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Mathematical Challenges

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1. Given a vector field

$$F\,:\,\mathbb{R}^3\longrightarrow\mathbb{R}^3\,,\;F(x,y,z)=\begin{pmatrix}x-y^2\\x^2\\z\end{pmatrix}$$

Calculate the circulation of F along the mathematically positive oriented unit circle in the (x,y)-plane

(a) by using the curl of a vector field.

(b) directly by an appropriate path.

Reason: Circulation Of A Vector Field.

Solution: Let D be the unit disk around the origin with boundary ∂D. By Stoke's theorem we get

$$\int_{\partial D}F(r)\cdot dr=\int_D \operatorname{curl}(F)\cdot n\,dS$$

The curl of F is given by

$$\operatorname{curl}(F)=\begin{pmatrix}\partial_x\\\partial_y\\\partial_z\end{pmatrix}\wedge\begin{pmatrix}x-y^2\\x^2\\z\end{pmatrix}=\begin{pmatrix}0\\0\\2x+2y\end{pmatrix}$$

and has only a z-component. Parametrization of the unit disk in polar coordinates $\psi(r,\varphi)=(r\cos\varphi,r\sin\varphi,0)$ with a normal vector obeying the right hand rule, i.e. pointing to the positive z-direction

$$n(r,\varphi)=\partial_r\psi\wedge\partial_\varphi\psi=\begin{pmatrix}\cos\varphi\\\sin\varphi\\0\end{pmatrix}\wedge\begin{pmatrix}-r\sin\varphi\\r\cos\varphi\\0\end{pmatrix}=\begin{pmatrix}0\\0\\r\end{pmatrix}$$

Thus the circulation can be expressed as surface integral

$$\begin{aligned}\int_{\partial D}F(r)\,dr&=\int_D\operatorname{curl}(F)\cdot n\,dS\\&=\int_0^1 dr\int_0^{2\pi}d\varphi\,\begin{pmatrix}0\\0\\r\end{pmatrix}\cdot\begin{pmatrix}0\\0\\2r\cos\varphi+2r\sin\varphi\end{pmatrix}\\&=\int_0^1 dr\int_0^{2\pi}d\varphi\;2r^2(\cos\varphi+\sin\varphi)=0\end{aligned}$$

Alternatively, we can calculate the circulation directly, too. We therefore parameterize the boundary of the unit disk by $\gamma(\varphi) = (\cos\varphi, \sin\varphi)$, $\dot\gamma(\varphi) = (-\sin\varphi, \cos\varphi, 0)$ and calculate

$$\begin{aligned}
\int_{\partial D} F(r)\,dr &= \int_0^{2\pi} F(\gamma(\varphi)) \cdot \dot{\gamma(\varphi)}\,d\varphi \\
&= \int_0^{2\pi} \begin{pmatrix} \cos\varphi - \sin^2\varphi \\ \cos^2\varphi \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} d\varphi \\
&= \int_0^{2\pi} (\cos^3\varphi + \sin^3\varphi - \sin\varphi\cos\varphi)\,d\varphi \\
&= \frac{1}{3}\left[\cos^2\varphi\sin\varphi + 2\sin\varphi\right]_0^{2\pi} \\
&- \frac{1}{3}\left[\sin^2\varphi\cos\varphi + 2\cos\varphi\right]_0^{2\pi} \\
&- \frac{1}{2}\left[\sin^2\varphi\right]_0^{2\pi} \\
&= 0 - 0 - \frac{2}{3} + \frac{2}{3} - 0 + 0 \\
&= 0
\end{aligned}$$

2. An odd prime p can be written as $p = x^2 + y^2$ with integers $x, y \in \mathbb{Z}$ if and only is $p \equiv 1 \mod 4$.

   Reason: Fermat's Theorem About The Sum Of Two Squares.

   Solution:

   - Let $p = 4k+1$ and $S := \{\,(x,y,z) \in \mathbb{N}^3 \mid x^2+4yz = p\,\}$. Then S has two involutions $(x,y,z) \longmapsto (x,z,y)$ whose fixed points (x,y,y) correspond to representations of p as sum of two squares, and

     $$(x,y,z) \longmapsto \begin{cases} (x+2z, z, y-x-z), & \text{if } x \leq y-z \\ (2y-x, y, x-y+z), & \text{if } y-z < x < 2y \\ (x-2y, x-y+z, y), & \text{if } x \geq 2y \end{cases}$$

     which has exactly one fixed point $(1,1,k)$. Two involutions over the same finite set must have sets of fixed points with the same parity, which is odd in our case due to the second involution. (It is an easy proof by induction, that the order of the set of fixed points of an involution on a finite set has the same parity as the set has. Hence the parity is independent of a certain involution.)

So the first involution has a nonzero fixed point (x_0, y_0, y_0) which means $x_0^2 + (2y_0)^2 = p$.

If conversely $p = x^2 + y^2$, then $x^2 + y^2 \equiv r \mod 4$ with $r \in \{0, 1, 2\}$ and $p = 2n + 1 \equiv s \mod 4$ with $s \in \{1, 3\}$. Thus $p = x^2 + y^2 \equiv 1 \mod 4$.

- The above theorem can also be proven with the help of Minkowski's lattice theorem:

  Let $\Gamma \subseteq \mathbb{R}^d$ be a lattice and $C \subseteq \mathbb{R}^d$ a convex, bounded and with respect to the origin symmetric area. If $\operatorname{vol}(C) > 2^d \cdot \operatorname{vol}(\Gamma)$, where the volume of the lattice is meant to be the volume of the primitive cell, then C contains besides the origin at least one more lattice point.

  Let $p = 4k + 1$. Then by Euler's criterion

  $$\left(\frac{-1}{p}\right) \equiv (-1)^{\frac{p-1}{2}} \equiv ((-1)^2)^k \equiv 1 \mod p$$

  -1 is a quadratic residue modulo p, i.e. there is an integer m such that $-1 \equiv m^2 \mod p$ or $p \,|\, (m^2+1)$. Let $\hat{\imath}, \hat{\jmath}$ be the standard basis of $\mathbb{R}^2$. Set $u = \hat{\imath} + m\hat{\jmath}$, $v = p\hat{\jmath}$ and consider the lattice $\Gamma = \mathbb{Z}u + \mathbb{Z}v$. If $w = \alpha u + \beta v = \alpha i + (\alpha m + \beta p)j \in \Gamma$ then

  $$\|w^2\| \equiv \alpha^2 + (\alpha m + \beta p)^2 \equiv \alpha^2(1 + m^2) \equiv 0 \mod p$$

  and $p \,|\, \|w\|^2$ for any $w \in \Gamma$. Furthermore we have $\operatorname{vol}(\Gamma) = p$ and $\operatorname{vol}(C) = 2\pi p > 2^2 \operatorname{vol}(\Gamma)$ for the open disc $C = U(0; \sqrt{2p})$. Then by Minkowski's theorem there exists a nonzero vector $w \in \Gamma$ with $w \in C$. Hence $\|w\|^2 < 2p$ and $p \,|\, \|w\|^2$ so

  $$p = \|w\|^2 = \alpha^2 + (\alpha m + \beta p)^2$$

  is the sum of two squares. The other direction follows as above.

3. Let $\mathbb{T}^n := \mathbb{R}^n / \mathbb{Z}^n$ equipped with the quotient topology according to the projection

   $$\pi : \mathbb{R}^n \longrightarrow \mathbb{T}^n \,,\, \pi(a) = a + \mathbb{Z}^n.$$

   Show that $\mathbb{T}^n$ is a topological manifold.

   Reason: Torus.

   Solution: We have to show that $\mathbb{T}^n$ is Hausdorff, second countable, and that every point has an open neighborhood which is homeomorphic to an open subset of $\mathbb{R}^n$.

$\mathbb{R}^n$ is second countable, because we can choose open balls of rational points with rational radius as basis. Quotient building doesn't change this property, i.e. the torus is second countable, too.

Let $P, Q \in \mathbb{T}^n$ be two distinct points. As π is surjective, there are $x, y \in \mathbb{R}^n$ such that $\pi(x) = P, \pi(y) = Q$. The preimages of Q under π is the set $\pi^{-1}(Q) = y + \mathbb{Z}^n$. Choose $\varepsilon > 0$ small enough, such that $B_\varepsilon(x) \subseteq \mathbb{R}^n$ and $B_\varepsilon(y+k) \subseteq \mathbb{R}^n$ are disjoint for all $k \in \mathbb{Z}^n$. Then $P \in \pi(B_\varepsilon(x))$ and $Q \in B_\varepsilon(y)$ are disjoint sets. Thus $\mathbb{T}^n$ is Hausdorff, if π is open.

Let $U \subseteq \mathbb{R}^n$ be open. Then $\pi(U)$ is open by the definition of the quotient topology, if and only if $\pi^{-1}(\pi(U)) \subseteq \mathbb{R}^n$ is open.

$$\begin{aligned}
x \in \pi^{-1}(\pi(U)) &\Longleftrightarrow \pi(x) \in \pi(U) \\
&\Longleftrightarrow (\exists\, y \in U)\, \pi(x) = \pi(y) \\
&\Longleftrightarrow (\exists\, y \in U) x - y \in \mathbb{Z}^n \\
&\Longleftrightarrow (\exists\, y \in U) x \in y + \mathbb{Z}^n \\
&\Longleftrightarrow x \in U + \mathbb{Z}^n \\
&\Longleftrightarrow (\exists\, k \in \mathbb{Z}^n)\, x \in U + k \\
&\Longleftrightarrow x \in \bigcup_{k \in \mathbb{Z}^n} U + k
\end{aligned}$$

Thus $\pi^{-1}(\pi(U)) = \bigcup_{k\in\mathbb{Z}^n} U + k$, and this is an open set in $\mathbb{R}^n$.

Now let us fix a point $a \in \mathbb{R}^n$ and set $V_a := B_{1/2}(a) \subseteq \mathbb{R}^n$, the open ball around a, and $U_a := \pi(V_a) \subseteq \mathbb{T}^n$ which is open as well by the previous argument that π is open. The mapping $\rho : V_a \longrightarrow U_a$ with $y \longmapsto \pi(y)$ is continuous, since it is the restriction of the continuous function π, surjective by definition, and open as restriction of an open function on open sets in domain and codomain. Hence we must show, that ρ is injective, too. Assume two points $x \neq y$ in V_a such that $\rho(x) = \pi(x) = x + \mathbb{Z}^n = y + \mathbb{Z}^n = \pi(y) = \rho(y.)$ Since the diameter of V_a is 1, we have $\|x - y\| < 1$. But $x - z \in \mathbb{Z}^n$ which implies $\|x - y\| \geq 1$, a contradiction. Thus ρ is continuous, open and bijective, i.e. a homeomorphism. Its inverse $\rho_a^{-1} : U_a \longrightarrow V_a$ is therefore a chart at a and $\mathcal{A} := \{\, \rho_a \,|\, a \in \mathbb{R}^n \,\}$ an atlas, because each point in $\mathbb{T}^\ltimes$ is at least in one chart. Hence $\mathbb{T}^n$ is an n-dimensional topological manifold.

4. Let $\alpha \in \mathbb{R} - \{0\}$. Determine all functions $f : \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{>0}$ which satisfy

for all $x, y \in \mathbb{R}_{>0}$

$$f(f(x)+y) = \alpha x + \frac{1}{f\left(\dfrac{1}{y}\right)}$$

Reason: Functional Equation.

Hint: Use Cauchy's first functional equation.

Solution: We will show that only $\alpha = 1$ allows a solution, namely $f(x) = x$.

We obviously have $\alpha > 0$ for the expression on the left would otherwise become negative for large x. Moreover

$$\begin{aligned} f(x) = f(y) &\Longrightarrow \\ \alpha x + \frac{1}{f\left(\dfrac{1}{z}\right)} &= f(f(x)+z) = f(f(y)+z) = \alpha y + \frac{1}{f\left(\dfrac{1}{z}\right)} \\ &\Longrightarrow x = y \end{aligned}$$

$f(x)$ is not bounded from above, since $f(f(x)+1) = \alpha x + \dfrac{1}{f(1)}$ and we can simply choose x large enough. Let $\beta \in (0, \infty)$. Then

$$f\left(f\left(\frac{\beta f(y^{-1})-1}{\alpha f(y^{-1})}\right)+y\right) = \alpha \cdot \frac{\beta f(y^{-1})-1}{\alpha f(y^{-1})} + \frac{1}{f(y^{-1})} = \beta$$

for any $y > 0$. With an y chosen such that $f(y^{-1}) > \beta^{-1}$ we get $\dfrac{\beta f(y^{-1})-1}{\alpha f(y^{-1})} > 0$ and so a preimage of β in the domain of $f(x)$.

This means that $f(x)$ is bijective, that is there is a one-to-one correspondence between all $y > 0$ and all $f(y)$. Hence we can write

$$f(f(x)+f(y)) = \alpha x + \frac{1}{f\left(\dfrac{1}{f(y)}\right)} = \alpha y + \frac{1}{f\left(\dfrac{1}{f(x)}\right)}$$

for the symmetry on the left hand side. If we fix y and set $C := \dfrac{1}{f\left(\dfrac{1}{f(y)}\right)} - \alpha y$ we get $\alpha x + C = \dfrac{1}{f\left(\dfrac{1}{f(x)}\right)} > 0$. We also have $C > 0$

since otherwise we could choose x small enough and get a negative function value. Now $f(f(x)+f(y)) = \alpha y + \alpha x + C$.

$$\begin{aligned} x+y=z+w &\Longrightarrow \alpha x + \alpha y + C = \alpha z + \alpha w + C \\ &\Longrightarrow f(f(x)+f(y)) = f(f(z)+f(w)) \\ &\Longrightarrow f(x)+f(y) = f(z)+f(w) \\ &\Longrightarrow f(x+1)+f(y+1) = f(x+y+1)+f(1) \end{aligned}$$

We thus get for the function $g : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ defined by $g(x) := f(x+1)$ the functional equation $g(x)+g(y) = g(x+y)+g(0)$. We now set $h(x) := g(x) - g(0) \geq -g(0)$ and get Cauchy's functional equation

$$h(x)+h(y) = g(x)-g(0)+g(y)-g(0) = g(x+y)-g(0) = h(x+y).$$

If there was a function value $h(t) < 0$ then $h(nt) = nh(t)$ would each any negative number, which is impossible as $-g(0)$ is a fixed lower bound. So $h(x) \geq 0$ for all $x \geq 0$, and for $0 < uyv$ we have $h(v) = h(v-u)+h(u) \geq h(u)$, i.e. $h(x)$ is a monotone function.The only solutions to Cauchy's functional equation which are monotone are linear functions $h(x) = cx$.

(see https://en.wikipedia.org/wiki/Cauchy%27s_functional_equation)

Hence we get for $x > 1$ that $f(x) = g(x-1) = h(x-1) - g(0) = cx+d$ for suitable constants c, d. Let $0 < x \leq 1$ and set $y = 3, z = 2, w = x+1$ so

$$f(x) = f(z)+f(w)-f(y) = (2c+d)+c(x+1)+d-3c-d = cx+d$$

which means that $f(x) = cx+d$ on its entire domain. As $f(x)$ reaches all positive values, and is positive itself, we conclude that $c > 0$ and $d = 0$ for we would get negative or missing values otherwise. This means

$$f(f(x)+y) = f(cx+y) = c^2x + cy = \alpha x + \frac{1}{\frac{c}{y}} = \alpha x + \frac{y}{c}$$

Comparing coefficients yields $c^2 = \alpha$ and $c^2 = 1$, hence $f(x) = x$.

5. Show that the quaternion group $G = \{\pm 1, \pm i, \pm j, \pm k\}$ is a Hamilton group, and cannot be written as a semidirect product in a nontrivial way.

   Reason: Group Theory.

   Solution: Let's consider the subgroups of G. We clearly have the subgroup $\{\pm 1\}$, which is normal, since $m(-1)m^{-1} = -1$ for all $m \in G$. Any other subgroup U contains at least one pure quaternion or its negative, say w.l.o.g. $k \in U$. Then $k^2 = -1$ and $-k = k^3$ are also in U, i.e. $\langle k \rangle = \mathbb{Z}_4 \subseteq U$ and thus $U = \mathbb{Z}_4$, since any bigger subgroup would already have to be the entire group. Now $jkj^{-1} = ij^{-1} = -ij = -k \in U$ and analogue $iki^{-1} \in U$, i.e. U is a normal subgroup, too. Hence G is a Dedekind group (all subgroups are normal), and a Hamilton group (Dedekind and non abelian). Therefore any semidirect product in G is already a direct product. Thus $G \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ would be the only possible split, however, $\mathbb{Z}_2 \subseteq \mathbb{Z}_4$ in all possible combinations, and the product cannot be direct.

6. (a) Calculate

$$\frac{1}{2} - \frac{1}{12} + \frac{1}{40} - \frac{1}{112} \mp \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)2^{k+1}} \,.$$

   (b) Prove $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots = 1\,.$

   Reason: Trick For Infinite Series.

   Solution:

   (a) We introduce a parameter and define

$$f(x) := \frac{x}{2} - \frac{x^3}{12} + \frac{x^5}{40} - \frac{x^7}{112} \mp \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)2^{k+1}}$$

$$f'(x) = \frac{1}{2} - \frac{x^2}{4} + \frac{x^4}{8} - \frac{x^6}{16} \mp \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{2^{k+1}} = \frac{1}{2+x^2}$$

As $f(0)=0$ we get

$$\begin{aligned}
f(1) &= \sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k+1)2^{k+1}}\\
&= f(1)-f(0)=\int_0^1 f'(x)\,dx\\
&= \int_0^1 \frac{dx}{2+x^2}=\left[\frac{1}{\sqrt{2}}\cdot\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^1\\
&= \frac{1}{\sqrt{2}}\cdot\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)\approx 0.43521
\end{aligned}$$

(b)

$$\begin{aligned}
F(x) &:= \frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+\ldots=\frac{1}{x}e^x-1-\frac{1}{x}\\
F'(x) &= -\frac{1}{x^2}e^x+\frac{1}{x}e^x+\frac{1}{x^2}\\
&\Longrightarrow \frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots\\
&= \left(\frac{1}{2!}+\frac{2}{3!}x+\frac{3}{4!}x^2+\ldots\right)(1)\\
&= F'(1)=\left(-\frac{1}{x^2}e^x+\frac{1}{x}e^x+\frac{1}{x^2}\right)(1)\\
&= 1
\end{aligned}$$

7. Prove

$$\tan^{-1}(1/2)+\tan^{-1}(1/3)=\pi/4=4\tan^{-1}(1/5)-\tan^{-1}(1/239)$$

and use the power series representation

$$\tan^{-1}(x)=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}\mp\ldots=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}$$

to determine how many terms would it take to compute π up to 100 digits by $\pi=4\tan^{-1}(1)$ and by the formulas above.

Reason: Algorithmic Precision.

Solution:

$$z=a+ib=re^{i\varphi}=r(\cos\varphi+i\sin\varphi)\,,\;\tan\varphi=\frac{b}{a}$$

From $(2+i)(3+i) = 5(1+i)$ we get that the angles on the left $\tan^{-1}(1/2), \tan^{-1}(1/3)$ add up to the one on the right, $\pi/4$.

For the next identity we consider

$$\begin{aligned} (5+i)^4 &= (24+10i)^2 = 476+480i \\ (5+i)^4(-239+i) &= -114244(1+i) \\ 4\tan^{-1}(1/5) + (\pi - \tan^{-1}(1/239)) &= 5/4\pi \\ 4\tan^{-1}(1/5) - \tan^{-1}(1/239) &= \pi/4 \end{aligned}$$

In order to compute π to 100 digits with the alternating series, we need the nth term being smaller than 10^{-100}. That is, with $x=1$, $2k > 10^{100}$ or $k > \frac{1}{2}$ Googol.

$\tan^{-1}(1/2)\ :\ 2^{2k+1} \cdot 2k > 10^{100}$ or $(2k+1)\log(2) + \log(2k) > 100\log(10)$

We concentrate on the slower first term and get

$$k > \lceil -\frac{1}{2} + 50\frac{\log 10}{\log 2} \rceil = 166$$

and improve it to

$$k > 166 - \frac{1}{2}\log(2\cdot 166) > 166 - \lfloor \frac{\log 332}{2} \rfloor = 164$$

$$\begin{aligned} \tan^{-1}(1/3)\ &: 3^{2k+1}\cdot 2k > 10^{100} \text{ or } (2k+1)\log(3) + \log(2k) > 100\log(10) \\ k &> \lceil -\frac{1}{2} + 50\frac{\log 10}{\log 3} \rceil = 105 \\ k &> 105 - \lfloor \frac{\log 210}{2} \rfloor = 103 \end{aligned}$$

$$\begin{aligned} \tan^{-1}(1/5)\ &: 5^{2k+1}\cdot 2k > 10^{100} \text{ or } (2k+1)\log(5) + \log(2k) > 100\log(10) \\ k &> \lceil -\frac{1}{2} + 50\frac{\log 10}{\log 5} \rceil = 72 \\ k &> 72 - \lfloor \frac{\log 144}{2} \rfloor = 70 \end{aligned}$$

$$\begin{aligned} \tan^{-1}(1/239)\ &: 239^{2k+1}\cdot 2k > 10^{100} \text{ or } (2k+1)\log(39) + \log(2k) > 100\log(10) \\ k &> \lceil -\frac{1}{2} + 50\frac{\log 10}{\log 239} \rceil = 21 \\ k &> 21 - \lfloor \frac{\log 42}{2} \rfloor = 20 \end{aligned}$$

Thus we need a Googol steps to compute π with the standard tangent series up to 100 digits from $\tan^{-1}(1)$, $164+103=267$ steps with the second formula and $70+20=90$ steps with the third.

