have repulsive force it is $Qq < 0$. That's exactly the opposite of the case we just have the Coulomb force, because then when it is $Qq < 0$ the force is attractive.

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