



Mathematical Challenges

July 2019 - December 2019

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1 December 2019

1. Let (X, d) be a metric space. The open ball with center $z \in X$ of radius $r > 0$ is defined as

$$B_r(z) := \{x \in X \mid d(x, z) < r\}$$

- (a) Give an example for

$$\overline{B_r(z)} \neq K_r(z) := \{x \in X \mid d(x, z) \leq r\}$$

Does at least one of the inclusions \subseteq or \supseteq always hold?

- (b) What are the answers in the previous case, if we additionally assume that (X, d) has an inner metric?

An inner metric d_0 associated to d is defined as the infimum of all lengths of rectified curves between two points:

Let $\sigma : [0, 1] \rightarrow X$ with $\sigma(0) = x$, $\sigma(1) = y$ a rectified curve with length

$$L(\sigma) = \sup \left\{ \sum_{k=1}^n d(\sigma(t_{k-1}), \sigma(t_k)) \mid 0 = t_0 < t_1 < \dots < t_n = 1, n \in \mathbb{N} \right\}$$

Then $d_0(x, y) = \inf L(\sigma)$.

Reason: Exceptions in Metric Spaces.

Solution: The function $d_z := d(z, \cdot) : X \rightarrow \mathbb{R}$ is Lipschitz continuous with constant 1 and thus continuous: w.l.o.g. we may assume $d_z(x) \geq d_z(y)$ so

$$\begin{aligned} |d_z(x) - d_z(y)| &= d_z(x) - d_z(y) = d(z, x) - d(z, y) \\ &\leq d(z, y) + d(y, x) - d(z, y) = d(y, x) = d(x, y) \end{aligned}$$

Hence $d^{-1}([0, r]) = K_r(z)$ is closed. As $B_r(z) \subseteq K_r(z)$ we have

$$\overline{B_r(z)} \subseteq K_r(z)$$

in any case. Now we define a metric space (\mathbb{R}^n, d) by

$$d(x, y) := \begin{cases} \|x - y\| & , \|x - y\| \leq 1 \\ 1 & , \|x - y\| > 1 \end{cases}$$

As $\|\cdot\|$ is the ordinary Euclidean norm, we have

$$\overline{B_1(z)} = \{x \in \mathbb{R}^n \mid \|x-z\| \leq 1\} \subsetneq \mathbb{R}^n = \{x \in \mathbb{R}^n \mid d(x, z) \leq 1\} = K_r(z)$$

Let (X, d) now be an inner metric space and $x \in K_r(z)$.

Then $\overline{B_r(z)} = K_r(z)$ if there is a sequence $(x_n)_{n \in \mathbb{N}} \subseteq B_r(z)$ such that $\lim_{n \rightarrow \infty} x_n = x$.

If $x \in B_r(z)$ we can simply choose the constant sequence, hence we may assume $d(x, z) = r$. We choose a monotone decreasing sequence $(\varepsilon_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}_{>0}$ of positive real numbers which converges to 0. By assumption there are rectified curves σ_n with length $L(\sigma_n) = r + \varepsilon_n$ for every $n \in \mathbb{N}$ with $\sigma_n(0) = z$, $\sigma_n(1) = x$. Let's assume the curves are parameterized by their paths so we can choose $x_n := \sigma_n(r - \varepsilon_n)$. Then

$$d(z, x_n) = d(z, \sigma_n(r - \varepsilon_n)) = L(\sigma_n) = r + \varepsilon_n < r$$

which means that all $x_n \in B_r(z)$ and $x_n \xrightarrow{n \rightarrow \infty} x$ follows from

$$d(x, x_n) = d(x, z) - d(z, x_n) = r + \varepsilon_n - (r - \varepsilon_n) \leq 2\varepsilon_n \xrightarrow{n \rightarrow \infty} 0$$

2. Let $f(z) = \frac{7z - 51}{z^2 - 12z + 27}$ be a complex function.

- (a) Determine the Laurent series of $f(z)$ and their radius of convergences around $z = 3$ in the cases where 0 is in the area of convergence, and 10 is in the area of convergence.
- (b) Determine $\lim_{z \rightarrow 3} f(z)$, $\text{Res}(f, 3)$ and the kind of singularity in $z = 3$.