8. Calculate the following integrals

(a) $\displaystyle\int_0^{2\pi} e^{(e^{it})}\,dt$

(b) $\displaystyle\int_{|z|=1} \frac{\sin(z^2)}{(\sin z)^2}\,dz$

(c) $\displaystyle\int_{|z|=1} \sin\left(e^{1/z}\right)\,dz$

(d) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2-2x+2}\,dx$

Reason: Function Theory.

Solution:

(a) Consider the closed path $\gamma\,:\,[0,2\pi]\longrightarrow\mathbb{C}\,,\,\gamma(t)=e^{it}$ such that

$$\int_0^{2\pi} e^{(e^{it})}\,dt=\int_0^{2\pi}\frac{e^{(e^{it})}}{ie^{it}}\,ie^{it}\,dt=\frac{1}{i}\int_\gamma\frac{e^z}{z}\,dz=\frac{2\pi i}{i}\,e^0=2\pi$$

by Cauchy's integral formula.

(b) Consider the entire functions $f(z)=\sin(z^2)$ and $g(z)=(\sin z)^2$ where the zeros of $g(z)$ are all $\pi\mathbb{Z}$. Then f/g is holomorphic in $\{z\in\mathbb{C}\,:\,|z|<\pi\}-\{0\}$. Both functions have a twofold zero at $z=0$, since $f'(0)=g'(0)=0$ and $f''(0),g''(0)\neq 0$. Thus f/g has a removable singularity at $z=0$ and $\int_{|z|=1}(f(z)/g(z))dz=0$.

(c) (Residue Theorem)

$$\int_{|z|=1}\sin\left(e^{1/z}\right)\,dz=2\pi i\ \operatorname{Res}(z=0)\ \sin\left(e^{1/z}\right)$$

We develop $\sin\left(e^{1/z}\right)$ into a Laurent series at $z=0$ to calculate the residue. The function $w\longmapsto\sin\left(e^w\right)$ is holomorphic everywhere on $\mathbb{C}$ and can be developed into a power series with infinite radius of convergence, say $\sin\left(e^w\right)=\sum_{k=0}^\infty a_kw^k$. Thus $\sin\left(e^{1/z}\right)=\sum_{k=0}^\infty a_kz^{-k}$ and $\operatorname{Res}(z=0)\ \sin\left(e^{1/z}\right)=a_1=\sin\left(e^w\right)'\big|_{w=0}=\cos(1)$, all in all

$$\int_{|z|=1}\sin\left(e^{1/z}\right)\,dz=2\pi i\ \cos(1).$$

(Cauchy's integral formula)

$$\begin{aligned}\int_{|z|=1} \sin\left(e^{1/z}\right)\,dz &= \int_{-\pi}^{\pi} \sin\left(e^{(e^{-is})}\right)\,ie^{is}\,ds = \int_{-\pi}^{\pi} \frac{\sin\left(e^{(e^{it})}\right)}{e^{2it}}\,ie^{it}\,dt\\ &= \int_{|w|=1} \frac{\sin(e^{w})}{w^2}\,dw = 2\pi i\,(\sin(e^w))'\big|_{w=0}\\ &= 2\pi i\,\cos(1)\end{aligned}$$

(d) Let $f(z) = \dfrac{1}{z^2-2z+2}$ where the denominator $z^2-2z+2 = (z-1)^2+1 = (z-(1+i))(z-(1-i))$ has zeros $z_j = 1\pm i$, hence f is holomorphic on $\mathbb{C}-\{z_1,z_2\}$ with first order poles z_j.

Let $r>1$ and $\gamma_r\,:\,[0,\pi]\longrightarrow\mathbb{C}$ given as $\gamma_r(t) = 1+re^{it}$. Then we get from the residue theorem

$$\int_{1-r}^{1+r} f(x)\,dx\,+\int_{\gamma_r} f(z)\,dz = 2\pi i\;\mathrm{Res}(z=1+i)f(z).$$

As the pole $z=1+i$ is of first order, we have

$$\mathrm{Res}(z=1+i)f(z) = \lim_{z\to 1+i}(z-1-i)f(z) = \lim_{z\to 1+i}\frac{1}{z-(1-i)} = \frac{1}{2i}$$

Furthermore

$$\int_{\gamma_r} f(z)\,dz = \int_0^{\pi} f(1+re^{it})rie^{it}\,dt = \int_0^{\pi}\frac{rie^{it}}{(1+re^{it}-1)^2+1}\,dt$$

so

$$\left|\int_{\gamma_r} f(z)\,dz\right| \leq \int_0^{\pi}\frac{r}{|r^2e^{2it}+1|}\,dt \leq \int_0^{\pi}\frac{r}{r^2-1}\,dt = \frac{\pi r}{r^2-1}\stackrel{r\to\infty}{\longrightarrow} 0$$

f is a rational real function which hasn't any real poles. The degree of the denominator polynomial is two less than that of the numerator, so the integral we are looking for exists as improper Riemannian integral, i.e.

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{1}{x^2-2x+2}\,dx &= \lim_{r\to\infty}\int_{1-r}^{1+r} f(x)\,dx\\ &= 2\pi i\,\mathrm{Res}(z=1+i)f(z) - \lim_{r\to\infty}\int_{\gamma_r} f(z)\,dz\\ &= \pi\end{aligned}$$

9. Suppose someone gives you a coin and claims that this coin is biased; that it lands on heads only 48% of the time with an error margin of 2%. You decide to test the coin for yourself. If you want to be 95% confident that this coin is indeed biased, how many times must you flip the coin? Compare the estimations by the weak law of large numbers and by the central limit theorem!

   Reason: Coin Flips.

   Solution: Let X be the random variable such that $X = 1$ if the coin lands on heads and $X = 0$ for tails. Thus $\mu = 0.48 = p$ and $\sigma^2 = p(1-p) = 0.48 \cdot 0.52 = 0.2496$. To test the coin flip we flip it n times and allow for a 2% error of precision, i.e. $\varepsilon = 0.02$. This means we are testing the probability of the coin landing on heads being between $(0.46, 0.50)$.

   (a) (WLLN) By the law of large numbers, we want n such that

   $$P\left(\left|\bar{X} - 0.48\right| > 0.02\right) \leq \frac{0.2496}{n(0.02)^2}$$

   So for a 95% confidence interval we need

   $$\frac{0.2496}{n(0.02)^2} = 0.05 \Longleftrightarrow n = 12,480$$

   (b) (CLT)

   $$\begin{aligned} P\left(\frac{S_n}{n}; 0.50\right) &= P\left(\frac{S_n - 0.48n}{n} < 0.02\right) \\ &= P\left(\frac{S_n - 0.48n}{\sqrt{n \cdot 0.2496} < \frac{0.02\sqrt{n}}{\sqrt{0.2496}}}\right) \\ &\geq P\left(\frac{S_n - 0.48n}{\sqrt{n \cdot 0.2496}} \leq 0.04\sqrt{n}\right) \\ &\approx \Phi(0.04\sqrt{n}) \geq 0.95 \end{aligned}$$

   which means $0.04\sqrt{n} = 1.645$, i.e. $n = 1,692$.

   As we can see, the weak law of large numbers is not as powerful or accurate as the central limit theorem. However, it can still be used to a certain degree of accuracy.

10. A hat-check boy at a congress held at Hilbert's hotel completely loses track of which of hats belong to which owners, and hands them back at random to their owners as the latter leave. What is the probability P that nobody receives their own hat back?

Reason: Combinatorics.

Solution: Let D_n denote the number of derangements on a finite ordered set S of cardinality n. If s_m is the m-th element of S. If $A_m := \{\sigma \in \operatorname{Sym}(S) \,|\, \sigma(s_m) = s_m\}$, then the number W of permutations, with at least one fixed element is

$$W = \left| \bigcup_{m=1}^{n} A_m \right|$$

By the inclusion-exclusion-principle we get

$$\begin{aligned} W = \sum_{m_1=1}^{n} |A_{m_1}| \quad &- \sum_{1\leq m_1<m_2\leq n} |A_{m_1} \cap A_{m_2}| \\ + \sum_{1\leq m_1<m_2<m_3\leq n} &|A_{m_1} \cap A_{m_2} \cap A_{m_3}| \quad \mp \ldots \end{aligned}$$

Each value $A_{m_1} \cap \ldots \cap A_{m_k}$ represents the set of permutations which fix p values $m_1, \ldots, m_k$. Note that the number of permutations which fix k values only depends on k, not on the particular values of m. There are thus $\binom{n}{k}$ terms in each summation

$$W = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} |A_1 \cap \ldots \cap A_k|$$

$|A_1 \cap \ldots \cap A_k|$ is the number of permutations fixing k elements in position. This is equal to the number of permutations which rearrange the remaining $n-k$ elements, which is $(n-k)!$, i.e.

$$W = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (n-k)! = n! \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!}$$

So finally we have

$$D_n = |\operatorname{Sym}(S)| - W = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

and from that

$$P = \lim_{n\to\infty} \frac{D_n}{n!} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1} = \frac{1}{e} \approx 36.8\%$$

11. (HS-1) Prove that

$$\frac{1}{1+x+\dfrac{1}{y}}+\frac{1}{1+y+\dfrac{1}{z}}+\frac{1}{1+z+\dfrac{1}{x}}\leq 1$$

for all positive real numbers x, y, z

Reason: Inequality.

Solution: $0 \leq (xyz-1)^2 = x^2y^2z^2 - 2xyz + 1$ and thus $2 \leq xyz + \dfrac{1}{xyz}$. Therefore

$$\begin{aligned}
&6+2x+2y+2z+\frac{2}{x}+\frac{2}{y}+\frac{2}{z}+xy+yz+zx\\
&+\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}+\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\\
&\leq xyz+\frac{1}{xyz}+4+2x+2y+2z+\frac{2}{x}+\frac{2}{y}+\frac{2}{z}+xy+yz+zx\\
&+\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}+\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\\
&\Longleftrightarrow\\
&\left(1+x+\frac{1}{y}\right)\cdot\left(1+y+\frac{1}{z}\right)+\left(1+y+\frac{1}{z}\right)\cdot\left(1+z+\frac{1}{x}\right)\\
&+\left(1+z+\frac{1}{x}\right)\cdot\left(1+x+\frac{1}{y}\right)\\
&\leq\left(1+x+\frac{1}{y}\right)\cdot\left(1+y+\frac{1}{z}\right)\cdot\left(1+z+\frac{1}{x}\right)
\end{aligned}$$

12. (HS-2) Which is the smallest natural number greater than one such that the following statement holds:
In any set of n natural numbers are at least two numbers, whose sum or difference is dividable by seven.

Reason: Pigeon Hole Principle.

Solution: We can exclude $n = 2$ by $\{1, 2\}$ and $n = 3$ by $\{1, 2, 3\}$. We also exclude $n = 4$ by $\{4, 5, 6, 7\}$ which has sums $\{9, 10, 11, 12, 13\}$ and differences $\{3, 2, 1\}$. We will now show that $n = 5$ has the required property. If a set of five natural numbers contains two numbers with the same remainder by division by seven, them their difference is dividable by seven. Hence we may assume that all remainders are pairwise different: five out of $\{0,1,2,3,4,5,6\}$. Hence there are at most two remainders in $R=\{1,2,3,4,5,6\}$ which do not occur. However, we have three pairs $(1,6),(2,5),(3,4)$ whose sum is dividable by seven. Since we can exclude at most two of them, the statement follows.

13. (HS-3) Determine

$$\left[\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\frac{1}{\sqrt{5}+\sqrt{6}}+\ldots+\frac{1}{\sqrt{n^2-4}+\sqrt{n^2-3}}+\frac{1}{\sqrt{n^2-2}+\sqrt{n^2-1}}\right]$$

for any odd natural number $n\geq 3$ where $[n]=\lfloor n\rfloor$ is the greatest integer smaller or equal n.

Reason: Arithmetics.

Solution: Set

$$a=\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\frac{1}{\sqrt{5}+\sqrt{6}}+\ldots+\frac{1}{\sqrt{n^2-4}+\sqrt{n^2-3}}+\frac{1}{\sqrt{n^2-2}+\sqrt{n^2-1}}$$

and

$$b=\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{4}+\sqrt{5}}+\frac{1}{\sqrt{6}+\sqrt{7}}+\ldots+\frac{1}{\sqrt{n^2-3}+\sqrt{n^2-2}}+\frac{1}{\sqrt{n^2-1}+\sqrt{n^2}}$$

Note that

$$\frac{1}{\sqrt{k}+\sqrt{k+1}}>\frac{1}{\sqrt{k+1}+\sqrt{k+2}}$$

for any positive k, so $a>b$ or

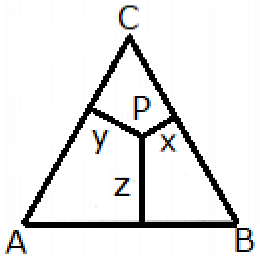
$$\begin{aligned}0<a-b&=\frac{1}{\sqrt{1}+\sqrt{2}}-\left(\frac{1}{\sqrt{2}+\sqrt{3}}-\frac{1}{\sqrt{3}+\sqrt{4}}\right)-\left(\frac{1}{\sqrt{4}+\sqrt{5}}-\frac{1}{\sqrt{5}+\sqrt{6}}\right)-\ldots\\ &\ldots-\left(\frac{1}{\sqrt{n^2-3}+\sqrt{n^2-2}}-\frac{1}{\sqrt{n^2-2}+\sqrt{n^2-1}}\right)-\frac{1}{\sqrt{n^2-1}+\sqrt{n^2}}\\ &<\frac{1}{\sqrt{1}+\sqrt{2}}-\frac{1}{\sqrt{n^2-1}+\sqrt{n^2}}<\frac{1}{\sqrt{1}+\sqrt{2}}<1\end{aligned}$$

On the other hand $\dfrac{1}{\sqrt{k}+\sqrt{k+1}}=\sqrt{k+1}-\sqrt{k}$ so

$$\begin{aligned}a&=\left(\sqrt{2}-1\right)+\left(\sqrt{4}-\sqrt{3}\right)+\ldots+\left(\sqrt{n^2-1}-\sqrt{n^2-2}\right)\\ b&=\left(\sqrt{3}-\sqrt{2}\right)+\left(\sqrt{5}-\sqrt{4}\right)+\ldots+\left(\sqrt{n^2}-\sqrt{n^2-1}\right)\\ a+b&=n-1\end{aligned}$$

Thus $n-1<2a<n$ or $\frac{n-1}{2}<a<\frac{n}{2}$, and since n is odd we get $[a]=\lfloor a\rfloor=\dfrac{n-1}{2}$.

14. (HS-4) Given a point P inside an equilateral triangle $\triangle ABC$ with area 1, show that for the lengths x, y, z of the perpendiculars of P onto the sides of the triangle holds



$$x + y + z = \sqrt[4]{3}$$

Reason: Geometry.

Solution: The area of an equilateral triangle of side length a is $F = 1 = \dfrac{a^2}{4} \cdot \sqrt{3}$, i.e. $a = \dfrac{2}{\sqrt[4]{3}}$. The straights $\overline{AP}, \overline{BP}, \overline{CP}$ divide the triangle into three smaller triangles with areas $\dfrac{az}{2}, \dfrac{ax}{2}, \dfrac{az}{2}$. Thus

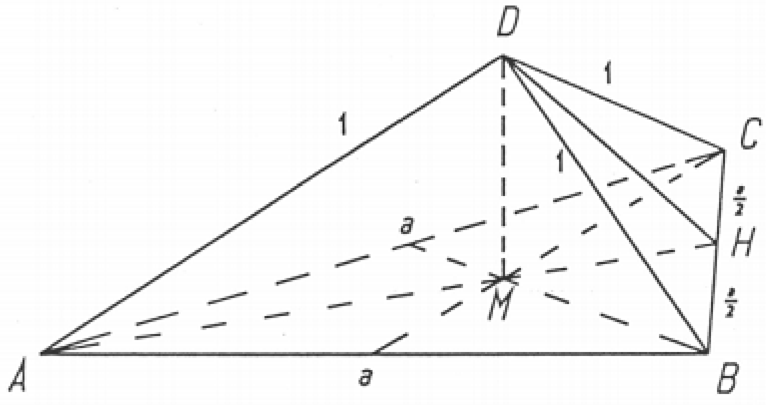
$$1 = \frac{az}{2} + \frac{ax}{2} + \frac{az}{2} = \frac{x + y + z}{\sqrt[4]{3}}$$

15. (HS-5) Prove: If for the edges of a tetrahedron $ABCD$ holds

$$\overline{AD} = \overline{BD} = \overline{CD} = 1 \text{ and } \overline{AB} = \overline{BC} = \overline{CA}\,,$$

then its surface is smaller than $\dfrac{3\sqrt{3}}{2}$.

Reason: Geometry.



Solution:

Set $a = \overline{AB} = \overline{BC} = \overline{CA}$ and $\overline{AH}$ the height of the triangle $\triangle ABC$

on $\overline{BC}$ with length $h = \dfrac{a}{2} \cdot \sqrt{3}$. Since the intersection M of all heights in $\triangle ABC$ is the barycenter which divides the height $2 : 1$, we have $\overline{AM} = \dfrac{2}{3}h = \dfrac{a}{3} \cdot \sqrt{3}$ and an area $F_1 = \dfrac{a}{2} \cdot h = \dfrac{a^2}{4} \cdot \sqrt{3}$.

Each of the triangles $\triangle ABD, \triangle BCD, \triangle CAD$ has isosceles of length 1 and a base line of length a. The corresponding height is also the median, so Pythagoras yields a height $\sqrt{1 - \dfrac{a^2}{4}}$, hence each of these triangles has an area $F_2 = \dfrac{a}{2}\sqrt{1 - \dfrac{a^2}{4}}$. The surface area of the tetrahedron is thus

$$F = F_1 + 3F_2 = \frac{a^2}{4} \cdot \sqrt{3} + \frac{3a}{2}\sqrt{1 - \frac{a^2}{4}}$$

The points D and M have both the same distances to A, B, C, which means that DM is perpendicular to the plane $\triangle ABC$. This makes $\triangle AMD$ a right triangle at M and so

$$0 < \overline{AM} < \overline{AD} \iff 0 < \frac{a}{3} \cdot \sqrt{3} < 1 \iff 0 < a < \sqrt{3}$$

Hence

$$\begin{aligned} 0 < (3 - a^2)^2 &\Longrightarrow 0 < 36 - 24a^2 + 4a^4 \\ &\Longrightarrow -3a^4 + 12a^2 < 36 - 12a + a^4 = (6 - a^2)^2 \\ &\Longrightarrow 3a^2(4 - a^2) < (6 - a^2)^2 \end{aligned}$$

As all factors are positive $(0 < a^2 < 3)$, we may apply the root function and get

$$a \cdot \sqrt{3} \cdot \sqrt{4 - a^2} = a \cdot 2 \cdot \sqrt{3} \cdot \sqrt{1 - \frac{a^2}{4}} < 6 - a^2$$

$$\Longrightarrow$$

$$\frac{a^2}{4} \cdot \sqrt{3} + a \cdot \frac{3}{2} \cdot \sqrt{1 - \frac{a^2}{4}} = F < \frac{3}{2} \cdot \sqrt{3}$$

2 November 2020

1. Let $u(x,t)$ satisfy the one dimensional diffusion equation $u_t = Du_{xx}$ in a space-time rectangle $R = \{0 \leq x \leq l, 0 \leq t \leq T\}$, then the maximum value of $u(x,t)$ is assumed either on the initial line ($t = 0$) or on the boundary lines ($x = 0 \, or \, x = l$). $D > 0$.

   Reason: (Weak) maximum principle for the diffusion equation.

   Solution: From Analysis we know: For a maximum in the inner of the definition area, the first derivatives have to vanish, and the second derivatives have to satisfy certain inequalities, e.g. $u_{xx} \leq 0$. If we knew (which is not the case), that $u_{xx} \neq 0$ at the maximum, then we have $u_{xx} < 0$ and simultaneously $u_t = 0$, i.e. $u_t \neq Du_{xx}$, a contradiction. But $u_{xx} = 0$ is possible, so we need some more effort.

   Let M be the maximum of $u(x,t)$ on the three boundaries $t = 0$, $x = 0$ and $t = l$. Note that a continuous function which is defined on a bounded, closed set, is bounded and assumes its maximum on this set, so M exists. We have to show that $u(x,t) \leq M$ on the whole rectangle R. Let $\varepsilon > 0$ and $v(x,t) := u(x,t) + \varepsilon x^2$. (Next goal is to show that $v(x,t) \leq M + \varepsilon l^2$ in R.) We have for $t = 0$, $x = 0$ and $x = l$

   $$v(x,t) \leq M + \varepsilon l^2$$

   Furthermore

   $$v_t - Dv_{xx} = u_t - D(u + \varepsilon x^2)_{xx} = u_t - Du_{xx} - 2\varepsilon D = -2\varepsilon D < 0$$

   which corresponds to a *diffusion inequality*. Assume that v assumes its maximum at an inner point (x_0, t_0), i.e. $0 < x_0 < l$ and $0 < t_0 < T$. Then $v_t = 0$ and $v_{xx} \leq 0$ at (x_0, t_0), but this contradicts the inequality above. Hence there is no maximum possible for $v(x,t)$ in the interior of R.

   Next assume that $v(x,t)$ has a maximum on the upper boundary of R ($t_0 = T, 0 < x < l$). Again, $v_{xx}(x_0, t_0) \leq 0$. As $v(x_0, t_0) > v(x_0, t_0 - \delta)$, we get

   $$v_t(x_0, t_0) = \lim_{\delta \downarrow 0} \frac{v(x_0, t_0) - v(x_0, t_0 - \delta)}{\delta} \geq 0$$

   and thus again a contradiction to the above inequality. But somewhere in R, there must be a maximum of $v(x,t)$. Thus, it has to be on the basic line or on the boundaries of R, and $v(x,t) \leq M + \varepsilon l^2$ is valid for the whole R. Thus

   $$u(x,t) = v(x,t) - \varepsilon x^2 \leq M + \varepsilon(l^2 - x^2).$$

Since this is true for all $\varepsilon > 0$, we get for all $(x,t) \in R$

$$u(x,t) \leq M.$$

2. Show that $M = \{(a_n) \in \ell^2(\mathbb{C}) \,|\, \forall\, n : |a_n| \leq n^{-1}\} \subseteq \ell^2(\mathbb{C})$ is bounded and compact.

   Reason: Compactness in $\ell^2(\mathbb{C})$.

   Solution: Let $(x^{(n)})_n \subseteq M$ be a sequence, then $|x_k^{(n)}| \leq k^{-1}$ and $(x_k^{(n)})_n$ is a bounded sequence of complex numbers for any $k \in \mathbb{N}$. By Cantor's diagonalisation method we can choose a subsequence $(x^{(n_j)}) \subseteq (x^{(n)})$ such that for all $k \in \mathbb{N}$

$$\lim_{j\to\infty} x_k^{(n_j)} = x_k$$

   for some $x_k \in \mathbb{C}$. Since $|x_k^{(n_j)}| \leq k^{-1}$, we also have $|x_k| \leq k^{-1}$ for all $k \in \mathbb{N}$. This means $x := (x_k)_k \in M$ and

$$\|x^{(n_j)} - x\|_2^2 \leq \underbrace{\sum_{k=1}^{s} |x_k^{(n_j)} - x_k|^2}_{\stackrel{j\to\infty}{\longrightarrow} 0 \text{ for all } s\in\mathbb{N}} + \underbrace{\sum_{k=s+1}^{\infty} \left(\frac{2}{k}\right)^2}_{\stackrel{s\to\infty}{\longrightarrow} 0}$$

   Thus $\|x^{(n_j)} - x\|_2 \longrightarrow 0$ for $j \to \infty$ and M is sequence compact and therefore bounded.

3. Show by two different methods that the normed space $\mathcal{C} := (C^1([0,1]), \|.\|_\infty)$ is not a Banach space.

   Reason: Banach Space.

   Solution:

   (a) Solution 1.

$$\left[\left|x - \frac{1}{2}\right| + \sqrt{\frac{1}{n}}\right]^2 \geq \left(x - \frac{1}{2}\right)^2 + \frac{1}{n} \geq 0$$

$$\Longrightarrow$$

$$\left(\left(x - \frac{1}{2}\right)^2 + \frac{1}{n}\right)^{1/2} - \left|x - \frac{1}{2}\right| \leq \sqrt{\frac{1}{n}}$$

Therefore the differentiable functions $f_n(x) := \left(\left(x-\frac{1}{2}\right)^2+\frac{1}{n}\right)^{1/2} \in$ $C^1([0,1])$ converge to the function $f(x) = \left|x-\frac{1}{2}\right|$ which is not differentiable at $x = 1/2 \in [0,1]$. Hence $\mathcal{C}$ is not complete, i.e. no Banach space.

(b) Solution 2.

Let's consider the differential operator $D : (C^1([0,1]), \|.\|_\infty) \longrightarrow (C([0,1]), \|.\|_\infty)$ and a sequence $(f_n) \overset{n\to\infty}{\longrightarrow} f$ in $C^1([0,1])$ with $f_n' \overset{n\to\infty}{\longrightarrow} g \in C([0,1])$. Since we have a uniform convergence

$$\begin{aligned}\int_0^x g(s)\,ds &= \int_0^x \lim_{n\to\infty} f_n'(s)\,ds = \lim_{n\to\infty}\int_0^x f_n'(s)\,ds\\ &= \lim_{n\to\infty}(f_n(x)-f(0)) = f(x)-f(0)\end{aligned}$$

Thus $Df = f' = g$ for all $x \in (0,1]$, and because g is continuous, $Df(0) = f'(0) = g(0)$, i.e. $Df = g$. We have therefore shown that the graph

$$\Gamma(D) = \{(f, Df)\,|\, f \in (C^1([0,1]), \|.\|_\infty)\}$$

is closed, which is equivalent to the boundedness of D. However, the differential operator isn't bounded. This contradiction implies that $(C^1([0,1]), \|.\|_\infty)$ cannot be a Banach space.

4. Let

$$A := \begin{bmatrix} 5 & 0 & 1 & 6\\ 3 & 3 & 5 & 2\\ 0 & 0 & 3 & 0\\ 6 & 0 & 3 & 0 \end{bmatrix} \in \mathbb{M}_4(\mathbb{Z}_7)$$

(a) Determine the characteristic polynomial $\chi_A(x)$ of A.

(b) Determine bases of the eigenspaces.

(c) Determine a matrix $S \in \mathrm{GL}_4(\mathbb{Z}_7)$ such that $S^{-1}AS$ is a diagonal matrix. Which one?

(d) Calculate A^{31}.

Reason: Finite Fields.

Solution:

(a)

$$\begin{aligned}
\det(A-xI) &= \det\begin{bmatrix} 5-x & 0 & 1 & 6 \\ 3 & 3-x & 5 & 2 \\ 0 & 0 & 3-x & 0 \\ 6 & 0 & 3 & -x \end{bmatrix} \\
&= (3-x)\det\begin{bmatrix} 5-x & 1 & 6 \\ 0 & 3-x & 0 \\ 6 & 3 & -x \end{bmatrix} \\
&= (3-x)^2\det\begin{bmatrix} 5-x & 6 \\ 6 & -x \end{bmatrix} \\
&= (3-x)^2((x-5)x-1) = (3-x)^2(x^2-5x-1) \\
&= (3-x)^3(2-x)
\end{aligned}$$

(b) For the eigenvalue 3 we have to solve the linear equation system

$$A-3I = \left[\begin{array}{cccc|c} 2 & 0 & 1 & 6 & 0 \\ 3 & 0 & 5 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 3 & 4 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

and we see immediately the three eigenvectors $(0,1,0,0)^\tau$, $(1,0,5,0)^\tau$ and $(1,0,0,2)^\tau$ as basis of $E_A(3)$.

For the eigenvalue 2 we have to solve the linear equation system

$$A-2I = \left[\begin{array}{cccc|c} 3 & 0 & 1 & 6 & 0 \\ 3 & 1 & 5 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 6 & 0 & 3 & 5 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

which is solved by the basis eigenvector $(2,3,0,6)^\tau$ of $E_A(2)$. Note that A is diagonalizable since the dimensions of the eigenspaces coincide with the algebraic multiplicities of the eigenvalues.

(c) The eigenvectors provide us the transformation matrix

$$S = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & 3 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$

$$S^{-1}AS = \begin{bmatrix} 3 & 1 & -3\cdot 5^{-1} & -3\cdot 2^{-1} \\ 0 & 0 & 5^{-1} & 0 \\ 3 & 0 & -3\cdot 5^{-1} & -1 \\ -1 & 0 & 5^{-1} & 2^{-1} \end{bmatrix} \begin{bmatrix} 5 & 0 & 1 & 6 \\ 3 & 3 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 6 & 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & 3 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 3 & 0 & 5 & 6 \\ 6 & 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 3 & 3 & 4 \\ 3 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 6 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} =: D$$

(d)

$$\begin{aligned} A^{31} &= (SDS^{-1})^{31} = SD^{31}S^{-1} = S\,\mathrm{diag}(3^{31}, 3^{31}, 3^{31}, 2^{31})S^{-1} \\ &= S\,\mathrm{diag}(3\cdot(3^6)^5, 3\cdot(3^6)^5, 3\cdot(3^6)^5, 2\cdot(2^6)^5)S^{-1} \\ &= S\,\mathrm{diag}(3\cdot 1^5, 3\cdot 1^5, 3\cdot 1^5, 2\cdot 1^5)S^{-1} = SDS^{-1} = A \end{aligned}$$

5. Let $f(x,y) = 34x^2 + 24xy + 41y^2 + 20x + 110y + 50$. Determine the Euclidean normal form of the conic section

$$Q_f = \{(x,y)^\tau \in \mathbb{R}^2 \mid f(x,y) = 0\}.$$

What are its foci and vertices in the normal form?

Reason: Quadratic Forms.

Solution: Two compute the normal form we consider the two matrices

$$A = \begin{bmatrix} 34 & 12 \\ 12 & 41 \end{bmatrix},\ M = \left[\begin{array}{cc|c} 34 & 24/2 & 20/2 \\ 24/2 & 41 & 110/2 \\ \hline 20/2 & 110/2 & 50 \end{array}\right] = \left[\begin{array}{cc|c} 34 & 12 & 10 \\ 12 & 41 & 55 \\ \hline 10 & 55 & 50 \end{array}\right]$$

and compute the eigenvalues of A.

$$\chi_A(x) = (34-x)(41-x) - 144 = x^2 - 75x + 1250 = (x-50)(x-25)$$

To receive the eigenvector basis we solve

$$\left[\begin{array}{cc|c} 34-25 & 12 & 0 \\ 12 & 41-25 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 9 & 12 & 0 \\ 12 & 16 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 3 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

and

$$\left[\begin{array}{cc|c} -16 & 12 & 0 \\ 12 & -9 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} -4 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

and get $E_A(25) = \mathbb{R} \cdot (-4,3)^\tau$, $E_A(50) = \mathbb{R} \cdot (3,4)^\tau$. Now we norm the basis and define the orthonormal matrix

$$T := \frac{1}{5} \cdot \begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix} \in \mathrm{O}_2(\mathbb{R}).$$

Now we have

$$T^\tau A T = \frac{1}{25} \begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 34 & 12 \\ 12 & 41 \end{bmatrix} \begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 25 & 0 \\ 0 & 50 \end{bmatrix} = \operatorname{diag}(25, 50)$$

We set $D := \frac{1}{5} \left[\begin{array}{cc|c} -4 & 3 & 0 \\ 3 & 4 & 0 \\ \hline 0 & 0 & 5 \end{array}\right] \in \mathrm{O}_3(\mathbb{R})$ and finally get

$$D^\tau M D = \frac{1}{25} \left[\begin{array}{cc|c} -4 & 3 & 0 \\ 3 & 4 & 0 \\ \hline 0 & 0 & 5 \end{array}\right] \left[\begin{array}{cc|c} 34 & 12 & 10 \\ 12 & 41 & 55 \\ \hline 10 & 55 & 50 \end{array}\right] \left[\begin{array}{cc|c} -4 & 3 & 0 \\ 3 & 4 & 0 \\ \hline 0 & 0 & 5 \end{array}\right]$$
$$= \left[\begin{array}{cc|c} -4 & 3 & 5 \\ 6 & 8 & 10 \\ \hline 2 & 11 & 10 \end{array}\right] \left[\begin{array}{cc|c} -4 & 3 & 0 \\ 3 & 4 & 0 \\ \hline 0 & 0 & 5 \end{array}\right] = \left[\begin{array}{cc|c} 25 & 0 & 25 \\ 0 & 50 & 50 \\ \hline 25 & 50 & 50 \end{array}\right]$$
$$= \frac{1}{25} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 2 & 2 \\ \hline 1 & 2 & 2 \end{array}\right] =: \frac{1}{25} M'$$

We now need a vector $(x_0, y_0)^\tau \in \mathbb{R}^2$ such that $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is as simple as possible, so we choose $x_0 = y_0 = -1$. Next we define $D' = \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -1 \\ \hline 0 & 0 & 1 \end{array}\right]$ and compute

$$M'' = D'^{\,\tau} M' D' = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline -1 & -1 & 1 \end{array}\right] \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 2 & 2 \\ \hline 1 & 2 & 2 \end{array}\right] \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -1 \\ \hline 0 & 0 & 1 \end{array}\right]$$
$$= \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 2 & 2 \\ \hline 0 & 0 & -1 \end{array}\right] \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -1 \\ \hline 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 2 & 0 \\ \hline 0 & 0 & -1 \end{array}\right]$$

From that we get that the normal form of Q_f is described by the equation $x^2 + 2y^2 - 1 = 0$ or

$$x^2 + \frac{y^2}{\left(\frac{1}{\sqrt{2}}\right)^2} = 1$$

which is an ellipse with $a = 1$ and $b = \dfrac{1}{\sqrt{2}}$. Hence the foci are $(1/\sqrt{2}, 0), (-1/\sqrt{2}, 0)$, and the vertices are $(0, \pm\sqrt{a^2 - b^2}) = (0, \pm 1/\sqrt{2})$, and $(1, 0), (-1, 0)$.

6. Let $u(x,t)$ be a solution of the one dimensional diffusion equation $u_t = Du_{xx}$. Assume that

$$C := \int_{-\infty}^{\infty} u(x,t)\,dx$$

is independent of t, which corresponds to a constant *population*, and u is small at infinity, which means that

$$\lim_{x\to\pm\infty} xu(x,t) = 0 = \lim_{x\to\pm\infty} x^2 \frac{\partial}{\partial x} u(x,t)$$

If

$$\sigma^2(t) = \frac{1}{C}\int_{-\infty}^{+\infty} x^2 u(x,t)\,dx$$

then

$$\sigma^2(t) = 2Dt \, + \, \sigma^2(0)$$

In the special case of an initial population (i.e. for $t = 0$) which is concentrated near $x = 0$ (like a δ-function) then we get $\sigma^2(t) \approx 2Dt$.

Reason: Population Distribution.

Solution: Using the fact that u is a solution of the diffusion equation and integrating by parts twice yields:

$$\begin{aligned}
\frac{C}{D}\frac{d\sigma^2}{dt} &= \frac{1}{D}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} x^2 u\,dx = \frac{1}{D}\int_{-\infty}^{\infty} x^2 \frac{\partial u}{\partial t}\,dx = \int_{-\infty}^{\infty} x^2 \frac{\partial^2 u}{\partial x^2} dx \\
&= \underbrace{\left[x^2 \frac{\partial u}{\partial x}\right]_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} 2x \frac{\partial u}{\partial x}\,dx = \underbrace{-[2xu]_{-\infty}^{\infty}}_{=0} + \int_{-\infty}^{\infty} 2u\,dx \\
&= 2\int_{-\infty}^{\infty} u(x,t)\,dx = 2C
\end{aligned}$$

thus
$$\frac{d\sigma^2}{dt} = 2D \Longrightarrow \sigma^2(t) = 2Dt + \sigma^2(0)$$
Last, we consider the special case that the particles start in a small interval around $x = 0$, e.g. such that $u(x,0) = 0$ for all $|x| > \varepsilon$. Then we get automatically
$$\int_{-\infty}^{\infty} x^2 u(x,0)\,dx = \int_{-\varepsilon}^{\varepsilon} x^2 u(x,0)\,dx \leq \varepsilon^2 \int_{-\varepsilon}^{\varepsilon} u(x,0)\,dx = \varepsilon^2 C\,,$$
i.e. $\sigma^2(0) = \varepsilon^2 \approx 0$.

7. Let G be a group of order 351. Show that G has a non trivial normal subgroup.

   Reason: Sylow Subgroups.

   Solution: $351 = 13 \cdot 3^3$. By the third Sylow theorem we get that the number n_{13} of $13-$Sylow subgroups P is congruent one modulo 13 and a divisor of $[G : P] = 351 : 13 = 27$. Thus $n_{13} \in \{1, 27\}$. In case $n_{13} = 1$ we are done, since this is equivalent to $P \trianglelefteq G$ being normal by the second Sylow theorem. Let's consider the case $n_{13} = 27$. Each of the 27 $13-$Sylow subgroups is of prime order, so they intersect each other only trivially. This means we have $27 \cdot 12$ elements of order 13, and the remaining 27 elements generate the $3-$Sylow subgroups. Each of these subgroups Q has the order 27 by the first Sylow theorem, i.e. the number n_3 of $3-$Sylow subgroups is $n_3 = 1$ which again by the second Sylow theorem means, that $Q \trianglelefteq G$ is a normal subgroup.

   In any case, their is a normal subgroup in G.

8. Show that the diffusional Lotka-Volterra system ($a > 0$)
$$u_t = u(1-v) + D\Delta u \tag{1}$$
$$v_t = av(u-1) + D\Delta v \tag{2}$$
   with equal diffusion coefficient $D > 0$ and homogeneous Neumann boundary conditions
$$\frac{\partial u}{\partial n}(x,t) = 0 = \frac{\partial v}{\partial n}(x,t)$$
   for $x \in \partial\Omega\,,\ \Omega \subseteq \mathbb{R}^n$ of finite volume and n outward normal, Δ the Laplace operator, tends to a spatially uniform state for $t \to \infty$, i.e.
$$\lim_{t\to\infty} \nabla u = \lim_{t\to\infty} \nabla v = 0$$

Hint: Consider the *energy* of the system $s = a(u - \log u) + (v - \log v)$.

Reason: Murray, 1975, Lotka-Volterra.

Solution: Let the initial conditions be $u(x,0) = u_0(x)$ and $v(x,0) = v_0(x)$ for $x \in \Omega$. We define $s(x,t)$ as

$$s = a(u - \log u) + (v - \log v)$$

i.e. for $D = 0$ it would satisfy

$$\begin{aligned} s_t &= a\left(u_t - \frac{u_t}{u}\right) + \left(v_t - \frac{v_t}{v}\right) = au_t(1-u^{-1}) + v_t(1-v^{-1}) \\ &= au(1-v)(1-u^{-1}) + av(u-1)(1-v^{-1}) \\ &= a(1-v)(u-1) + a(u-1)(v-1) = 0 \end{aligned}$$

so the question is: How does the corresponding differential equation for s look like for $D > 0$?

In this case we get by differentiation

$$\begin{aligned} s_t &= a(1-u^{-1})(u(1-v) + D\Delta u) + (1-v^{-1})(av(u-1) + D\Delta v) \\ &= aD(1-u^{-1})\Delta u + D(1-v^{-1})\Delta v \\ &= aD\left(\Delta u - \frac{\Delta u}{u}\right) + D\left(\Delta v - \frac{\Delta v}{v}\right) \\ \Delta s &= a(\Delta u - \Delta \log u) + (\Delta v - \Delta \log v) \\ &= a\left(\Delta u - \frac{\Delta u}{u} + \frac{|\nabla u|^2}{u^2}\right) + \left(\Delta v - \frac{\Delta v}{v} + \frac{|\nabla v|^2}{v^2}\right) \end{aligned}$$

Thus

$$s_t - D\Delta s = -D\left(a\frac{|\nabla u|^2}{u^2} + \frac{|\nabla v|^2}{v^2}\right) \leq 0$$

This can be interpreted in such a way that the energy is dissipated by the diffusion terms. The boundary conditions for s are

$$\frac{\partial s}{\partial n}(x,t) = 0 \text{ for } x \in \partial\Omega\,,$$

the initial conditions $s_0(x) := s(x,0) = a(u_0 - \log u_0) + (v_0 - \log v_0)$.

Via integration over Ω we can define the *total amount of energy* in the system at time t by

$$S(t) = \int_\Omega s(x,t)\,dx$$

Using the Neumann boundary condition and Green formula yields

$$\begin{aligned}\dot S(t)=\frac{dS}{dt}&=\int_\Omega s_t\,dx=\int_\Omega D\Delta s-D\left(a\frac{|\nabla u|^2}{u^2}+\frac{|\nabla v|^2}{v^2}\right)dx\\ &=\underbrace{D\int_{\partial\Omega}\frac{\partial S}{\partial n}\,dS}_{=0}-D\int_\Omega a\frac{|\nabla u|^2}{u^2}+\frac{|\nabla v|^2}{v^2}\,dx\leq 0\end{aligned}$$

Obviously, S is monotone non-increasing; there are two possibilities: it tends to a finite limit or it tends to $-\infty$ for $t\to\infty$. By definition, $s(x,t)\geq a+1$, so

$$S(t)=\int_\Omega s(x,t)\,dx\geq(a+1)|\Omega|\,,$$

so S indeed tends to a finite limit, which requires

$$\lim_{t\to\infty}\dot S(t)=-D\lim_{t\to\infty}\int_\Omega a\frac{|\nabla u|^2}{u^2}+\frac{|\nabla v|^2}{v^2}\,dx=0$$

The only possibility to satisfy this, is that $\nabla u,\nabla v$ both tend to 0 for $t\to\infty$, i.e. the system tends to a spatially uniform state.

9. (a) Let R be a Notherian local commutative ring with 1 and maximal ideal M. If $A\trianglelefteq R$ is an ideal in R such that $A/MA\cong_R\{0\}$, then $A=(0)$.

   (b) Let R be an integral domain, and $\dim R_P=0$ for all $P\in\operatorname{Spec}(R)$, then R is a field. The dimension is the Krull dimension.

Reason: Ring Theory.

Solution:

(a) The Jacobson radical of a local ring is its maximal ideal. A as ideal of a Noetherian ring is a finitely generated submodule. Thus we can conclude by Nakayama's Lemma that $MA\neq A$ or $A=\{0\}$. We excluded the first possibility, so $A=\{0\}$.