Reason: Laurent Series.

Solution: We write

$$\begin{aligned} f(z) &= \frac{7z - 51}{z^2 - 12z + 27} = \frac{5}{z - 3} + \frac{2}{z - 9} = \frac{5}{z - 3} + \frac{2}{(z - 3) - 6} \\ &= \frac{5}{z - 3} - \frac{2}{6} \cdot \frac{1}{1 - \left(\frac{z - 3}{6}\right)} = \frac{5}{z - 3} - \frac{2}{6} \cdot \sum_{n=0}^{\infty} \left(\frac{z - 3}{6}\right)^n \\ &= \frac{5}{z - 3} - \frac{1}{3} - \frac{1}{18}(z - 3) - \frac{1}{108}(z - 3)^2 - \frac{1}{648}(z - 3)^3 - \frac{1}{3888}(z - 3)^4 - \dots \end{aligned}$$

which converges for $\left|\frac{z - 3}{6}\right| < 1 \iff 0 < |z - 3| < 6$ and includes $z = 0$.

In the other case we write

$$\begin{aligned}
 f(z) &= \frac{7z - 51}{z^2 - 12z + 27} = \frac{5}{z - 3} + \frac{2}{(z - 3) - 6} \\
 &= \frac{5}{z - 3} + \frac{2}{z - 3} \cdot \frac{1}{1 - \left(\frac{6}{z - 3}\right)} \\
 &= \frac{5}{z - 3} + \frac{2}{z - 3} \cdot \sum_{n=0}^{\infty} \left(\frac{6}{z - 3}\right)^n \\
 &= \frac{7}{z - 3} + \frac{12}{(z - 3)^2} + \frac{72}{(z - 3)^3} + \frac{432}{(z - 3)^4} + \frac{2592}{(z - 3)^5} + \dots
 \end{aligned}$$

which converges for $\left|\frac{6}{z - 3}\right| < 1 \iff 6 < |z - 3|$ and includes $z = 10$.

To determine the singularity at $z = 3$ we can only use the first expansion due to the area of convergence. The Laurent series

$$f(z) = \frac{5}{z - 3} - \frac{1}{3} - \frac{1}{18}(z - 3) - \frac{1}{108}(z - 3)^2 - \frac{1}{648}(z - 3)^3 - \frac{1}{3888}(z - 3)^4 - \dots$$

has only one power -1 and all others are higher. Hence $f(z)$ has a first order singularity at $z = 3$. It also implies $\lim_{z \rightarrow 3} f(z) = \infty$ and $\text{Res}(f, 3) = c_{-1} = 5$.

3. Write the following groups as amalgamated products of cyclic groups:

- (a) $G = \langle x, y \mid x^3 y^{-3}, y^6 \rangle$
- (b) $H = \langle x, y \mid x^{30}, y^{70}, x^3 y^{-5} \rangle$

Reason: Amalgamations.

Solution: By definition we have groups $F(x, y)/N$ where $F(x, y)$ is the free group generated by two elements and N the normal subgroup generated by the given relations. We prove

- (a) $G \cong \mathbb{Z}/6\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$

Since $N \trianglelefteq G$ is normal, we get from $x^3 y^{-3}, y^6 \in N$

$$x^6 = (x^3 y^{-3}) \cdot y^3 x^3 = (x^3 y^{-3}) \cdot (y^3 (x^3 y^{-3}) y^{-3} \cdot y^6) \in N$$

and so

$$G = \langle x, y \mid x^3 y^{-3}, y^6 \rangle = \langle x, y \mid x^3 y^{-3}, y^6, x^6 \rangle$$

Hence it is sufficient to show that the amalgamated product

$$\mathbb{Z}/6\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$$

has this presentation, too.

We have the free product

$$\mathbb{Z}/6\mathbb{Z} * \mathbb{Z}/6\mathbb{Z} \cong \langle x \mid x^6 \rangle * \langle y \mid y^6 \rangle = \langle x, y \mid x^6, y^6 \rangle$$

With $\mathbb{Z}/2\mathbb{Z} = \langle t \mid t^2 \rangle$ we have two inclusions

$$\iota_1 : t \longmapsto x^3, \iota_2 : t \longmapsto y^3$$

of $\mathbb{Z}/2\mathbb{Z}$ into the two factors $\mathbb{Z}/6\mathbb{Z}$. By definition of the amalgamated product we thus have

$$\begin{aligned} \mathbb{Z}/6\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z} &= (\mathbb{Z}/6\mathbb{Z} * \mathbb{Z}/6\mathbb{Z}) / \langle \iota_1(u) \iota_2^{-1}(u) \mid u \in \mathbb{Z}/2\mathbb{Z} \rangle \\ &\cong \langle x, y \mid x^6, y^6 \rangle / \langle \iota_1(t) \iota_2^{-1}(t) \rangle \\ &= \langle x, y \mid x^6, y^6 \rangle / \langle x^3 y^{-3} \rangle \\ &= \langle x, y \mid x^6, y^6, x^3 y^{-3} \rangle = G \end{aligned}$$

(b) $H \cong \mathbb{Z}/6\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/10\mathbb{Z}$

As before with $\mathbb{Z}/2\mathbb{Z} = \langle t \mid t^2 \rangle$ we have two inclusions

$$\iota_1 : t \longmapsto a^3, \iota_2 : t \longmapsto b^5$$

of $\mathbb{Z}/2\mathbb{Z}$ into the two groups $\mathbb{Z}/6\mathbb{Z}$, $\mathbb{Z}/10\mathbb{Z}$. By definition of the amalgamated product we thus have

$$\begin{aligned} \mathbb{Z}/6\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/10\mathbb{Z} &= (\mathbb{Z}/6\mathbb{Z} * \mathbb{Z}/10\mathbb{Z}) / \langle \iota_1(u) \iota_2^{-1}(u) \mid u \in \mathbb{Z}/2\mathbb{Z} \rangle \\ &\cong \langle a, b \mid a^6, b^{10} \rangle / \langle \iota_1(t) \iota_2^{-1}(t) \rangle \\ &= \langle a, b \mid a^6, b^{10} \rangle / \langle a^3 b^{-5} \rangle \\ &= \langle a, b \mid a^6, b^{10}, a^3 b^{-5} \rangle =: H' \end{aligned}$$

We now have to find an isomorphism $\varphi : H \longrightarrow H'$.

The mapping $x \longmapsto a$, $y \longmapsto b$ induces a homomorphism $\tilde{\varphi} : F(x, y) \longrightarrow H'$ by the universal property of the free group. Now

$$\begin{aligned} \tilde{\varphi}(x^{30}) &= \tilde{\varphi}(x)^{30} = (a^6)^5 = 1^5 = 1 \\ \tilde{\varphi}(y^{70}) &= \tilde{\varphi}(y)^{70} = (b^{10})^7 = 1^7 = 1 \\ \tilde{\varphi}(x^3 y^{-5}) &= \tilde{\varphi}(x)^3 \tilde{\varphi}(y)^{-5} = a^3 b^{-5} = 1 \end{aligned}$$

Since all relations of H are mapped onto 1, there is a homomorphism $\varphi : H \rightarrow H'$ by the universal property of the presentation of $H = \langle x, y \mid x^{30}, y^{70}, x^3y^{-5} \rangle$ such that $\varphi \circ \pi = \tilde{\varphi}$ with the canonical projection $\pi : F(x, y) \rightarrow H$. In order to show that φ is actually an isomorphism, we construct a homomorphism $\psi : H' \rightarrow H$ which is inverse to φ .

As before, this time by mapping $a \mapsto x$, $b \mapsto y$, we get a homomorphic mapping $\psi : H' \rightarrow H$. In order that this mapping is well-defined, we must ensure that the relations in H' are mapped onto $1 \in H$, i.e. we must show $x^6 = y^{10} = 1$.