(b) The Krull dimension of an integral domain is defined by

$$\begin{aligned}\dim R&=\max\{n\in\mathbb{N}\,|\,P_0\subsetneq\ldots\subsetneq P_n\,,\,P_j\trianglelefteq R\text{ prime ideal }\}\\ &\stackrel{(*)}{=}\sup\{\dim R_M\,|\,M\trianglelefteq R\text{ maximal ideal }\}\\ &=0\end{aligned}$$

per given condition. So every prime ideal is maximal. Particularly $\{0\}\subsetneq R$ is a prime, hence maximal, and $R=R/\{0\}$ a field.

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(*) locality of the dimension:
https://www.mathematik.uni-kl.de
/~gathmann/class/commalg-2013/commalg-2013-c11.pdf
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10. Let $\alpha \in \mathbb{C}$ a root of the polynomial $f(x) = x^3 - 3x - 1 \in \mathbb{Q}[x]$. Show that $f(x)$ is irreducible, and that there is an automorphism $\sigma \in \operatorname{Aut}(\mathbb{Q}(\alpha)/\mathbb{Q})$ with $\sigma(\alpha) = 2 - \alpha^2$. If α is chosen closest to zero, what is $+\sqrt{12 - 3\alpha^2}$ in the splitting field of $f(x)$? This means in terms of a polynomial in α, not numerical.

    Reason: Field Extension.

    Solution: Assume we have a rational root of $f(x)$. Then we get $p^3 - 3pq^2 = q^3$ with coprime integers $p, q \in \mathbb{Z}$, which cannot be both odd or both even. p even and q odd is also impossible, hence $p = 2k + 1$ and $q = 2l$. But now we get $(2k + 1)^3 \equiv 0 \mod 4$ which is not possible. This shows that $f(x)$ is irreducible over $\mathbb{Q}$.

$$\begin{aligned} f(2-\alpha^2) &= (2-\alpha^2)^3 - 3(2-\alpha^2) - 1 \\ &= 8 - 12\alpha^2 + 6\alpha^4 - \alpha^6 - 6 + 3\alpha^2 - 1 \\ &= -(\alpha^6 - 6\alpha^4 - 2\alpha^3 + 9\alpha^2 + 1 + 6\alpha) - 2\alpha^3 + 6\alpha + 2 \\ &= -(\alpha^3 - 3\alpha - 1)^2 - 2(\alpha^3 - 3\alpha - 1) \\ &= 0 \end{aligned}$$

Assume $\alpha = 2 - \alpha^2$. Then $2\alpha = -1 \pm \sqrt{3}$ and $f(\alpha) \neq 0$. Hence we have found two distinct roots $\alpha, 2 - \alpha^2$ of $f(x)$, i.e. $\mathbb{Q}(a)$ is the decomposition field of $f(x)$, because $\mathbb{Q}(\alpha)[x]$ contains two of three and therewith all linear factors of $f(x)$. Thus $\mathbb{Q}(\alpha) \supsetneq \mathbb{Q}$ is a Galois extension and the automorphism group operates transitive on the roots of $f(x)$, which proves the existence of σ.

$f(x)$ has local extrema at $x = \pm 1$ with $f(-1) = 1$ and $f(1) = -3$. This implies that all roots are real. With $f(-1/3) = -1/27 \approx 0$ we have a root near $x = -1/3$. The other roots must be greater than 1 and less than -1. Long division by $x - \alpha$ yields

$$x^3 - 3x - 1 = (x - \alpha) \cdot \left(x + \frac{1}{2} \left(\alpha + \sqrt{12 - 3\alpha^2} \right) \right) \cdot \left(x + \frac{1}{2} \left(\alpha - \sqrt{12 - 3\alpha^2} \right) \right)$$

Since we know that $1/3 \approx \alpha$ and $f(2 - \alpha^2) = 0$, we have

$$2 - \alpha^2 = -\frac{1}{2} \left(\alpha \pm \sqrt{12 - 3\alpha^2} \right) \Longleftrightarrow \pm\sqrt{12 - 3\alpha^2} = 2\alpha^2 - 4 - \alpha$$

Since the right hand side is negative for our choice of α, we have

$$+\sqrt{12-3\alpha^2} = -2\alpha^2 + \alpha + 4\,.$$

11. (HS-1) Determine all $a \in \mathbb{R}$ such that

$$x(x+1)(x+2)(x+3) = a$$

has no real solution, a unique real solution, exactly two, three, or four real solutions, more than four real solutions.

Reason: Equation Solving.

Solution: $(0,2,4,3,2)$. If we set $z := x + (0+1+2+3)/4 = x + 3/2$ then the equation reads

$$\begin{aligned}
a &= x(x+1)(x+2)(x+3) = \left(z-\frac{3}{2}\right)\left(x-\frac{1}{2}\right)\left(z+\frac{1}{2}\right)\left(z+\frac{3}{2}\right)\\
&= \left(z^2-\frac{9}{4}\right)\left(z^2-\frac{1}{4}\right) = z^4 - \frac{5}{2}z^2 + \frac{9}{16}\\
&\Longleftrightarrow\\
z^2 &= \frac{5}{4} \pm \sqrt{\frac{25}{16}-\frac{9}{16}+a} = \frac{5}{4} \pm \sqrt{1+a}
\end{aligned}$$

(a) $a = 9/16 \Longrightarrow z \in \{0, -\sqrt{5/2}, +\sqrt{5/2}\} \Longrightarrow x \in \left\{-\frac{3}{2}, -\frac{3\pm\sqrt{10}}{2}\right\}$

(b) $a < -1$ doesn't allow any real solution.

(c) $a = -1 \Longrightarrow x \in \left\{-\frac{3\pm\sqrt{5}}{2}\right\}$

(d) $-1 < a < 9/16 \Longrightarrow x = -\frac{3}{2} \pm \sqrt{\frac{5}{4} \pm \sqrt{1+a}}$

(e) $a > 9/16 \Longrightarrow z^2 = \frac{5}{4} + \sqrt{1+a} \Longrightarrow x = -\frac{3}{2} \pm \sqrt{\frac{5}{4} + \sqrt{1+a}}$

12. (HS-2) An international conference has 30 scientists who speak English, Russian or Spanish. The number of people who speak exactly two languages is more than twice as big, but less than thrice as much as the number of people who speak only one language, which are as many as speak all three languages. Those who speak only English are more than those who speak only Russian, but less than those who speak

only Spanish. The number of those who speak only English is less than thrice the number of people who speak only Russian. How many people do speak only English, Russian, Spanish, and how many all three languages? (The conference language is French.)

Reason: Combinatorics.

Solution: Let's denote the number of people who speak only Russian by R, only English by E, only Spanish by S, and people who speak one language by U, two languages by T and all three languages by A. Thus we are given the conditions:

(a) $3U > T > 2U$
(b) $U = A$
(c) $S > E > R$
(d) $3R > E$

We are not interested in T, so we eliminate it by the condition $30 = U + T + A = T + 2U$ and get $5U > 30 > 4U$ which is only possible for $U = E + R + S = A = 7$. Since $S > E > R$ we must have $S = 4 > E = 2 > R = 1$ which can be seen by checking $R = 1$ first.

13. (HS-3) Calculate (manually!)

$$z = \frac{65533^3 + 65534^3 + 65535^3 + 65536^3 + 65537^3 + 65538^3 + 65539^3}{32765 \cdot 32766 + 32767 \cdot 32768 + 32768 \cdot 32769 + 32770 \cdot 32771}$$

Reason: Calculation.

Solution: Set $n := 2^{15} = 32768$

$$\begin{aligned}
z &= \frac{(2n-3)^3 + (2n-2)^3 + (2n-1)^3 + (2n)^3 + (2n+1)^3 + (2n+2)^3 + (2n+3)^3}{(n-3)(n-2) + (n-1)n + n(n+1) + (n+2)(n+3)} \\
&= \frac{7 \cdot (2n)^3 + 3 \cdot (2n)^2 \cdot (-3-2-1+1+2+3)}{4n^2 + n \cdot (-3-2-1+1+2+3) + 6 + 6} \\
&\quad + \frac{3 \cdot (2n) \cdot ((-3)^2 + (-2)^2 + (-1)^2 + 1^2 + 2^2 + 3^2) - 3^3 - 2^3 - 1^3 + 1^3 + 2^3 + 3^3}{4n^2 + n \cdot (-3-2-1+1+2+3) + 6 + 6} \\
&= \frac{56n^3 + 168n}{4n^2 + 12} = \frac{4 \cdot 14 \cdot n \cdot (n^2+3)}{4 \cdot (n^2+3)} = 14n = 7 \cdot 2^{16} = 458752
\end{aligned}$$

14. (HS-4) Show that ($n \in \mathbb{N}_0$)

$$f_n(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}$$

has at most one real zero.

Reason: Exponential Function.

Solution: We show by induction the stronger statement: If n is even, then $f_n(x) > 0$; and if n is odd, then $f_n(x)$ has exactly one zero. This is true for $n = 0, 1$ so we may assume $n \geq 2$. Note that $f_n'(x) = f_{n-1}(x)$.

If n is odd, then by induction hypothesis $f_n'(x) = f_{n-1}(x) > 0$ and $f_n(x)$ is strictly monotone increasing. But $\lim_{x\to-\infty} f_n(x) = -\infty$ and $\lim_{x\to+\infty} f_n(x) = +\infty$, so there is exactly one zero for $f_n(x)$.

If n is even, then by induction hypothesis $f_n'(x_0) = f_{n-1}(x_0) = 0$ for exactly one point x_0. Now $f_n''(x_0) = f_{n-2}(x_0) > 0$, hence x_0 is a global minimum. As $f_n(x_0) = f_{n-1}(x_0) + \dfrac{x_0^n}{n!} = \dfrac{x_0^n}{n!} > 0$ for even n, we have shown that $f_n(x) > 0$ everywhere.

15. (HS-5) Find all $\lambda \in \mathbb{R}$ such that

$$\sin^4 x - \cos^4 x = \lambda(\tan^4 x - \cot^4 x)$$

has no, exactly one, exactly two, more than two real solutions in $\left(0, \dfrac{\pi}{2}\right)$

Reason: Trigonometry.

Solution: The equation holds for all $\lambda \in \mathbb{R}$ in case $x = \pi/4$. Furthermore we have an invariance $x \longleftrightarrow (\pi/2) - x$ on the interval given, i.e. every solution in $\left(0, \dfrac{\pi}{4}\right)$ corresponds uniquely to a solution in $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$. This already excludes the possibilities of *no solution* and *exactly two solutions*, plus we may assume $x \in \left(0, \dfrac{\pi}{4}\right)$.

Define $L : \left(0, \dfrac{\pi}{4}\right) \longrightarrow \mathbb{R}$ by the quotient

$$L(x) = \frac{\sin^4 x - \cos^4 x}{\tan^4 x - \cot^4 x} = \frac{\sin^4 x - \cos^4 x}{\dfrac{\sin^4 x}{\cos^4 x} - \dfrac{\cos^4 x}{\sin^4 x}} = \frac{\sin^4 x \cos^4 x}{\sin^4 x + \cos^4 x}$$

for $\sin^4 x \neq \cos^4 x$ which is given on the interval $\left(0, \dfrac{\pi}{4}\right)$. Now

$$L(x) = \frac{\sin^4(2x)}{16(1 - 2\sin^2 x \cos^2 x)} = \frac{\sin^4(2x)}{8(2 - \sin^2(2x))}$$

This shows that $L(x)$ is strictly monotone increasing on $\left(0, \dfrac{\pi}{4}\right)$ and assumes every value in $\left(0, \dfrac{1}{8}\right)$ exactly once.

We have exactly three solutions for any value $\lambda \in \left(0, \frac{1}{8}\right)$, and exactly one solution for any value $\lambda \in \mathbb{R} - \left(0, \frac{1}{8}\right)$.

3 October 2020

1. Let $(a_n) \subseteq \mathbb{R}$ be a sequence of real numbers such that $a_n \leq n^{-3}$ for all $n \in \mathbb{N}$. Given the family $\mathcal{A}$ of functions $f_n : [0,1] \longrightarrow \mathbb{R}$ defined by $f_n(x) = \sum_{k=n}^{\infty} a_k \sin(kx)$ for $n \in \mathbb{N}$, show that every sequence $(g_n) \subseteq \mathcal{A}$ contains a subsequence (g_{n_k}) which converges uniformly on $[0,1]$.

   Reason: Arzelà-Ascoli.

   Solution:
$$|g_n(x)| \leq \sum_{k=n}^{\infty} \left|\frac{\sin(kx)}{k^3}\right| \leq \sum_{k=1}^{\infty} \frac{kx}{k^3} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} =: L < \infty$$
   i.e. $\|g_n\|_\infty \leq L$ for all $n \in \mathbb{N}$.

   We show that (g_n) is uniformly continuous. Let $\varepsilon > 0$ and $\delta = \varepsilon/L$. Then we have for all $x, y \in [0,1]$ with $|x-y| < \delta$ and all $n \in \mathbb{N}$
$$|g_n(x) - g_n(y)| \leq \sum_{k=n}^{\infty} \left|\frac{\sin(kx) - \sin(ky)}{k^3}\right| \leq \sum_{k=1}^{\infty} \frac{|x-y|}{k^2} = \frac{\pi^2}{6}|x-y| < \varepsilon$$
   Since $g_n \in C([0,1])$ for all $n \in \mathbb{N}$, we may apply the theorem of Arzelà-Ascoli. Thus there is a subsequence $(g_{n_k}) \subseteq (g_n)$ which converges uniformly on $[0,1]$.

2. Let $\pi : \mathbb{R}^n \longrightarrow \mathbb{T}^n$ be the canonical projection and $f := \pi|_{[0,1]^n}$ its restriction on the closed unit cube. Show with the help of $f : [0,1]^n \longrightarrow \mathbb{T}^n$, that a quotient map in general doesn't have to be open.

   Solution: The set $U_0 := \left]-\frac{1}{2}, \frac{1}{2}\right[^n \subseteq \mathbb{R}^n$ is open, so $U := U_0 \cap [0,1]^n = \left[0, \frac{1}{2}\right[^n \subseteq [0,1]^n$ is open in the subspace topology. However, $f(U) \subseteq \mathbb{T}^n$ is not open, since $\pi^{-1}(f(U)) = \pi^{-1}\left(\left[0, \frac{1}{2}\right[^n + \mathbb{Z}^n\right) = \left[0, \frac{1}{2}\right[^n \subseteq \mathbb{R}^n$ is not open. Hence f isn't open.

   It remains to show that $f : [0,1]^n \longrightarrow \mathbb{T}^n$ is a quotient map, i.e that a set $U \subseteq \mathbb{T}^n$ is open if and only if $f^{-1}(U) \subseteq [0,1]^n$ is open. Hence we must show
$$f^{-1}(U) \subseteq [0,1]^n \text{ open } \Longleftrightarrow \pi^{-1}(U) \subseteq \mathbb{R}^n \text{ open}$$
   which is equivalent to
$$f^{-1}(U) \subseteq [0,1]^n \text{ closed } \Longleftrightarrow \pi^{-1}(U) \subseteq \mathbb{R}^n \text{ closed}$$

The implication that $f^{-1}(A) \subseteq [0,1]^n$ is closed for closed sets $A \subseteq \mathbb{T}^n$ is the continuity of π and the definition of the subspace topology. So let us conversely assume that $f^{-1}(A) \subseteq [0,1]^n$ is closed for some set $A \subseteq \mathbb{T}^n$. Then $f^{-1}(A)$ is compact, because $[0,1]^n$ is compact. Since continuous functions map compact sets on compact sets,

$$\pi(f^{-1}(A) = f(f^{-1}(A)) = A \subseteq \mathbb{T}^n$$

is also compact. However, $\mathbb{T}^n$ is Hausdorff, so compact subsets are closed. Hence A is closed what had to be shown.

3. Let $D = \{ z \in \mathbb{C} \, : \, |z| < 1 \}$ be the complex open unit disk and let $0 < a < 1$ be a real number. Suppose $f \, : \, D \longrightarrow \mathbb{C}$ is a holomorphic function such that $f(a) = 1$ and $f(-a) = -1$.

   (a) Prove that $\sup_{z\in D}\{|f(z)|\} \geq \dfrac{1}{a}$.

   (b) Prove that if f has no root, then $\sup_{z\in D}\{|f(z)|\} \geq \exp\left(\dfrac{1-a^2}{4a}\,\pi\right)$.

   Reason: Holomorphic Function.

   Solution:

   (a) Consider $g(z) = \dfrac{f(z)-f(-z)}{2z}$ for $z \neq 0$ and let $g(0) = f'(0)$. Then g is a holomorphic function, too, with $g(a) = 1/a$. By triangle inequality and the maximum principle we have for $a < r < 1$

   $$\begin{aligned}\sup_{z\in D}|f(z)| \geq \max_{|z|=r}|f(z)| &\geq r \cdot \max_{|z|=r} \frac{|f(z)|+|f(-z)|}{2r}\\ &\geq r \cdot \max_{|z|=r}|g(z)| = r \cdot |g(a)| = \frac{r}{a}\end{aligned}$$

   from which the statement follows for $r \to 1-0$.

   (b) Let $M := \sup_{z\in D}|f(z)|$. Since f is not constant, $|f| < M$ everywhere in D. And from $f(a) = 1$ we know, that $M > 1$. The function f is nonzero on the simply connected set D, so it has a logarithm; i.e. there is a holomorphic function $g(z) \, : \, D \longrightarrow \mathbb{C}$ such that $f(z) = \exp(g(z))$. W.l.o.g. we assume $g(a) = 0$. From $f(-a) = -1$ we get $g(-a) = k\pi\, i$ with some odd integer k, and from $|f| < M$ we get $\Re(g) < \log M$. Denote by H the half-plane $\Re(z) < \log M$. Then $g \, : \, D \longrightarrow H$. Next we define the linear fractional transformations

   $$\varphi \, : \, D \longrightarrow D, \;\; \varphi(z) = \frac{z+a}{1+az}, \;\; \varphi^{-1}(z) = \frac{z-a}{1-az}$$

and

$$\psi \,:\, H \longrightarrow D,\;\; \psi(z) = \frac{z}{2\log M - z}.$$

Now $h := \psi \circ g \circ \varphi \,:\, D \longrightarrow D$ with $h(0) = 0$. Schwarz's lemma applied to h and the point $\varphi^{-1}(-a) = \dfrac{-2a}{1+a^2}$ gives us $\left| h\left(\dfrac{-2a}{1+a^2}\right)\right| \leq \dfrac{2a}{1+a^2}$. Thus

$$\begin{aligned}\frac{2a}{1+a^2} &\geq \left|h(\varphi^{-1}(-a))\right| = |\psi(g(-a))| = \left|\frac{k\pi\, i}{2\log M - k\pi i}\right| \\ &= \frac{1}{\sqrt{\left(\dfrac{2\log M}{|k|\pi}\right)^2 + 1}}\end{aligned}$$

So

$$\begin{aligned}\log M &\geq \frac{|k|\pi}{2}\sqrt{\left(\frac{1+a^2}{2a}\right)^2 - 1} \\ &= \frac{|k|\pi}{2}\cdot\frac{1-a^2}{2a} \geq \frac{1-a^2}{4a}\,\pi.\end{aligned}$$

Remark: The estimates in the problem are sharp. For example, we have equality for $f(z) = \dfrac{z}{a}$ in part (a), and in part (b) for

$$f(z) = -i\,\exp\left(\frac{iz - a^2}{iz+1}\cdot\frac{\pi}{2a}\right).$$

4. Let $0 < p \leq a, b, c, d, e \leq q$ and show that

$$(a+b+c+d+e)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right) \leq 25 + 6\left(\sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}}\right)^2.$$

This is a special case of a general inequality. Which is the general case and how is it proven?

Reason: Inequality.

Solution:

$$f(a,b,c,d,e) := (a+b+c+d+e)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right)$$

is a convex function of each of the variables. Hence the maximum is taken on one of the 32 vertices of the 5-cube given by $p \leq a, b, c, d, e \leq q$. If there are $n \;\; p's$ and $5-n \;\; q's$, then we have to maximize the quadratic function

$$f(n) = (np + (5-n)q)\left(\frac{n}{p} + \frac{5-n}{q}\right) = 25 + n(5-n)\left(\sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}}\right)^2.$$

So $f(n)$ takes its maximum at $n = 5/2$, i.e. $n \in \{2,3\}$ where it has the value $25 + 6\left(\sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}}\right)^2$.

The general theorem (Kantorovich's inequality) is: Let $x_1, \ldots, x_n \in [a,b]$, where $0 < a < b$, then

$$(x_1 + \ldots + x_m)\left(\frac{1}{x_1} + \cdots + \frac{1}{x_m}\right) \leq \frac{(a+b)^2}{4ab}\, m^2.$$

The same argumentation as above results in the quadratic function

$$f(n) = -n^2\left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^2 + mn\left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^2 + m^2$$

with a maximum at $n = m/2$ and a value

$$\begin{aligned} f(m/2) &= -\frac{m^2}{4}\left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^2 + \frac{m^2}{2}\left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^2 + m^2 \\ &= \frac{m^2}{4}\left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^2 + \frac{4m^2}{4} \\ &= \frac{m^2}{4}\left(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}\right)^2 \\ &= \frac{m^2}{4} \cdot \frac{(a+b)^2}{ab} \end{aligned}$$

5. Let $n > 1$ be an integer. There are n lamps $L_0, \ldots, L_{n-1}$ arranged in a circle. Each lamp is either ON (1) or OFF (0). A sequence of steps $S_0, \ldots, S_i, \ldots$ is carried out. Step S_j affects the state of L_j only (leaving the states of all other lamps unaltered) as follows:

   If L_{j-1} is ON, S_j changes the state of L_j from ON to OFF or from

OFF to ON;

If L_{j-1} is OFF, S_j leaves the state of L_j unchanged.

The lamps are labeled modulo n, that is $L_{-1} = L_{n-1}, L_0 = L_n$, etc. Initially all lamps are ON. Show that

(a) there is a positive integer $M(n)$ such that after $M(n)$ steps all the lamps are ON again;

(b) if $n = 2^k$, then all lamps are ON after $(n^2 - 1)$ steps;

(c) if $n = 2^k + 1$, then all lamps are ON after $(n^2 - n + 1)$ steps.

Reason: Algorithm.

Solution:

(a) Let $x_j \in \{0, 1\}$ represent the state of lamp L_j. Operation S_j affects the state of L_j, which in the previous round has been set to the value x_{j-n}. At the moment when S_j is being performed, lamp L_{j-1} is in the state x_{j-1}. Consequently,

$$x_j \equiv x_{j-n} + x_{j-1} \mod 2, \quad (1)$$

This is true for all $j \geq 0$. Note that the initial state (all lamps ON) corresponds to

$$x_{-n} = x_{-n+1} = \ldots = x_{-2} = x_{-1} = 1. \quad (2)$$

The state of the system at instant j can be represented by the vector $v_j = (x_{j-n}, \ldots, x_{j-1})$, $v_0 = (1, \ldots, 1)$. Since there are only n feasible vectors, repetitions must occur in the sequence $v_0, v_1, v_2, \ldots$ The operation that produces v_{j+1} from v_j is invertible. Hence, the equality $v_{j+m} = v_j$ implies $v_m = v_0$; the initial state recurs in at most 2^n steps proving the first part.

Let's consider equation (1):

$$\begin{aligned} x_j &\equiv x_{j-n} + x_{j-1} \mod 2 \\ &\equiv (x_{j-2n} + x_{j-n-1}) + (x_{j-1-n} + x_{j-2}) \mod 2 \\ &\equiv x_{j-2n} + 2x_{j-n-1} + x_{j-2} \mod 2 \\ &\equiv x_{j-3n} + 3x_{j-2n-1} + 3x_{j-n-2} + x_{j-3} \mod 2 \\ &\equiv \ldots \end{aligned}$$

After r iterations we arrive at the equality

$$x_j = \sum_{i=0}^{r} \binom{r}{i} x_{j-(r-i)n-i} \mod 2 \quad (3)$$

holding for all j, r such that $j-(r-i)n-i \geq -n$.
If $r = 2^k$, then the binomial coefficients are all even except the two outer ones, and we obtain

$$x_j \equiv x_{j-rn} + x_{j-r} \quad (\text{for } r = 2^k), \quad (4)$$

provided the subscripts do not go below $-n$, i.e., for $j \geq (r-1)n$.

(b) If $n = 2^k$, choose $j \geq n^2 - n = (2^k - 1)2^k$, and with $r = n$, we obtain from (4)

$$x_j \equiv x_{j-n^2} + x_{j-n} \equiv x_{j-n^2} + (x_j - x_{j-1}) \mod 2.$$

Hence $x_{j-n^2} \equiv x_{j-1} \mod 2$, showing that the sequence (x_j) is periodic with period $n^2 - 1$.

(c) If $n = 2^k + 1$, choose $j \geq n^2 - n = (2^k + 1)2^k$, and set in (4) $r = n - 1$, obtaining

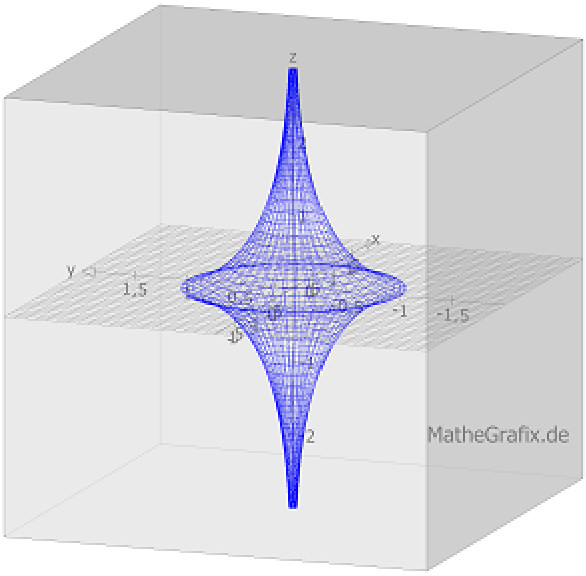
$$\begin{aligned} x_j &\equiv x_{j-rn} + x_{j-r} \mod 2 \\ &\equiv x_{j-n^2+n} + x_{j-n+1} \mod 2 \\ &\equiv x_{j-n^2+n} + (x_{j+1} - x_j) \mod 2 \\ &\equiv x_{j-n^2+n} + x_{j+1} + x_j \mod 2 \end{aligned}$$

Hence $x_{j-(n^2-n+1)} \equiv x_j$, showing the sequence is periodic with period $n^2 - n + 1$.

6. The pseudosphere is the rotational surface of the tractrix, e.g. parameterized by

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \, , \; f(x,y) = \begin{bmatrix} \cos(y)/\cosh(x) \\ \sin(y)/\cosh(x) \\ x - \tanh(x) \end{bmatrix} .$$

Show that the pseudosphere has a constant negative Gauß curvature.



Reason: Curvature Pseudo-Sphere.

Solution:

$$f_x = \begin{bmatrix} -\dfrac{\cos(y)\sinh(x)}{\cosh^2(x)} \\ -\dfrac{\sin(y)\sinh(x)}{\cosh^2(x)} \\ \dfrac{\sinh^2(x)}{\cosh^2(x)} \end{bmatrix}, \; f_y = \begin{bmatrix} \dfrac{-\sin(y)}{\cosh(x)} \\ \dfrac{\cos(y)}{\cosh(x)} \\ 0 \end{bmatrix}, \; f_x \times f_y = \begin{bmatrix} -\dfrac{\cos(y)\sinh^2(x)}{\cosh^3(x)} \\ -\dfrac{\sin(y)\sinh^2(x)}{\cosh^3(x)} \\ -\dfrac{sinh(x)}{\cosh^3(x)} \end{bmatrix}$$

Thus we get

$$\|f_x \times f_y\|^2 = \frac{\sinh^4(x)}{\cosh^6(x)} + \frac{\sinh^2(x)}{\cosh^6(x)} = \frac{\sinh^2(x)}{\cosh^4(x)} \left(\frac{\sinh^2(x)+1}{\cosh^2(x)} \right) = \frac{\sinh^2(x)}{\cosh^4(x)}$$

and in case $x \geq 0$

$$N = \frac{f_x \times f_y}{\|f_x \times f_y\|} = \begin{bmatrix} -\dfrac{\cos(y)\sinh(x)}{\cosh(x)} \\ -\dfrac{\sin(y)\sinh(x)}{\cosh(x)} \\ -\dfrac{1}{\cosh(x)} \end{bmatrix} = \begin{bmatrix} -\cos(y)\tanh(x) \\ -\sin(y)\tanh(x) \\ -\dfrac{1}{\cosh(x)} \end{bmatrix}$$

Now we calculate

$$N_x = \begin{bmatrix} -\dfrac{\cos(y)}{\cosh^2(x)} \\ -\dfrac{\sin(y)}{\cosh^2(x)} \\ \dfrac{\sinh(x)}{\cosh^2(x)} \end{bmatrix} \, , \, N_y = \begin{bmatrix} \sin(y)\tanh(x) \\ -\cos(y)\tanh(x) \\ 0 \end{bmatrix}$$

Observe that $D_pN = D_pf \cdot A_p$ with $A_p = \begin{bmatrix} \dfrac{1}{\sinh(x)} & 0 \\ 0 & -\sinh(x) \end{bmatrix}$ such that the Gauß curvature is given by

$$\kappa_p = \det A_p = -1 \text{ for all } p \in \mathbb{R}^2.$$

The sign of the determinant of A_p does not change, if x changes its sign and so skips signs in D_pN, because A_p is a 2×2 matrix. The Gauß curvature is thus negative, too, in case $x < 0$.

7. Let $\mathfrak{g}$ be a Lie algebra with trivial center $\mathfrak{Z}(\mathfrak{g}) = \{0\}$ over a field of characteristic not equal two and

$$\begin{aligned} \mathfrak{A}(\mathfrak{g}) &= \{\varphi : \mathfrak{g} \stackrel{linear}{\longrightarrow} \mathfrak{g} \,|\, [\varphi(X), Y] = [\varphi(Y), X] \text{ for all } X, Y \in \mathfrak{g}\} \\ &= \operatorname{lin}\{\alpha, \beta \neq 0 \,|\, [\alpha, \beta] = \alpha\beta - \beta\alpha = \beta\} \end{aligned}$$

Show that image $\operatorname{im}\beta$ and kernel $\ker\beta$ of β are ideals in $\mathfrak{g}$.

Hint: $\mathfrak{A}(\mathfrak{g})$ is a $\mathfrak{g}$-module by $X.\varphi = [\operatorname{ad} X, \varphi]$.

Reason: Lie Algebras.

Solution: Let $L, K \in \ker\beta$, $B = \beta(A), D = \beta(C) \in \operatorname{im}\beta$.

$$\begin{aligned} \beta(\alpha(K)) &= \alpha(\beta(K)) - [\alpha, \beta](K) = -\beta(K) = 0 \\ \beta(A + \alpha(A)) &= B + \alpha(\beta(A)) - [\alpha, \beta](A) = B + \alpha(B) - \beta(A) = \alpha(B) \\ [L, B] &= [L, \beta(A)] = -[\beta(L), A] = 0 \\ [X, \beta([L, K])] &= -[\beta(X), [L, K]] = [\beta(L), [K, X]] + [\beta(K), [X, L]] = 0 \\ &\Longrightarrow \beta([L, K]) \in \mathfrak{Z}(\mathfrak{g}) = \{0\} \\ &\Longrightarrow [L, K] \in \ker\beta \\ [X, [\beta(A), \beta(C)]] &= -[X, [\beta^2(A), C]] = -[\beta^2(X), [A, C]] = -[X, \beta^2([A, C])] \\ &\Longrightarrow [\beta(A), \beta(C)] + \beta^2([A, C]) \in \mathfrak{Z}(\mathfrak{g}) = \{0\} \\ &\Longrightarrow [\beta(A), \beta(C)] = \beta(-\beta([A, C])) \in \operatorname{im}\beta \end{aligned}$$

The calculations show that $\mathfrak{K} := \ker \beta$ and $\mathfrak{I} := \operatorname{im} \beta$ are $\mathfrak{A}(\mathfrak{g})$ invariant, commuting subalgebras of $\mathfrak{g}$.

Let $\gamma \in \mathfrak{A}(\mathfrak{g})$ such that $X.\gamma = 0$ for all $X \in \mathfrak{g}$. Then

$$\begin{aligned} 0 &= (X.\gamma)(Y) = [X,\gamma(Y)] - \gamma([X,Y]) = [X,\gamma(Y)] + \gamma([Y,X]) \\ &= [X,\gamma(Y)] - (Y.\gamma)(X) + [Y,\gamma(X)] = 2[X,\gamma(Y)] \\ &\Longrightarrow \gamma(Y) \in \mathfrak{Z}(\mathfrak{g}) = \{0\} \Longrightarrow \gamma = 0 \end{aligned}$$

Hence $A.\alpha \neq 0$ and $B.\beta \neq 0$ for some $A, B \in \mathfrak{g}$. Since $X.\alpha, X.\beta \in \mathfrak{A}(\mathfrak{g})$ we can write

$$\begin{aligned} X.\alpha &= \lambda(X)\alpha + \mu(X)\beta \\ X.\beta &= \nu(X)\alpha + \omega(X)\beta \end{aligned}$$

$$\begin{aligned} X.\beta = X.[\alpha,\beta] = [X.\alpha,\beta] + [\alpha, X.\beta] = \lambda(X)\beta + \omega(X)\beta = \nu(X)\alpha + \omega(X)\beta \\ \Longrightarrow \nu(X) = \lambda(X) = 0 \Longrightarrow X.\alpha = \mu(X)\beta\,,\; X.\beta = \omega(X)\beta \end{aligned}$$

$$X.\alpha = \mu(X)\beta\,,\; X.\beta = \omega(X)\beta$$

This implies especially that $\mathfrak{I} \trianglelefteq \mathfrak{g}$ and $\ker \omega \supseteq \mathfrak{K} \trianglelefteq \mathfrak{g}$ are ideals:

$$\begin{aligned} [X,\beta(Y)] &= (X.\beta)(Y) + \beta([X,Y]) = \omega(X)\beta(Y) + \beta([X,Y]) \in \operatorname{im}\beta \\ \beta([X,K]) &= (X.\beta)(K) - [X,\beta(K)] = \omega(X)\beta(K) = 0 \\ \omega(K)\beta(X) &= (K.\beta)(X) = \beta([X,K]) = 0 \end{aligned}$$

8. (HS-1) Given a positive integer n. Assume that 4^n and 5^n start with the same digit in the decimal system. Show that this digit has to be 2 or 4.

   Reason: Numbers.

   Solution: Let z be the leading digit of 4^n and 5^n, so

   $$\begin{aligned} z \cdot 10^r &\leq 4^n = 2^{2n} < (z+1)\cdot 10^r \\ z \cdot 10^s &\leq 5^n < (z+1)\cdot 10^s \end{aligned}$$

   If we square the second and multiply both, we get

   $$z^3 \cdot 10^{r+2s} \leq 10^{2n} < (z+1)^3 \cdot 10^{r+2s} \Longrightarrow 1 \leq z^3 \leq 10^{2n-r-2s} < (z+1)^3 \leq 1000$$

   This means that $2n - r - 2s \in \{0,1,2\}$.

(a) $2n - r - 2s = 0$

In this case we have $z = 1$. If $1 \cdot 10^r < 4^n$, then we would get by the procedure above that $z^3 \cdot 10^{r+2s} = 1 \cdot 10^{2n} < 10^{2n}$ which is impossible. Hence $4^n = 10^r$ which is only possible for $n = r = 0$ since $5 \nmid 4^n$. But this contradicts our choice of n.

(b) $2n - r - 2s = 1$

Here we get $z^3 \leq 10^1 = 10$ which means $z = 2$.

(c) $2n - r - 2s = 2$

Here we get $z^3 \leq 10^2 = 100 < (z+1)^3$ and so $z \leq \sqrt[3]{100} = 4.64\ldots < z + 1$ which means $z = 4$.

9. (HS-2) A parcel service charges a price proportional to the sum height plus length plus width per box. Could it be, that there is a case where it is cheaper to put a more expensive package into a cheaper box?

Reason: Optimization.

Solution: Assume we have boxes $B = B(a,b,c) \subseteq A = A(x,y,z)$. We define the sets $A_\delta := \{x \in \mathbb{R}^3 \,|\, d(A,x) \leq \delta\}$ and similar B_δ of all points not farther away from the boxes than δ. Each of these sets consists of the box itself, 6 boxes of height δ above each surface, 12 quarter cylinders of radius δ along each edge, and eight eighth of a ball of radius δ above each vertex. Hence the volumes are

$$|A_\delta| = xyz + 2(xy + xz + yz)\delta + \pi(x + y + z)\delta^2 + \frac{4}{3}\pi\delta^3$$
$$|B_\delta| = abc + 2(ab + ac + bc)\delta + \pi(a + b + c)\delta^2 + \frac{4}{3}\pi\delta^3$$

Since $B \subseteq A$ we have $B_\delta \subseteq A_\delta$ for any positive real number δ, too. Thus

$$\frac{abc}{\delta^2} + \frac{2(ab + ac + bc)}{\delta} + \pi(a + b + c) \leq \frac{xyz}{\delta^2} + \frac{2(xy + xz + yz)}{\delta} + \pi(x + y + z)$$

Since this has to hold for any δ, we can take the limit to infinity and see, that the inequality only holds if

$$a + b + c \leq x + y + z$$

which means our answer is NO: We cannot save money by using cheaper boxes.

10. (HS-3) Let a be a positive integer and $(a_n)_{n\in\mathbb{N}_0}$ the sequence defined by
$$a_0 := 1\ ,\ a_n := a + \prod_{k=0}^{n-1} a_k \quad (n \geq 1)$$
    (a) There are infinitely many primes which divide at least one number of the sequence.
    (b) There is a prime which does not divide any of the numbers in the sequence.

    Reason: Primes.

    Solution: $\gcd(a, a_0) = 1$ and
$$\gcd(a, a_n) = \gcd\left(a, a + \prod_{k=0}^{n-1} a_k\right) = \gcd\left(a, \prod_{k=0}^{n-1} a_k\right) = 1$$
    by induction.

    (a) Let $p_1, \ldots, p_N$ be primes each dividing at least one a_n. Then there is a minimal M, such that all these primes are divisors of some numbers of $a_0, \ldots, a_M$. This means however, that all $p_i \mid \prod_{k=0}^{M-1} a_k$. Thus we get from the above consideration, that none of the p_i divides a, hence none of them divides $e_M > 1$. We thus get a prime factor p_{N+1} of e_M which wasn't on the list. But if we can always add a prime to the list, it cannot be finite.
    (b) If $a > 1$ then it has a prime factor which does not divide any a_n because we saw that $\gcd(a, a_n) = 1$.

    Now let $a = 1$ and set $m_i := \prod_{k=0}^{i} a_k$. That is
$$m_0 = a_0 = 1\ ,\ m_{k+1} = m_k a_{k+1} = m_k(a + m_k) = m_k(1 + m_k)$$
    We observe that $m_0 \equiv 1 \mod 5\,,\ m_1 \equiv 2 \mod 5\,,\ m_2 \equiv 1 \mod 5\,, \ldots$ As m_{k+1} only depends on m_k, we see that all remainders have to be 1 or 2, and the m_k are never divisible by 5. But $a_k \,|\, m_k$ so 5 can never be a divisor of any a_n.

11. (HS-4) Let a, b, c be positive real numbers such that $a+b+c+2 = abc$. Show that $(a + 1)(b + 1)(c + 1) \geq 27$. Under which condition does equality hold?

    Reason: Inequality.

Solution: We set $x = a+1, y = b+1, z = c+1$ and have to show that $xyz \geq 27$ under the assumption that

$$\begin{aligned} xyz &= (a+1)(b+1)(c+1) \\ &= (ab+a+b+1)(c+1) \\ &= abc+ac+bc+ab+a+b+c+1 \\ &= ac+bc+ab+2a+2b+2c+3 \\ &= (ab+a+b+1)+(ac+a+c+1)+(bc+b+c+1) \\ &= (a+1)(b+1)+(a+1)(c+1)+(b+1)(c+1) \\ &= xy+yz+xz \end{aligned}$$

With $AM \geq GM$ we get $xyz = xy + yz + xz \geq 3\sqrt[3]{x^2y^2z^2}$ which is equivalent to $xyz \geq 27$ since all numbers are positive.

Equality holds if and only if $xy = yz = xz$, i.e. $x = y = z$. This is true for $x = y = z = 3$ or $a = b = c = 2$.

4 September 2020

1. Given a group G then the intersection of all maximal subgroups of G is called Frattini subgroup $\Phi(G)$. If G hasn't a maximal subgroup, we set $\Phi(G) = G$. Show that $\Phi(G) \trianglelefteq G$ is a normal subgroup, and that $\Phi(G)$ is nilpotent in case G is finite.

   Reason: Frattini Subgroup.

   Solution: The intersection of all maximal subgroups of G is invariant under group automorphisms

$$\varphi(\Phi(G)) = \varphi\left(\bigcap_{\substack{M \leq G \\ maximal}} M\right) \subseteq \bigcap_{\substack{M \leq G \\ maximal}} \varphi(M) = \bigcap_{\substack{M \leq G \\ maximal}} M = \Phi(G)$$

   and thus especially under inner automorphisms, i.e. conjugation, i.e. $\Phi(G) \triangleleft G$.

   Assume $|G| =: n$ is finite and $P \leq \Phi(G)$ a nontrivial p-group, i.e. the order of any element of P is a power of the prime $p \,|\, n$. Such subgroups exist by Sylow's first theorem for prime factors of n, or by Cauchy's theorem below.

   (a) Lemma: If a group H of order p^n (p prime) acts on a finite set S and if $S_0 := \{\, x \in S \,|\, h.x = x \text{ for all } h \in H \,\}$ denotes the set of fixed points of S under the action, then $|S| \equiv |S_0| \mod p$.

   Proof: An orbit $\tilde{x} = H.x$ contains exactly one element if and only if $x \in S_0$. Hence S can be written as a disjoint union $S = S_0 \cup \tilde{x}_1 \cup \tilde{x}_2 \cup \ldots \cup \tilde{x}_m$ with $|\tilde{x}_k| > 1$ for all k. Hence $|S| = |S_0| + |\tilde{x}_1| + |\tilde{x}_2| + \cdots + |\tilde{x}_m|$. Now $p \,|\, |\tilde{x}_k|$ for each k since $|\tilde{x}_k| > 1$ and $|\tilde{x}_k| = [H : H.x_k] \,|\, |H| = p^n$ by the orbit-stabilizer theorem. Thus $|S| \equiv |S_0| \mod p$.

   (b) Cauchy's Theorem.

   If G is a finite group whose order $|G| = n$ is divisible by a prime p, then G contains an element of order p.