From $x^3y^{-5} = 1$ we get $H \ni 1 = x^{30} = y^{50}$, hence

$$y^{10} = y^{-200}y^{210} = (y^{50})^{-4} \cdot (y^{70})^3 = 1^{-4} \cdot 1^3 = 1$$

and

$$x^6 = (x^3)^2 = (y^5)^2 = y^{10} = 1$$

So ψ is a well-defined homomorphism. By their mappings of the generators, it is obvious that they are inverse to one another and

$$H \cong H' = \mathbb{Z}/6\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/10\mathbb{Z}$$

4. Prove that there are uncountably many groups, which are generated by two elements, and not finitely presented.

Hint: There are uncountably many non-isomorphic groups with two generators [Bernhard Neumann, 1937].

Reason: Group Presentations.

Solution: Assume there are countably many groups, which are generated by two elements, and not finitely presented. We show that the set X of groups, which are generated by two elements, and are finitely presented, is countable. If both those sets are countable, then so is their union, contradicting the given hint.

Let $G \in X$ with generators a, b . The set of possibly not reduced words of length k over the alphabet $\{a, b, a^{-1}, b^{-1}\}$ has $4^k < \infty$ elements. Thus the set of those words of length not greater than k is also finite. Hence there are only finitely many possibilities for m many relations r of length $l(r) \leq k$. So there are only finitely many possible groups

$$G(m, k, R) = \langle a, b \mid R; |R| \leq m \text{ and } \max_{r \in R} l(r) \leq k \rangle$$

and the set of those groups $X_{m,k} := \{ G(m,k,R) \mid R \subseteq F(a,b) \}$ is finite, and so is the countably infinite union

$$X = \bigcup_{m,k \in \mathbb{N}} X_{m,k}$$

5. Let $f : [1, \infty) \rightarrow [0, \infty)$ be a continuously differentiable function. Write S for the solid of revolution of the graph $y = f(x)$ about the x -axis. If the surface area of S is finite, then so is the volume.

Reason: Gabriel's Horn.

Solution: Since the surface area is finite we get

$$\begin{aligned} \limsup_{t \rightarrow \infty} \sup_{x \geq t} f(x)^2 - f(1) &= \limsup_{t \rightarrow \infty} \int_1^t (f(x)^2)' dx \\ &\leq \int_1^\infty |(f(x)^2)'| dx \\ &= 2 \int_1^\infty f(x) |f'(x)| dx \\ &\leq 2 \int_1^\infty f(x) \sqrt{1 + f'(x)^2} dx \\ &= \frac{A}{\pi} < \infty \end{aligned}$$

Hence there is a $t_0 \geq 1$ such that $\sup_{x \geq t_0} f(x) < \infty$ and so is $L := \sup_{x \geq 1} f(x) < \infty$ because $f(x)$ is continuous with values in $[0, \infty)$, i.e. bounded on $[1, \infty)$. For the volume we have

$$\begin{aligned} V &= \int_1^\infty f(x) \cdot \pi f(x) dx \\ &\leq \int_1^\infty \frac{L}{2} \cdot 2\pi f(x) dx \\ &\leq \frac{L}{2} \int_1^\infty 2\pi f(x) \sqrt{1 + f'(x)^2} dx \\ &= \frac{L}{2} \cdot A \\ &< \infty \end{aligned}$$

6. Calculate $\sum_{k,j=1}^\infty \frac{1}{kj(k+j)^2}$

Reason: Harmonic Series.

Solution: We will make use of the Taylor expansion at $x = 0$

$$\frac{1}{1-x} \log\left(\frac{1}{1-x}\right) = \sum_{n=1}^{\infty} H_n x^n \text{ where } H_n = \sum_{k=0}^n \frac{1}{k} \text{ and } |x| < 1 \quad (*)$$