   Proof: Let S be the set of p-tuples of group elements

$$\{\, (a_1, a_2, \ldots, a_p) \,|\, a_k \in G \text{ and } a_1 a_2 \cdots a_p = 1 \,\}.$$

   Since $a_p = (a_1 a_2 \cdots a_{p-1})^{-1}$ is uniquely determined by the other elements, if follows that $|S| = n^{p-1}$. As $p \,|\, n$, $|S| \equiv 0 \mod p$. The cyclic group $\mathbb{Z}_p$ acts on S by

$$k.(a_1, a_2, \ldots, a_p) = (a_{1+k}, a_{2+k}, \ldots, a_p, a_1, \ldots, a_k) \quad (k \in \mathbb{Z}_p)$$

(With $ba = (a^{-1}a)(ba) = a^{-1}(ab)a = 1$ if $ab = 1$ we see by induction, that $(a_{k+1}, a_{k+2}, \ldots, a_k) \in S$. It's easy to verify for $x \in S,\ 0, k, k' \in \mathbb{Z}_p$ that $0.x = x$ and $(k + k').x = k.(k'.x)$, and that the action is well-defined.)

Now $x = (a_1, \ldots, a_p) \in S_0$ is a fixed point if and only if $a_1 = a_2 = \ldots = a_p$ and $(1, 1, \ldots, 1) \in S_0$, so $S_0 \neq 0$. By the previous Lemma we get $|S_0| \equiv |S| \equiv 0 \mod p$ and at least p elements in S_0, that is, there is $a \neq 1$ such that $(a, a, \ldots, a) \in S_0$ and hence $a^p = 1$. Since p is prime, $|a| = p$.

(c) Corollary: A finite group P is a p-group if and only if $|P|$ is a power of p.

Proof: If P is a p-group and $q \,|\, |P|$ a prime, then P contains an element of order q by Cauchy's theorem. Since every element has order a power of p, $q = p$. Hence $|P|$ is a power of p. The converse is an immediate consequence of Lagrange's theorem, that the order of every group element divides the order of the (finite) group.

(d) P has a nontrivial center: $C(P) \neq \{1\}$.

Proof: Consider the class equation of P:

$$|P| = |C(P)| + \sum |P : C_P(x_i)|$$

where $C_P(x) = \{\, p \in P \,|\, px = xp \,\}$ is the centralizer of $x \in P$, and the action is conjugation. If $|P : C_P(x_i)| = 1$ then $P = C_P(x_i)$ and $x_i \in C(P)$, and we are done. Otherwise each $|P : C_P(x_i)| > 1$ and divides $|P| = p^n (n \geq 1)$, so p divides each $|P : C_P(x_i)|$ and divides $|P|$, and therefore divides $|C(P)|$. As $|C(P)| \geq 1$ because $1 \in C(P)$, $C(P)$ has at least p elements.

(e) Every finite p-group P is nilpotent.

Proof: Let G be a group. The center $C(G)$ of G is a normal subgroup. Let $C_2(G)$ be the inverse image of $C(G/C(G))$ under the canonical projection $G \twoheadrightarrow G/C(G)$. Then $C_2(G)$ is normal in G and contains $C(G)$. Continue this process by defining inductively: $C_1(G) = C(G)$ and $C_i(G)$ is the inverse image of $C(G/C_{i-1}(G))$ under the canonical projection $G \twoheadrightarrow G/C_{i-1}(G)$. Thus we obtain a sequence of normal subgroups of G, called the ascending central series of G : $1 \leq C_1(G) \leq C_2(G) \leq \ldots$ A group is called *nilpotent* if $G = C_n(G)$ for some $n \in \mathbb{N}$.

An equivalent definition can be given by commutator groups: Let $G^{(0)} = G$ and $G^{(i)} = [G, G^{(i-1)}]$. Then G is nilpotent if $G^{(n)} = 1$ for some $n \in \mathbb{N}$. The series $G \geq G^{(1)} \geq G^{(2)} \geq \ldots$ is called descending central series.

Now P and all its nontrivial quotients are p-groups, and therefore have nontrivial centers. If P is Abelian, then it is nilpotent. Otherwise $P \neq C(P)$. If $P \neq C_i(P)$, then $C_i(P)$ is strictly contained in $C_{i+1}(P)$. Since P is finite, $C_n(P)$ must equal P for some n.

(f) Frattini's argument: $\Phi(G)N_G(P) = G$.

Proof: $N_G(P) = \{\, g \in G \mid gPg^{-1} \subseteq P \,\}$ is the normalizer of P in G. Recall $P \leq \Phi(G)$ has been chosen as a p-subgroup of $\Phi(G)$.

Let $g \in G$. Then gPg^{-1} is a subgroup of $\Phi(G)$. By Sylow's second theorem there is an element $f \in \Phi(G)$ such that $f(gPg^{-1})f^{-1} \subseteq P$. So $x := fg \in N_G(P)$ and $G \ni g = f^{-1}x \in \Phi(G)N_G(P)$.

(g) $P \trianglelefteq \Phi(G)$ is normal.

Proof: Let $N_G(P) \subseteq M \subsetneq G$ be contained in a proper subgroup M of G. Then $\Phi(G) \subseteq M \cap \Phi(G)N_G(P) \subseteq M$ which is a contradiction. Hence $G = N_G(P)$ and P is normal in G, and especially normal in $\Phi(G)$.

(h) $\Phi(G)$ is nilpotent.

Proof: We will show that $\Phi(G)$ is the direct sum of its p-groups. Thus we have a direct sum of normal, nilpotent subgroups, which is therefore nilpotent, too. This follows e.g. from Fitting's theorem, but can also be proven directly.

Proof: Let $p_1, p_2, ..., p_s$ be the distinct primes dividing the order of $\Phi(G)$, and let P_i be p_i-groups for $1 \leq i \leq s$. For any t, $1 \leq t \leq s$ we show inductively that $P_1P_2 \cdots P_t$ is isomorphic to $P_1 \times P_2 \times \cdots \times P_t$. As each P_i is normal in $\Phi(G)$ so $P_1P_2 \cdots P_t$ is a subgroup of $\Phi(G)$. Let H be the product $P_1P_2 \cdots P_{t-1}$ and let $K = P_t$,so by induction H is isomorphic to $P_1 \times P_2 \times \cdots \times P_{t-1}$. In particular, $|H| = |P_1|\Delta|P_2| \cdots |P_{t-1}|$. Since $|K| = |P_t|$, the orders of H and K are relatively prime. Lagrange's Theorem implies the intersection of H and K is equal to 1. By definition, $P_1P_2 \cdots P_t = HK$, hence HK is isomorphic to $H \times K$ which is equal to $P_1 \times P_2 \times \cdots \times P_t$. This completes the induction. Now we take $t = s$ to obtain the result.

Remark: The subgroup of a group G which is generated by all nilpotent normal subgroups is called Fitting subgroup $F(G)$. If G is finite, we have

$$[F(G), F(G)] \leq \Phi(G) \leq F(G)$$
$$F(G)/\Phi(G) = F(G/\Phi(G))$$

2. The n-th Fermat number $F_n = 2^{2^n} + 1$ is prime for $n \in \mathbb{N}$ if and only if

$$3^{(F_n - 1)/2} \equiv -1 \mod F_n.$$

3 is a primitive root modulo F_n in this case.

Reason: Fermat Primes (Pépin, 1877).

Solution: We see from $3^{2^{2^n-1}} \equiv -1 \mod F_n$ that the remainder class of 3 in $(\mathbb{Z}/(F_n))^*$ has the order 2^{2^n}, i.e. $(\mathbb{Z}/(F_n))^*$ has at least $2^{2^n} = F_n - 1$ elements. This is only possible, if F_n is prime and 3 is a primitive root modulo F_n.

Now we show that this condition is necessary, too. Let F_n be prime. From $F_n \equiv 1 \mod 4$ we get

$$\left(\frac{3}{F_n}\right) = \left(\frac{F_n}{3}\right) = \left(\frac{2}{3}\right) = -1$$

where we used quadratic reciprocity and $F_n \equiv 2^{2^n} + 1 \equiv (-1)^{2^n} + 1 \equiv 2 \mod 3$. With Euler's criterion we now find

$$3^{(F_n - 1)/2} \equiv -1 \mod F_n.$$

3. Show that none of the numbers

$$11\,,\,111\,,\,1111\,,\,11111\,,\,111111\,,\,\ldots$$

can be written as a sum of two squares.

Reason: Number Theory.

Solution: The numbers $n = 11\,,\,111\,,\,1111\,\ldots$ are all congruent 11 mod 100 and so congruent $3 \mod 4$. Such a number has at least one odd prime factor $p \equiv 3 \mod 4$ which occurs in an odd power, since otherwise we would only have prime factors congruent $1 \mod 4$ and even powers of prime factors congruent $3 \mod 4$. Pairing two prime factors congruent $3 \mod 4$ results in a factor $1 \mod 4$. But all factors congruent $1 \mod 4$ remain congruent $1 \mod 4$ by multiplication. In order for $n \equiv 3 \mod 4$, n has to have a prime factorization

$$n = p \cdot \prod_{i=1}^{I} p_i^{2\nu_i} \cdot \prod_{j=1}^{J} q_j^{\mu_j} \qquad (*)$$

with primes $p, p_i \equiv 3 \mod 4$, $q_j \equiv 1 \mod 4$. Let's assume now n can be written as $n = x^2 + y^2$. If $d = \gcd(x, y) > 1$, then $d^2 \,|\, n$ and we can cancel it out without affecting p since all prime divisors of d occur twice. Hence we may assume that x, y are coprime, to the expense that we changed the value of n, but we still have $n = x^2 + y^2$ with x, y coprime, and a factorization $(*)$.

Since $p \nmid x$, because otherwise $p \,|\, n - x^2 = y^2$ and $p \,|\, y$, but we assumed them to be coprime, x is a unit modulo p, say $tx \equiv 1 \mod p$. From $p \,|\, n = x^2 + y^2$ we get $y^2 \equiv -x^2 \mod p$ and thus

$$(ty)^2 = t^2y^2 \equiv -t^2x^2 \equiv -1 \mod p.$$

With Euler's criterion we calculate

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} = (-1)^{\frac{4k+2}{2}} = (-1)^{2k+1} = -1$$

i.e. -1 is no quadratic residue modulo p, which means there is no number $z^2 \equiv -1 \mod p$, contradicting $z = ty$ which we just found.

4. Let $G = \langle a, b \,|\, a^p = b^q = 1, (aba) = b^r, a^s = b^t \rangle$ be a group of order twelve which operates on $\mathbb{R}^4$ by

$$a.v = \frac{1}{2} \cdot \begin{bmatrix} 1 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 1 & 0 & 0 \\ 0 & 0 & 1 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 1 \end{bmatrix} .v, \quad b.v = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} .v.$$

(a) Determine the group G and its presentation (p, q, r, s, t).

(b) Which group is

$$H = \langle a, b \,|\, a^6 = b^2 = 1, (aba) = b \rangle\,?$$

(c) The above groups are obviously not Abelian. There is another non Abelian group L of order twelve. Which one and what is (p, q, r, s, t) in that case?

Reason: Group Theory.

Solution:

(a) Since $a^{\operatorname{ord} a}(v) = 1(v)$ for all $v \in \mathbb{R}^4$ we have to calculate the order of the given, regular matrices:

$$\begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}^2 = \begin{bmatrix} -2 & 2\sqrt{3} \\ -2\sqrt{3} & -2 \end{bmatrix}, \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}^3 = \begin{bmatrix} -8 & 0 \\ 0 & -8 \end{bmatrix}$$
$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}^2 = \begin{bmatrix} -I & 0 \\ 0 & -I \end{bmatrix}$$

so $\operatorname{ord} a = 6$ and $\operatorname{ord} b = 4$ and $a^3 = b^2$. Now

$$\frac{1}{4}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \cdot \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \cdot \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 & AB \\ -BA & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

hence $aba.v = b.v$ for all $v \in \mathbb{R}^4$ and thus $aba = b$ and

$$G = \langle a, b \,|\, a^6 = b^4 = 1, (aba) = b^1, a^3 = b^2 \rangle.$$

The elements are $G = \{ 1, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b \}$ This means that $G = \operatorname{Dic}_3$, the dicyclic group of order 12.

(b) We have elements of order 6 and order 2 in

$$H = \langle a, b \,|\, a^6 = b^2 = 1, (aba) = b \rangle$$

The subgroup generated by a is normal: $ba^nb^{-1} = a^{-n}$, whereas the subgroup generated by b is not: $aba^{-1} = aba^5 = ba^4 \notin \langle b \rangle$. Hence $H \cong \mathbb{Z}_6 \rtimes \mathbb{Z}_2 \cong D_6$, the dihedral group of order 12.

(c) The third non Abelian group of order 12 is the alternating group $L = A_4 = \langle a, b \,|\, a^p = b^q = 1, (aba) = b^r, a^s = b^t \rangle$. It contains all even permutations of $\{ 1, 2, 3, 4 \}$. $(123), (234) \in A_4$, so there are at least two elements of order 3, from which we can choose one as generator, say $a = (123)$. The cycles of the Klein subgroup V_4 are all of order two. Now $(12)(34)(123)(12)(34) = (142)$ and $(123)(12)(34)(132) = (14)(23)$ show that $V_4 \triangleleft A_4$ is a normal subgroup, and that a $3-$cycle generates no normal subgroup, hence $A_4 \cong V_4 \rtimes \mathbb{Z}_3$. It can be shown that a and $(12)(34)$ generate A_4, but for our desired presentation, we need a generator b such that $aba = b^r$ is a relation. We can rule out $r = 0$ since it would imply $aba = 1 \Longrightarrow ab = a^{-1} = a^2 \Longrightarrow a = b$. But $aba = b$

cannot be true either, as $aV_4a \notin V_4$. Thus we have to choose another 3-cycle; $b := (234)$. Since $ab = (123)(234) = (12)(34)$ we already *know* by the omitted calculation above, that a and b generate the group. In addition we have $a^3 = b^3 = 1$ and $aba = (12)(34)(123) = (243) = b^{-1} = b^2$, so finally we get

$$A_4 \cong V_4 \rtimes \mathbb{Z}_3 \cong \langle a = (123), b = (234) | a^3 = b^3 = 1, (aba) = b^2, a^3 = b^3 \rangle$$

5. Let A be an associative, finite dimensional algebra with 1 over a field $\mathbb{F}$, $M \neq 0$ an A-module, and $0 \neq P \subseteq A_A$ a submodule of A as right A-module. Show that

   (a) M is irreducible if and only if 0 and 1 are the only idempotent elements of the endomorphism ring $\operatorname{End}_A(M)$.

   (b) P is a direct summand of A_A if and only if there is an idempotent element $e \in A$ such that $P = eA$.

   Reason: Modules.

   Solution:

   (a) If M is reducible, then there are submodules $0 \neq L, K \subseteq M$ such that $L \oplus K = M$. The projection $\pi_K \in \operatorname{End}_A(M)$ on K is idempotent and $\pi_K \notin \{0, 1\}$. If conversely $e \in \operatorname{End}_A(M)$ is idempotent, then $M = e(M) \oplus (1_M - e)(M)$. If $e \notin \{0, 1\}$, then $e(M), (1_M - e)(M) \neq 0$ and M is reducible.

   (b) Let P be a direct summand and $A = P \oplus Q$. Then we can write $A \ni 1 = e + f$.

   $$e - e^2 = (1 - e)e = fe \in P \cap Q = 0$$

   hence $e = e^2$ and $fe = 0$. By the same argument we get $f^2 = f$ and $ef = 0$. Moreover we have $eA \subseteq PA = P$ and for $p \in P$

   $$p = 1 \cdot p = (e + f) \cdot p = ep + fp = ep \in eA$$

   since $fp \in P \cap Q = 0$.
   Let conversely be $e \in A$ an idempotent, and set $P := eA$. Then

   $$f := 1 - e = 1 - 2e + e = 1 - 2e + e^2 = (1 - e)^2 = f^2$$

   is also an idempotent and we have $1 = e + f$ and $ef = fe = 0$. Hence $A = eA \oplus fA = P \oplus fA$.

6. We consider the topological space $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ equipped with distance

$$\chi(x,y) := \begin{cases} \dfrac{\|x-y\|_2}{\sqrt{1+\|x\|_2^2}\sqrt{1+\|y\|_2^2}} & \text{if } x,y \neq \infty \\ \dfrac{1}{\sqrt{1+\|x\|_2^2}} & \text{if } x \neq \infty, y = \infty \\ \dfrac{1}{\sqrt{1+\|y\|_2^2}} & \text{if } x = \infty, y \neq \infty \\ 0 & \text{if } x = y = \infty \end{cases}$$

Show that χ defines a metric such that $\mathcal{C} := (\mathbb{C}_\infty, \chi)$ is a compact topological space.

Reason: Compact Space.

Solution: The chordal metric χ as defined is half the Euclidean distance in $\mathbb{R}^3$ under the stereographic projection

$$\pi \,:\, \mathbb{R}^3 \supset \mathbb{S}^2_{(0,0,1)} - \{(0,0,2)\} \longrightarrow \mathcal{C}\;,\; \pi(a,b,c) = \left(\frac{2a}{2-c} + i \cdot \frac{2b}{2-c}\right)$$

of the Riemann sphere: $2\chi(x,y) \stackrel{(*)}{=} \|\pi^{-1}(x) - \pi^{-1}(y)\|_2$.

Let us consider the stereographic projection. A point $P = u + iv = (u,v,0)$ on the complex plane corresponds to the point on the sphere $\{\,(x,y,z) \,|\, x^2+y^2+(z-1)^2 \stackrel{(**)}{=} 1\,\}$ which is part of the line through P and the north pole $N = (0,0,2)$, hence the point $\pi^{-1}(P) = (x,y,z)$ that satisfies $(**)$ and $(x,y,z) = (0,0,2) + \lambda(u,v,-2)$ for some $\lambda \in \mathbb{R}$. Solving these for λ yields $\lambda = 4/(4+u^2+v^2)$ and

$$\pi^{-1}(P) = \pi^{-1}(u+iv) = \frac{1}{4+u^2+v^2} \cdot (4u, 4v, 2u^2+2v^2)$$

If we reduce the last coordinate by 1, we will get a sphere of radius 1 and the center at the origin. Thus

$$\bar{\pi} \,:\, \mathbb{R}^3 \supset \mathbb{S}^2_{(0,0,0)} - \{(0,0,1)\} \longrightarrow \mathcal{C}\;,\; \bar{\pi}(a,b,c) = \left(\frac{a}{1-c} + i \cdot \frac{b}{1-c}\right)$$

and

$$\bar{\pi}^{-1}(P) = \bar{\pi}^{-1}(u+iv) = \frac{1}{1+u^2+v^2} \cdot (2u, 2v, -1+u^2+v^2)$$

which doesn't affect the distances on $\mathbb{S}^2 \subseteq \mathbb{R}^3$. So in order to prove $(*)$ we set $P = u + iv$, $Q = x + iy$, $p := x^2 + y^2 + 1$, $q := u^2 + v^2 + 1$ and calculate

$$
\begin{aligned}
\|\bar{\pi}^{-1}(P) - \bar{\pi}^{-1}(Q)\|_2^2 &= \frac{1}{(pq)^2} \,\|(2up - 2xq, 2vp - 2yq, (p-2)q - (q-2)p)\|_2^2 \\
&= \frac{4}{(pq)^2} \left((up - xq)^2 + (vp - yq)^2 + (p - q)^2\right) \\
&= \frac{4}{(pq)^2}(u^2p^2 + v^2p^2 + x^2q^2 + y^2q^2 - 2uxpq - 2vypq \\
&\qquad + p^2 + q^2 - 2pq) \\
&= \frac{4}{(pq)^2}(p^2(u^2 + v^2 + 1) + q^2(x^2 + y^2 + 1) \\
&\qquad - 2pq(ux + vy + 1)) \\
&= \frac{4}{(pq)^2}(p^2q + q^2p - 2pq(ux + vy + 1)) \\
&= \frac{4}{pq}(p + q - 2(ux + vy + 1)) \\
&= \frac{4}{pq}(x^2 + y^2 + u^2 + v^2 - 2ux - 2vy) \\
&= \frac{4}{pq} \,\|(x - u) + i(y - v)\|_2^2 \\
&= 4\chi(P, Q)^2
\end{aligned}
$$

As χ can be expressed as a Euclidean distance, it is clear that the triangle inequality holds. χ is positive definite and symmetric which is more or less obvious. It is also clear that

$$\chi(x, y) \leq \chi(x, z) + \chi(z, y)$$

as soon as at least two of the points are at infinity. Hence it remains to check the cases $x = \infty$ or $z = \infty$.

- $\chi(x,y) \leq \chi(x,\infty) + \chi(\infty,y)$.

$$\begin{aligned}\chi(x,y) &\leq \frac{\|x\|_2}{\sqrt{1+\|x\|_2^2}\sqrt{1+\|y\|_2^2}} + \frac{\|y\|_2}{\sqrt{1+\|x\|_2^2}\sqrt{1+\|y\|_2^2}} \\ &= \frac{\|x\|_2}{\sqrt{1+\|x\|_2^2}} \cdot \frac{1}{\sqrt{1+\|y\|_2^2}} \\ &\quad + \frac{\|y\|_2}{\sqrt{1+\|y\|_2^2}} \cdot \frac{1}{\sqrt{1+\|x\|_2^2}} \\ &\leq \chi(y,\infty) + \chi(\infty,x)\end{aligned}$$

because $\dfrac{\|x\|_2}{\sqrt{1+\|x\|_2^2}}, \dfrac{\|y\|_2}{\sqrt{1+\|y\|_2^2}} \leq 1$.

- $\chi(\infty,z) \leq \chi(\infty,y) + \chi(y,z)$.

Set $y = a+ib,\ z = u+iv,\ p = \sqrt{1+|y|^2},\ q = \sqrt{1+|z|^2}$. Then

$$\begin{aligned}0 &\leq (av-bu)^2 + (a-u)^2 + (b-v)^2 \\ &= a^2+b^2+u^2+v^2+a^2v^2+b^2u^2-2abuv-2au-2bv \\ 1 &+ a^2u^2+b^2v^2+2abuv+2au+2bv \\ &\leq a^2+b^2+u^2+v^2+a^2v^2+b^2u^2+1+a^2u^2+b^2v^2\end{aligned}$$

So $(1+au+bv)^2 \leq (1+a^2+b^2)(1+u^2+v^2)$ and

$$\begin{aligned}1+au+bv &\leq pq \\ 2+a^2+b^2+u^2+v^2-2pq &\leq -2au-2bv+a^2+b^2+u^2+v^2 \\ (p-q)^2 = q^2+p^2-2pq &\leq (a-u)^2(b-v)^2 \\ p-q &\leq \|y-z\|_2 \\ \chi(\infty,z)-\chi(y,\infty) = \frac{1}{q}-\frac{1}{p} &\leq \frac{\|y-z\|_2}{pq} = \chi(y,z) \\ \chi(\infty,z) &\leq \chi(\infty,y)+\chi(y,z)\end{aligned}$$

$\mathcal{C}$ is compact if and only if it is sequentially compact. Let $(z_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}$.

- Case 1: There is an $N \in \mathbb{N}$ such that $(z_n)_{n\geq N}$ is a bounded sequence in $\mathbb{C}$.

In this case there is a convergent subsequence $(z_{n_k})_{n_k\geq N} \subseteq (z_n)_{n\geq N} \subseteq (\mathbb{C}, \|.\|_2)$, say $\lim_{k\to\infty} z_{n_k} = z$. Thus

$$0 \leq \chi(z_{n_k},z) = \frac{\|z_{n_k}-z\|_2}{\sqrt{1+\|z_{n_k}\|_2^2}\sqrt{1+\|z\|_2^2}} \leq \|z_{n_k}-z\|_2 \stackrel{n\to\infty}{\longrightarrow} 0$$

and $\lim_{k\to\infty} \chi(z_{n_k}, z) = 0$ by the sandwich principle, so $(z_{n_k})_{k\in\mathbb{N}}$ is a convergent subsequence of $(z_n)_{n\in\mathbb{N}}$ in $\mathcal{C}$, which had to be shown.

- Case 2: $(z_n)_{n\geq N}$ is for each $N \in \mathbb{N}$ an unbounded sequence in $\mathbb{C}$ or contains $\{\infty\}$.

  In this case there is an $N \leq M_N \in \mathbb{N}$ such that $\|z_{M_N}\|_2 > N$ for all $N \in \mathbb{N}$. If $z_{M_N} = \infty$ we write $\|z_{M_N}\|_2 = \infty$. Now if we define $C_N := \max\{\, M_k \,|\, 1 \leq k \leq N \,\}$ we will get an increasing list of natural numbers. Each natural number can occur at most finitely often on this list, since $M_N \geq N$. Thus $\lim_{N\to\infty} C_N = \infty$.

$$
\begin{aligned}
0 \leq \chi(z_{C_N}, \infty) &= \begin{cases} 0 & \text{if } z_{C_N} = \infty \\ \dfrac{1}{\sqrt{1+\|z_{C_N}\|_2^2}} & \text{if } \|z_{C_N}\|_2 \neq \infty \end{cases} \\
&\leq \frac{1}{\sqrt{1+N^2}} \\
&\stackrel{N\to\infty}{\longrightarrow} 0
\end{aligned}
$$

  and $(z_{C_N})_{N\in\mathbb{N}} \subseteq (z_n)_{n\in\mathbb{N}}$ is the subsequence we were looking for, and which converges to $0 \in \mathcal{C}$.

7. (a) Calculate $\displaystyle\int_{|z|=5} \frac{e^z}{z^2+\pi^2}\,dz$.

   (b) Determine all $z \in \mathbb{C}$ such that

$$f(z) = e^{z^7(\sin z)^{16}} + \bar{z}^2$$

   is complex differentiable.

Reason: Complex Integration And Differentiability.

Solution:

(a) We write $D(z_0, r) = \{\, z \in \mathbb{C} \,|\, |z - z_0| < r \,\}$ for a disk around z_0 with radius r. The zeros of the denominator are $z = \pm i\pi$ inside $D(0, 5)$. Let

$$\Omega := D(0,5) - (D(i\pi, 1) \cup D(-i\pi, 1))$$

Then

$$\begin{aligned}
\int_{|z|=5}\frac{e^z}{z^2+\pi^2}\,dz &= \int_{|z|=5}\frac{e^z}{(z-i\pi)(z+i\pi)}\,dz \\
&= \int_{\partial\Omega}\frac{e^z}{(z-i\pi)(z+i\pi)}\,dz \\
&+ \int_{|z-i\pi|=1}\frac{e^z}{(z-i\pi)(z+i\pi)}\,dz + \int_{|z+i\pi|=1}\frac{e^z}{(z-i\pi)(z+i\pi)}\,dz \\
&= 0 + \int_{|z-i\pi|=1}\frac{\frac{e^z}{z+i\pi}}{z-i\pi}\,dz + \int_{|z+i\pi|=1}\frac{\frac{e^z}{z-i\pi}}{z+i\pi}\,dz \\
&= 2i\pi\left(\frac{e^{i\pi}}{i\pi+i\pi}+\frac{e^{-i\pi}}{-i\pi-i\pi}\right) \\
&= e^{i\pi}-e^{-i\pi} \\
&= 0
\end{aligned}$$

(b) If $f(z)$ is complex differentiable, then $g(z) := f(z) - e^{z^7(\sin z)^{16}} = \bar{z}^2$ is complex differentiable at z, too, since it is the composition of two on $\mathbb{C}$ holomorphic functions.

$$g(z) = g(x+iy) = \overline{x+iy}^{\,2} = (x-iy)^2 = \underbrace{x^2-y^2}_{=:u(x,y)} + i\,\underbrace{(-2xy)}_{=:v(x,y)}$$

For the Cauchy Riemann equations we check

$$u_x = 2x \ , \ u_y = -2y \ , \ v_x = -2y \ , \ v_y = -2x$$

Now $u_x = v_y$ implies $x = 0$ and $u_y = -v_x$ implies $y = 0$. Since all derivatives are continuous on $\mathbb{R}^2$, the Cauchy Riemann equations are not only necessary, but sufficient as well. Hence $g(z)$ is only complex differentiable at $z = 0$ and so is $f(z)$.

8. Calculate
$$\int_1^\infty \frac{1+x^2-2x^2\log(x)}{x(1+x^2)^2}\,dx$$

Reason: Catalan's Constant.

Solution: The square in the denominator reminds us on the quotient rule, so we consider

$$\frac{f(x)}{g(x)} = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$$

and set $g(x) = (x^2+1)$. Hence

$$\begin{aligned}\frac{1+x^2-2x^2\log(x)}{x(1+x^2)^2} &= \frac{\frac{1}{x}+x-2x\log(x)}{(1+x^2)^2} \\ &= \frac{\frac{1}{x}(1+x^2)-2x\log(x)}{(1+x^2)^2} \\ &= \frac{f'(x)(x^2+1)-f(x)2x}{(x^2+1)^2}\end{aligned}$$

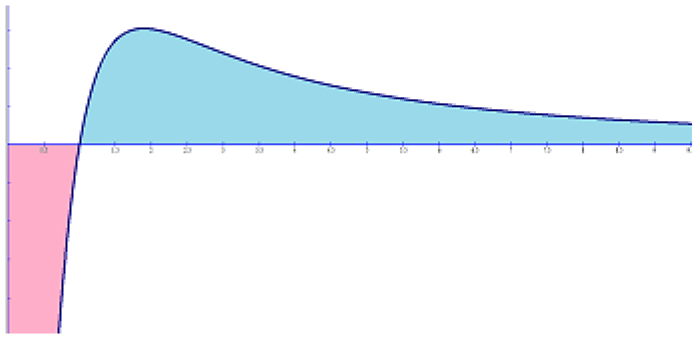
so $f(x) = \log x$ gives the desired solution and

$$\int_1^\infty \frac{1+x^2-2x^2\log(x)}{x(1+x^2)^2}\,dx = \left[\frac{\log(x)}{x^2+1}\right]_1^\infty = 0$$

The function $f(x) = \dfrac{\log(x)}{x^2+1}$ has an interesting property:

$$\left|\int_0^1 f(x)\,dx\right| = \int_1^\infty f(x)\,dx = C = 0.915965594177219 01\ldots$$

where C is Catalan's constant A006752 in the OEIS.



9. Determine the square root and the inverse matrix of

$$A = \begin{pmatrix} 5 & -4 & 2 \\ -4 & 7 & -8 \\ 1 & -4 & 6 \end{pmatrix}$$

What is the dimension of the simple Lie algebra whose Cartan matrix $\sqrt{A}$ is?

Reason: Matrix Calculations.

Solution:

$$\sqrt{A}=\begin{pmatrix}2&-1&0\\-1&2&-2\\0&-1&2\end{pmatrix}\;,\;A^{-1}=\frac{1}{4}\cdot\begin{pmatrix}10&16&18\\16&28&32\\9&16&19\end{pmatrix}$$

The Cartan matrix $\sqrt{A}$ belongs to the simple Lie algebra of type B_3 which is the 21 dimensional orthogonal Lie algebra $\mathfrak{o}(7,\mathbb{R})=\mathfrak{so}(7,\mathbb{R})$.

10. Let R be a commutative ring with 1. We define the nilradical $N(R)=N\subseteq R$ as intersection of all prime ideals of R, and the Jacobson radical $J(R)=J$ as intersection of all maximal ideals.

   (a) Show that $N(R)$ contains exactly all nilpotent Elements of R.

   (b) Assume R is Artinian. Show that all prime ideals are maximal, hence $N(R)=J(R)$ in an Artinian ring.

   (c) Assume R is Artinian. Show that $N(R)$ is a nilpotent Ideal.

   (d) Give an example of $N(R)\neq J(R)$ if R is not Noetherian and thus not Artinian either.

Reason: Ring Theory.

Solution:

(a) $N(R)$ is the set of all nilpotent elements of R.

   i. Let $r\in R$ be nilpotent and $P\subseteq R$ a prime ideal. Then $r^n=0\in P$ for some $n\in\mathbb{N}$ and $r\in P$ since P is prime. So all nilpotent elements of R are contained in all prime ideals.

   ii. Let $r\in R$ be not nilpotent. We consider the set of ideals

$$\Sigma:=\{\,I\trianglelefteq R\mid n>0\Longrightarrow r^n\notin I\}$$

   Since $0\in\Sigma$ we have $\Sigma\neq\emptyset$ and a maximal element $M\in\Sigma$ by inclusion as order and Zorn's Lemma (AC). We must show that M is a prime ideal, because from $r\notin M$ we get that any non nilpotent element cannot be in all prime ideals. Let $x\,,\,y\notin M$. Then we have to show that $x\cdot y\notin M$.

   $M+(x),M+(y)\supsetneq M$ so they cannot belong to Σ for the maximality of M. Thus there are numbers $n,m>0$ with $r^n\in M+(x)$ and $r^m\in M+(y)$, i.e. $r^{n+m}=r^n\cdot r^m\in M+(x\cdot y)$. As we have found a positive power of r which is in an ideal $M+(xy)$, we have shown that $M+(xy)\notin\Sigma$, i.e. $xy\notin M\in\Sigma$ which had to be shown. In other words, we have found a prime ideal M which doesn't contain r, so $r\notin N(R)$.

(b) Let R be Artinian and $P \subseteq R$ a prime ideal. Then $S := R/P$ is an Artinian integral domain. Let $s \in S - \{0\}$. Because of the descending chain condition on ideals, we have $(s^n) = s^{n+1}$ for some $n \in \mathbb{N}$ and thus $s^n = s^{n+1}r$ or $s^n(sr-1) = 0$. Since S is an integral domain, we may conclude $sr = 1$ as $s^n \neq 0$. But this means s is a unit, i.e. S is a field and P maximal.

(c) Let $N^k = N^{k+1} = \ldots =: A \triangleleft R$ for some $0 \neq k \in \mathbb{N}$. Now assume that $A \neq 0$, and consider the set Ξ of all ideals B such that $AB \neq 0$. Since $A^2 = A \neq 0$ we have $A \in \Xi \neq \emptyset$, and because R is Artinian we can choose a minimal ideal $C \in \Xi$. Then there is an element $x \in C$ such that $Ax \neq 0$. We even have $(x) = C$ by minimality of C. The same argument leads to $Ax = (x)$, because $A(Ax) = A^2x = Ax \neq 0$ means $Ax \in \Xi$ and $Ax \subseteq AC \subseteq C$, so minimality of C implies $x \in (x) = C = Ax$. Thus there is an element $a \in A$ such that $x = ax$, hence $x = ax = a^2x = a^3x = \ldots = a^nx = \ldots$
Now $a \in A = N^k \subseteq N$ is nilpotent by the previous part, so $x = 0$ which contradicts $Ax \neq 0$ and our assumption $A \neq 0$ was wrong, hence $0 = A = N^k$ and $N(R)$ is nilpotent.

(d) Let $R = \mathbb{R}[x_1, x_2, \ldots]$ be the ring of real polynomials with countably infinite many indeterminates. This ring is neither Noetherian (a.c.c.) as $(x_1) \subsetneq (x_1, x_2) \subsetneq (x_1, x_2, x_3) \subsetneq \ldots$ shows, nor Artinian (d.c.c.) as $(x_1) \supsetneq (x_1^2) \supsetneq (x_1^3) \supsetneq \ldots$ shows. (x_1) is a prime ideal, $R/(x_1) \cong R$ is an integral domain, but no field and (x_1) not maximal. $M := (x_1, x_2, \ldots)$ is the unique maximal ideal, so $J(R) = M$ with $R/J \cong \mathbb{R}$. On the other hand R doesn't contain any nilpotent elements, so $N(R) = 0$. Another way to see it is, that $\bigcap_{n\in\mathbb{N}} (x_n) = 0$ because there is no polynomial which is divided by all indeterminates, or that the intersection of the two prime ideals $(x_1) \cap (1 + x_1) = 0$ is zero.

11. (HS-1) Prove that the geometric mean of two numbers is less or equal the arithmetic mean of these numbers by three different methods.

Reason: Geometry - Algebra - Calculus.

Solution:

(a) Geometry.

The height h in a right triangle is square root of the product pq of the two sections of the diameter it separates: $h = \sqrt{pq}$. The height is also less or equal the radius r of the surrounding circle, which is $r = \frac{p+q}{2}$ and thus $\sqrt{pq} \leq \frac{p+q}{2}$.

(b) Algebra.

$$\begin{aligned} 0 &\leq (p-q)^2 = p^2 - 2pq + q^2 \\ \Longrightarrow 4pq &\leq p^2 + 2pq + q^2 = (p+q)^2 \\ \Longrightarrow \sqrt{pq} &\leq \frac{p+q}{2} \end{aligned}$$

(c) Calculus.

Assume a fixed length L such that $L = p+q$. We want to minimize

$$f(p) := \frac{p + (L-p)}{2} - \sqrt{p(L-p)} = \frac{L}{2} - \sqrt{pL - p^2}$$

$$\begin{aligned} \frac{df}{dp} &= -\frac{1}{2} \cdot \frac{L-2p}{\sqrt{pL-p^2}} \\ \frac{d^2f}{dp^2} &= \frac{1}{4} \cdot \frac{L^2}{\sqrt{pL-p^2}^3} \end{aligned}$$

$f'(p) = 0$ for $p = L/2$ and $f''(L/2) > 0$ so $p = L/2$ is a minimum, i.e. $f(L/2) = 0 \leq f(p) = \dfrac{p+q}{2} - \sqrt{pq}$ for any p.

12. (HS-2) Calculate the formula for the tangent at the unit circle at $p = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ by three different methods, or better points of view.

Reason: Tangent Spaces.

Solution:

(a) The point lies in the first quadrant, so we can take $y = \sqrt{1-x^2}$ as function for the circle segment. Then $y' = -\dfrac{x}{\sqrt{1-x^2}}$ and $y'(1/2) = -1/\sqrt{3}$. Solving $y_T = -\frac{1}{\sqrt{3}} \cdot x + b$ for p results in

$$b = \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{3}} \cdot \frac{1}{2} \text{ and } y_T = -\frac{1}{\sqrt{3}} \cdot x + \frac{2}{\sqrt{3}}$$

(b) The tangent is perpendicular to the (normal) position vector $\vec{p}$, hence has the direction $\vec{t} = \vec{p}^{\perp} = (-\sqrt{3}/2\,,\, 1/2)$. This results in the straight

$$T : \vec{p} + s \cdot \vec{t} \Longleftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \cdot \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} + \frac{s}{2} \cdot \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}$$

(c) The circle is parameterizable by $C = \left(\dfrac{1-s^2}{1+s^2}\,,\,\dfrac{2s}{1+s^2}\right)$. The corresponding parameter for p is $s_p = 1/\sqrt{3}$.

$$D_pC = \left(-4\frac{s}{(s^2+1)^2}\,,\,-2\frac{s^2-1}{(s^2+1)^2}\right)_p = \left(-\frac{3\sqrt{3}}{4}\,,\,\frac{3}{4}\right) \sim (-\sqrt{3}\,,\,1)$$

Thus we get the tangent

$$T : \vec{p} + s \cdot D_pC \Longleftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \cdot \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} + \frac{3s}{4} \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}$$

13. (HS-3) We are looking for the number $n = abc$, where a is the maximal number of rotations which are necessary to solve Rubik's cube out of any state, b is the largest natural number of Chicken McNuggets which cannot be bought by the usual box sizes of $6, 9$ or 20, and c is the smallest three digit emirp number.

    Reason: Riddle.

    Solution: $a = 20\,,\, b = 43\,,\, c = 107\,,\, n = 92020.$

14. (HS-4) Show that the following linear equation system with variables $x_1, \ldots, x_n$ has always a unique solution:

$$\begin{aligned} x_1 &= 2x_{n-m+1} + 3x_{n-m+2} + b_1 \\ x_2 &= 4x_{n-m+2} + 9x_{n-m+3} + b_2 \\ \cdots & \quad \cdots \\ x_{m-1} &= 2^{m-1}x_{n-1} + 3^{m-1}x_n + b_{m-1} \\ x_m &= 2^m x_n + b_m \\ x_{m+1} &= b_{m+1} \\ \cdots & \quad \cdots \\ x_n &= b_n \end{aligned}$$

for all positive integers $1 \leq m < n$ and any real numbers $b_1, \ldots, b_n$.

Reason: Nilpotent Matrix.

Solution: The coefficient matrix of the linear equation system is

$$\begin{bmatrix} 1 & 0 & \ldots & 0 & -2 & -3 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 & -4 & -9 & \ldots & 0 & 0 \\ & & \ldots & & & & & \ldots & & \\ 0 & 0 & \ldots & 1 & 0 & 0 & 0 & \ldots & -2^{m-1} & -3^{m-1} \\ 0 & 0 & \ldots & 0 & 1 & 0 & 0 & \ldots & 0 & -2^m \\ 0 & 0 & \ldots & 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ & & \ldots & & & & & \ldots & & \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix} = 1 + N$$

with a nilpotent matrix N, i.e. $N^m = 0$. Now

$$1 = \underbrace{(1 - N + N^2 - N^3 \pm \ldots + (-1)^{m-1}N^{m-1})}_{=:M} \cdot (1 + N)$$

The linear equation system now writes $(1 + N)\vec{x} = \vec{b}$ or $\vec{x} = M\vec{b}$.

15. (HS-5) Calculate the following derivatives:

(a) $\dfrac{dy}{dx}$ if $y = 1 + y^x$

(b) $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ if $y = x + \log y$

(c) $\left.\dfrac{dy}{dx}\right|_{x=1}$ and $\left.\dfrac{d^2y}{dx^2}\right|_{x=1}$ if $x^2 - 2xy + y^2 + x + y - 2 = 0$

Reason: Differentiation.

Solution:

(a)

$$\begin{aligned} y = 1 + y^x \Longrightarrow y' &= (y^x)' = (\exp(x \log y))' \\ &= (\exp(x \log y)) \cdot (x \log y)' = y^x \cdot \left(\log y + x \cdot \frac{y'}{y}\right) \\ &\Longrightarrow y'\left(1 - \frac{xy^x}{y}\right) = y^x \log y \\ &\Longrightarrow \frac{dy}{dx} = y' = \frac{y^x \log y}{1 - xy^{x-1}} \end{aligned}$$

(b)

$$
\begin{aligned}
y = x + \log y &\Longrightarrow y' = 1 + (\log y)' = 1 + \frac{y'}{y} \\
&\Longrightarrow \frac{dy}{dx} = y' = \frac{y}{y-1} \\
&\Longrightarrow y'' = (y(y-1)^{-1})' = y'(y-1)^{-1} - y(y-1)^{-2}y' \\
&= \frac{y(y-1)}{(y-1)^3} - \frac{y^2}{(y-1)^3} = -\frac{y}{(y-1)^3} \\
&\Longrightarrow \frac{d^2y}{dx^2} = y'' = \frac{y}{(1-y)^3}
\end{aligned}
$$

(c)

$$
\begin{aligned}
x^2 - 2xy + y^2 + x + y - 2 = 0 &\Longrightarrow (x-y)^2 + (x+y) = 2 \\
\Longrightarrow 0 = 2(x-y)(1-y') + 1 + y' &= y'(1+2(y-x)) + 1 + 2(x-y) \\
\Longrightarrow y' &= \frac{2(y-x)-1}{2(y-x)+1}
\end{aligned}
$$

At $x = 1$ we have $1 - 2y + y^2 + 1 + y - 2 = 0 = y^2 - y = y(y-1)$, i.e. $y = 0$ or $y = 1$.

This is $\left.\frac{dy}{dx}\right|_{x=1} = y'(1) = 3$ or $\left.\frac{dy}{dx}\right|_{x=1} = y'(1) = -1$.

$$
\begin{aligned}
0 &= y'(1+2(y-x)) + 1 + 2(x-y) \Longrightarrow \\
0 &= y''(1+2(y-x)) + y'(2(y'-1)) + 2(1-y') \\
y'' &= \frac{2(y'-1)^2}{2(x-y)-1}
\end{aligned}
$$

At $x = 1$ we have $(y, y') = (0, 3)$ or $(y, y') = (1, -1)$.

This is $\left.\frac{d^2y}{dx^2}\right|_{x=1} = y''(1) = 8$ or $\left.\frac{d^2y}{dx^2}\right|_{x=1} = y''(1) = -8$.

5 August 2020

1. Let F be a meromorphic function (holomorphic up to isolated poles) in $\mathbb{C}$ with the following properties:

   (a) F is holomorphic (complex differentiable) in the half plane $H(0) = \{z \in \mathbb{C} \,:\, \Re(z) > 0\}$.

   (b) $zF(z) = F(z+1)$.

   (c) F is bounded in the strip $\{z \in \mathbb{C} \,:\, 1 \leq \Re(z) \leq 2\}$.

   Show that $F(z) = F(1)\Gamma(z)$.

   Reason: Wielandt's theorem.

   Solution: The gamma function satisfies the first two properties and the third follows from $|\Gamma(z)| \leq \Gamma(\Re(z))$ and that $\Gamma(x)$ is bounded on the closed interval $1 \leq x \leq 2$ since it is continuous.

   Now consider
$$F_0(z) := F(z) - F(1)\Gamma(z).$$
   Then F_0 fulfills all three conditions, too, and $F_0(1) = 0$. The functional equation $F_0(z) = F_0(z+1)/z$ implies, that F_0 is holomorphic at $z = 0$, and that F_0 is bounded on the strip $S_0 := \{z \in \mathbb{C} \,:\, 0 \leq \Re(z) \leq 1\}$. Hence the function
$$\Phi(z) := F_0(z)F_0(1-z)$$
   is bounded in S_0.. We have
$$\Phi(z+1) = F_0(z+1)F_0(-z) = zF_0(z)F_0(-z) = -F_0(z)F_0(-z+1) = -\Phi(z)$$
   This means that Φ is periodic with period 2 and bounded in entire $\mathbb{C}$. Now Φ is constant by Liouville'S theorem. The constant must equal zero, as $\Phi(1) = -\Phi(0)$. Hence $0 = F_0(z)F_0(1-z)$ so $F_0 \equiv 0$ and $F_0(z) = 0 = F(z) - F(1)\Gamma(z)$.

2. Show that if f is any continuous real function and n any positive number,
$$I := \int_{n^{-1}}^{n} f\left(x + \frac{1}{x}\right) \frac{\log x}{x}\, dx = 0.$$
   Reason: Integration Trick.

Solution:

$$\begin{aligned}
I &\stackrel{y=1/x}{=} \int_n^{n^{-1}} f\left(\frac{1}{y}+y\right)\,\frac{-\log y}{1/y}\,\frac{-dy}{y^2}\\
&= \int_n^{n^{-1}} f\left(y+\frac{1}{y}\right)\,\frac{\log y}{y}\,dy\\
&= -\int_{n^{-1}}^{n} f\left(y+\frac{1}{y}\right)\,\frac{\log y}{y}\,dy\\
&= -I
\end{aligned}$$

Since $\operatorname{char}\mathbb{R} = 0 \neq 2$ we get $I = 0$.

3. (HS-1) Let $a < b < c < d$ be real numbers. Sort $x = ab + cd$, $y = bc + ad$, $z = ac + bd$ and prove it.

   Reason: Arithmetics.

   Solution: We suppose from examples that $y < z < x$.

   (a) $y < z$

   This is equivalent to

   $$\begin{aligned}
   bc + ad < ac + bd &\Longleftrightarrow (b-a)c < (b-a)d\\
   &\Longleftrightarrow (b-a)(c-d) < 0
   \end{aligned}$$

   which is true as $b - a > 0$ and $c - d < 0$.

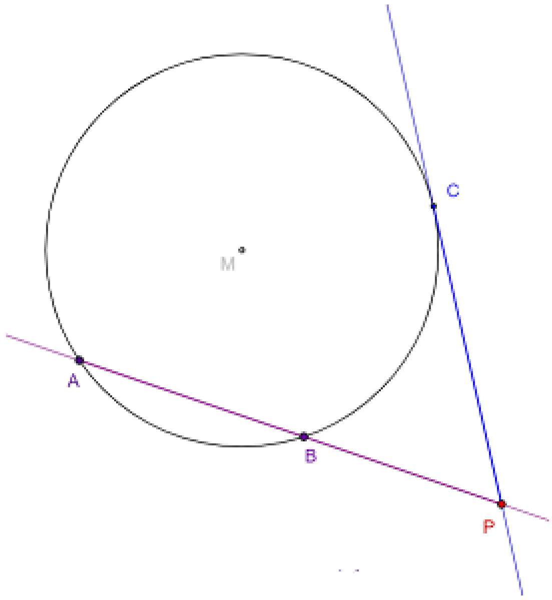
   (b) $z < x$

   This is equivalent to

   $$ac + bd < ab + cd \Longleftrightarrow (d-a)(b-c) < 0$$

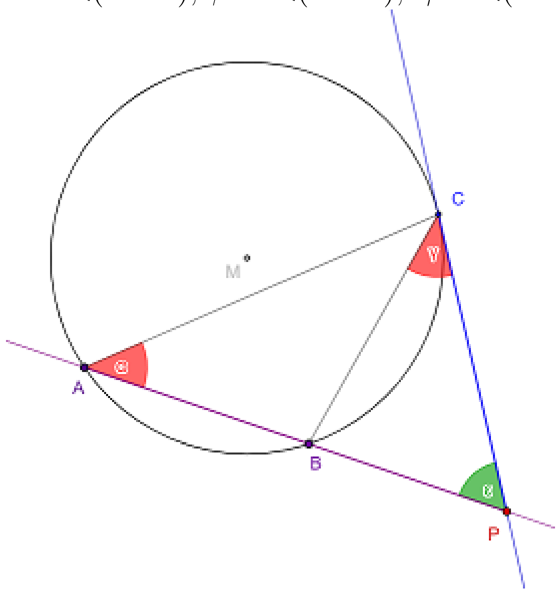
   which is true as $d - a > 0$ and $b - c < 0$.

4. (HS-2) Prove $\overline{CP}^2 = \overline{AP} \cdot \overline{BP}$



Reason: Geometry.

Solution: We connect the points AC and BC and define the angles $\alpha = \sphericalangle(BAC)$, $\beta = \sphericalangle(BPC)$, $\gamma = \sphericalangle(BCP)$



If we connect $\overline{CM}$ and elongate it to a diameter $\overline{CD}$ then $\gamma + \sphericalangle(DCB) = 90°$. Thales' theorem now gives us $\sphericalangle(CDB) + \sphericalangle(DCB) = 90°$ thus $\gamma = \sphericalangle(CDB)$. As all periphery angles over the same chord $\overline{CB}$ are equal $(= 1/2\sphericalangle(CMB))$, we get $\gamma = \alpha$. So with β we have two identical angles in $\triangle(PAC)$ and $\triangle(PBC)$, i.e. they are similar. This means

$$\frac{\overline{AP}}{\overline{CP}} = \frac{\overline{CP}}{\overline{BP}} \Longleftrightarrow \overline{AP} \cdot \overline{BP} = \overline{CP}^2$$

5. (HS-3) How big is the probability for two pocket aces in Texas Hold'em? Assume we have seen a show down in a heads-up. How many possible combinations are there, how many combinations of possible starting hands can the opponents have? How many possible community cards?

   Reason: Poker.

   Solution: There are $\binom{4}{2} = 6$ possible pocket aces among $\binom{52}{2} = 1,326$ possible hands, so the probability is $\frac{6}{1,326} = \frac{1}{221} \approx 0.45\%$. We have $\binom{52}{9} = 3,679,075,400$ combinations total after a show down in a heads-up. Community cards are $\binom{48}{5} = 1,712,304$ possibilities and $\binom{52}{2} \cdot \binom{50}{2} = 1,624,350$ possible starting hands in a heads-up.

6. (HS-4) Everybody knows that Schrödinger's cat is trapped in the box since 1935. Not well known is the fact, that the radioactive material was ten ^{14}C isotopes. Calculate the probability that the cat is still alive.

   Reason: Radiation.

   Solution: Half-life of ^{14}C are $5,730$ years. The decay rate is thus $\lambda = \dfrac{\log 2}{T_{1/2}} \approx 1.21 \cdot 10^{-4}\,\mathrm{a}^{-1}$. The probability for a single isotope to survive is $P = \exp(-\lambda T) \approx 98.977\%$. The cat survived, if all ten isotopes survived, i.e. with a probability of P^{10}. With significant figures provided by a modern calculator, this results in

   $$P^{10} = \left(\exp\left(-\frac{85 \cdot \log 2}{5730}\right)\right)^{10} = 0.902286772193 \approx 90\%$$

   So the cat has a $9 : 10$ chance to be still alive.

7. (HS-5) Show that there is no rational solution for $p^2 + q^2 + r^2 = 7$.

   Reason: Modular Arithmetic.

   Solution: The equation can be transformed into an equivalent integer equation $x^2+y^2+z^2 = 7w^2$ where $\gcd(x^2+y^2+z^2, w^2) = 1$. Given any integer n, then $n^2 \in \{\bar{0}, \bar{1}, \bar{4}\} \mod 8$. If $w^2 \equiv \bar{1} \mod 8$ then $7w^2 \equiv \bar{7} \mod 8$ but there is no way to get $\bar{7}$ as a result of three sums of elements from $\{\bar{0}, \bar{1}, \bar{4}\}$. Hence $4|w^2$ and w is even. Not all x, y, z can thus be odd.

Assume x is even and y, z are odd: $4a^2+(2b+1)^2+(2c+1)^2 \equiv \bar{2} \mod 4$ but $7w^2 \equiv \bar{0} \mod 4$ which cannot be equal. So x, y, z are all even. But now we have a divider 2 of $x^2 + y^2 + z^2$ and of w^2, in contradiction to our assumption of a primitive equation.

6 July 2020

1. Calculate the electrostatic potential $U(a)$ of a surface $S = \{\,(x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2=z^2,\, 0 \leq z \leq 1\,\}$ charged with a field of homogeneous density ρ at the point $a=(0,0,1)$.

   Reason: Electrostatic Potential.

   Solution: The general formula (Coulomb) for the potential is

   $$U(a) = \int\int_S \frac{\rho}{|x-a|}\,dO$$

   With the parameterization $\Phi(t,\varphi) = (t\cos\varphi, t\sin\varphi, t)$ we get $\Phi_t = (\cos\varphi, \sin\varphi, 1)$ and $\Phi_\varphi = (-t\sin\varphi, t\cos\varphi, 0)$. The fundamental quantities are

   $$E = \Phi_t \cdot \Phi_t\,,\; F = \Phi_t \cdot \Phi_\varphi\,,\; G = \Phi_\varphi \cdot \Phi_\varphi$$

   These are in our case $E=2$, $F=0$, $G=t^2$ and the scalar surface element is $dO = \sqrt{EG-F^2}\,dt\,d\varphi = \sqrt{2}t\,dt\,d\varphi$ since $t \geq 0$. Thus

   $$\begin{aligned}
   U(a) &= \int_0^1 \int_0^{2\pi} \frac{\rho}{\sqrt{(t\cos\varphi-0)^2+(t\sin\varphi-0)^2+(t-1)^2}}\sqrt{2}t\,dt\,d\varphi \\
   &= \int_0^1 dt \int_0^{2\pi} d\varphi\; \frac{\sqrt{2}\rho\, t}{\sqrt{t^2+(t-1)^2}} = 2\,\pi\rho \int_0^1 \frac{t}{\sqrt{t^2-t+\frac{1}{2}}}\,dt \\
   &= 2\,\pi\rho \int_0^1 \frac{1}{2}\left(\frac{2t-1}{\sqrt{t^2-t+\frac{1}{2}}} + \frac{1}{2\sqrt{t^2-t+\frac{1}{2}}}\right) \\
   &= \pi\rho\left[\sqrt{t^2-t+\frac{1}{2}}\right]_0^1 + \pi\rho\int_0^1 \frac{dt}{\sqrt{t^2-t+\frac{1}{2}}} \\
   &\overset{\tau=t-1/2}{=} \pi\rho \int_{-1/2}^{1/2} \frac{d\tau}{\sqrt{\tau^2+\frac{1}{4}}} = \pi\rho\int_{-1/2}^{1/2}\frac{d(2\tau)}{\sqrt{(2\tau)^2+1}} \\
   &= \pi\rho\,[\operatorname{arsinh}(2\tau)]_{-1/2}^{1/2} = \pi\rho\left[\log\left(2\tau+\sqrt{(2\tau)^2+1}\right)\right]_{-1/2}^{1/2} \\
   &= \pi\rho\log\left(\frac{1+\sqrt{2}}{-1+\sqrt{2}}\right) = (\log(3+2\sqrt{2}))\pi\rho
   \end{aligned}$$

2. (HS-1) Prove that the product of a finite number of sums of two integers

squares is again a sum of two integers squared.

$$(a_1^2+b_1^2)\cdot(a_2^2+b_2^2)\cdot\ldots\cdot(a_n^2+b_n^2)=a^2+b^2$$

Reason: Useful Trick.

Solution: We need to show that

$$\prod_{k=1}^{n}(a_k^2+b_k^2)=\prod_{k=1}^{n}\det\left(\begin{bmatrix}a_k & -b_k\\ b_k & a_k\end{bmatrix}\right)=\det\left(\begin{bmatrix}a & -b\\ b & a\end{bmatrix}\right)=a^2+b^2$$

which is true as the matrices of this form multiplied with each another have again such a form:

$$\begin{bmatrix}\alpha & -\beta\\ \beta & \alpha\end{bmatrix}\cdot\begin{bmatrix}\gamma & -\delta\\ \delta & \gamma\end{bmatrix}=\begin{bmatrix}\alpha\gamma-\beta\delta & -\alpha\delta-\beta\gamma\\ \alpha\delta+\beta\gamma & \alpha\gamma-\beta\delta\end{bmatrix}$$

and the determinant is a multiplicative function.

3. (HS-2) Given a positive integer in decimal representation without zeros. We build a new integer by concatenation of the number of even digits, the number of odd digits, and the number of all digits (the sum of the former two). Then we proceed with that number.

   Determine whether this algorithm always comes to a halt. What is or should be the criterion to stop?

   Reason: Algorithm.

   Solution: Let $|x|$ denote the number of digits of the integer x expressed in the decimal system, i.e. for $x=\sum_{k=0}^{n-1}x_k\cdot 10^k$ we have $|x|=n$. Say we have m even digits among the $\{x_i\}$, then one step of our algorithm transforms x to $f(x)=m\cdot 10^{|n|+|n-m|}+(n-m)\cdot 10^{|n|}+n$. Let us assume $n=|x|\geq 4$. Now $|x|\leq 1+\log_{10}x$ so

$$\begin{aligned}f(x)&\leq n\cdot 10^{2|n|}+n\cdot 10^{|n|}+n\\ &\leq n\cdot 100^{1+\log_{10}n}+n\cdot 10^{1+\log_{10}n}+n\\ &\leq 100n+n^3+10n+n^2+n\\ &=n^3+n^2+111n\\ &<10^{n-1}\\ &\leq x\end{aligned}$$

   if $n\geq 4$. Hence the algorithm decreases the input number on every single step, as long as there are at least four digits. If $n=4$, then

$f(x) \leq 999$ and we have a three digit number. This means $f(x) = 100m + 10(3-m) + 3$ with $m \in \{1, 2, 3\}$. The last step is thus one of the following: $f(303) = f(213) = f(123) = 123$, and 123 the stopping criterion for the algorithm. Since this is an endless loop, the algorithm doesn't stop and needs a stopping command at 123.

It remains to show what will happen on numbers smaller than 100.

- $n = 2$: The only possibilities are

$$\begin{aligned} f(x_1x_0) &= (022) \longrightarrow (303) \longrightarrow (123) \\ \text{or } &= (\ \ 22) \longrightarrow (202) \longrightarrow (303) \longrightarrow (123) \\ f(x_1x_0) &= (202) \longrightarrow (303) \longrightarrow (123) \\ f(x_1x_0) &\longrightarrow (112) \longrightarrow (123) \end{aligned}$$

- $n = 1$: In this case we will get the previous cases, too. Either $f(x_0) = (101) \longrightarrow (123)$ or $f(x_0) = (011) \longrightarrow (123)$ or if we do not allow leading zeros $f(x_0) = (11) \longrightarrow (022)$ (see case $n = 2$).

4. (HS-3) List all real functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ with the following properties:

$$\begin{aligned} f(xy) &= f(x)f(y) - f(x) - f(y) + 2 \\ f(x+y) &= f(x) + f(y) + 2xy - 1 \\ f(1) &= 2 \end{aligned}$$

Reason: Real Function.

Solution: Let f be such that the conditions hold. Then

$$\begin{aligned} f(2) &= f(1+1) = 2f(1) + 1 = 5 \\ f(2x) &= f(2)f(x) - f(2) - f(x) + 2 = 4f(x) - 3 \\ f(2x) &= f(x) + f(x) + 2x^2 - 1 = 2f(x) + 2x^2 - 1 \\ 0 &= 2f(x) - 2x^2 - 2 \\ f(x) &= x^2 + 1 \end{aligned}$$

Conversely we have to check that this function fulfills the conditions:

$$\begin{aligned} (xy)^2 + 1 &= (x^2+1)(y^2+1) - x^2 - 1 - y^2 - 1 + 2 && \checkmark \\ (x+y)^2 + 1 &= x^2 + 1 + y^2 + 1 + 2xy - 1 && \checkmark \\ 1^2 + 1 &= 2 && \checkmark \end{aligned}$$

5. (HS-4) Find all real solutions (x, y) such that
$$\sin^4 x = y^4 + x^2y^2 - 4y^2 + 4\,,\ \cos^4 x = x^4 + x^2y^2 - 4x^2 + 1$$

   Reason: Pigeon Hole Principle.

   Solution:
$$\sin^4 x + \cos^4 x = (x^2 + y^2 - 2)^2 + 1 \geq 1$$
   The values of sine and cosine are all in $[-1, 1]$, so $\sin^4 x \leq \sin^2 x$ and $\cos^4 x \leq \cos^2 x$. Hence $\sin^4 x + \cos^4 x \leq \sin^2 x + \cos^2 x = 1$, which is only possible if equality holds everywhere:
$$x^2 + y^2 = 2\,,\ 0 = \sin^2 x(\sin^2 x - 1) = \cos^2 x(\cos^2 x - 1).$$
   The equality of sines holds if and only if $x = k \cdot \pi/2$ for some $k \in \mathbb{Z}$. Now $k^2 \cdot \pi^2/4 = x^2 = 2 - y^2 \leq 2$ or $k^2 \leq 8/\pi^2 < 1$, i.e. $k = 0$. This $(0, \pm\sqrt{2})$ are the only possible solutions. It is easy to check that both points fulfill the given conditions.

6. (HS-5) Prove
$$\frac{(2n)!}{(n!)^2} > \frac{4^n}{n+1}$$
   for all natural numbers $n > 1$.

   Reason: Inequality.

   Solution: We proceed by induction and check for $n = 2$ the inequality
$$\frac{(2 \cdot 2)!}{(2!)^2} = \frac{4!}{4} = 3! = 6 > \frac{16}{3} = \frac{4^2}{2+1}.$$
   Now let us assume that $\dfrac{(2k)!}{(k!)^2} > \dfrac{4^k}{k+1}$.
$$\begin{aligned}\frac{(2(k+1))!}{((k+1)!)^2} &= \frac{(2k+2)!}{(k!(k+1))^2}\\ &= \frac{(2k)!(2k+1)(2k+2)}{(k!)^2(k+1)^2}\\ &= \frac{(2k)!}{(k!)^2} \cdot \frac{2(2k+1)}{k+1}\end{aligned}$$
   and
$$\frac{4^{k+1}}{(k+1)+1} = \frac{4^k}{k+1} \cdot \frac{4(k+1)}{k+2}$$

From $4k^2 + 10k + 4 = 2(2k+1)(k+2) > 4k^2 + 8k + 4 = 4(k+1)^2$ we get

$$\frac{2(2k+1)}{k+1} > \frac{4(k+1)}{k+2}$$

Combining those we get

$$\begin{aligned}\frac{(2(k+1))!}{((k+1)!)^2} &= \frac{(2k)!}{(k!)^2} \cdot \frac{2(2k+1)}{k+1}\\ &> \frac{4^k}{k+1} \cdot \frac{4(k+1)}{k+2}\\ &= \frac{4^{k+1}}{k+2}\end{aligned}$$

what had to be shown.