Let $S := \sum_{k,j=1}^{\infty} \frac{1}{kj(k+j)^2}$ which is

$$\begin{aligned} S &= \sum_{k,j=1}^{\infty} \frac{1}{kj} \int_0^1 x^{k+j} \frac{dx}{x} \int_0^1 y^{k+j} \frac{dy}{y} = \sum_{k,j=1}^{\infty} \frac{1}{kj} \int_0^1 \int_0^1 (xy)^{k+j} \frac{dx dy}{xy} \\ &= \int_0^1 \int_0^1 \sum_{k,j=1}^{\infty} \frac{(xy)^k}{k} \frac{(xy)^j}{j} \frac{dx dy}{xy} = \int_0^1 \int_0^1 \frac{\log^2(1-xy)}{xy} dx dy \\ &\stackrel{u=xy}{=} \int_0^1 \int_0^y \frac{\log^2(1-u)}{u} \frac{du}{y} dy = \int_0^1 \int_u^1 \frac{\log^2(1-u)}{u} \frac{dy}{y} du \\ &= \int_0^1 \frac{\log^2(1-u)}{u} \cdot (-\log(u)) du \stackrel{v=1-u}{=} \int_1^0 \frac{\log^2 v}{1-v} \log(1-v) dv \\ &= \int_0^1 \left(\frac{1}{1-v}\right) \log\left(\frac{1}{1-v}\right) (\log^2 v) dv = \int_0^1 \sum_{n=0}^{\infty} H_n v^n \log^2 v dv \\ &= \sum_{n=0}^{\infty} H_n \int_0^1 v^n \log^2 v dv \\ &= \sum_{n=0}^{\infty} H_n \left[\frac{v^{n+1} ((n+1)^2 \log^2 v - 2(n+1) \log v + 2)}{(n+1)^3} \right]_0^1 \\ &= 2 \sum_{n=1}^{\infty} H_{n-1} \frac{1}{n^3} = 2 \left(\sum_{n=1}^{\infty} \frac{H_n}{n^3} - \sum_{n=1}^{\infty} \frac{1}{n^4} \right) \\ &= 2 \left(\frac{\pi^4}{72} - \zeta(4) \right) = \frac{\pi^4}{36} - \frac{\pi^4}{45} = \frac{\pi^4}{180} \end{aligned}$$

$$7. \text{ Calculate } S := \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{3^k (2n-2k)! (2k)!}{2^k 8^n [(n-k)!]^2 [k!]^2 (2n(1+2k) + (1-4k^2))}$$

Reason: Cauchy Product.

Solution: We first clean up the various parts of the quotient. The faculties are $\binom{2n-2k}{n-k} \binom{2k}{k}$, the powers are $\frac{3^k}{16^k} \cdot \frac{1}{8^{n-k}}$, and the polynomial part is $(2n+1-2k)(1+2k)$. Thus we can write the series as a Cauchy

product

$$\begin{aligned}
 S &= \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{2k}{k} \frac{3^k}{16^k(2k+1)} \cdot \binom{2n-2k}{n-k} \frac{1}{8^{n-k}(2n-2k+1)} \\
 &= \left(\sum_{n=0}^{\infty} \binom{2n}{n} \frac{3^n}{16^n(2n+1)} \right) \left(\sum_{m=0}^{\infty} \binom{2m}{m} \frac{1}{8^m(2m+1)} \right) \\
 &= \left(\sum_{n=0}^{\infty} \binom{2n}{n} \frac{\left(\frac{\sqrt{3}}{2}\right)^{2n}}{4^n(2n+1)} \right) \left(\sum_{m=0}^{\infty} \binom{2m}{m} \frac{\left(\frac{1}{\sqrt{2}}\right)^{2m}}{4^m(2m+1)} \right) \\
 &= \frac{2}{\sqrt{3}} \arcsin\left(\frac{\sqrt{3}}{2}\right) \cdot \sqrt{2} \arcsin\left(\frac{1}{\sqrt{2}}\right) \\
 &= \frac{2}{\sqrt{3}} \cdot \frac{\pi}{3} \cdot \sqrt{2} \cdot \frac{\pi}{4} = \frac{1}{3\sqrt{6}} \pi^2
 \end{aligned}$$

8. Solve $y'x - y = \sqrt{x^2 - y^2}$

Reason: Jacobian Differential Equation.

Solution: $y'x - y = \sqrt{x^2 - y^2}$ can be transformed into a Jacobian differential equation. First we divide x and substitute $z = \frac{y}{x}$ so we get

$$y' = \frac{y}{x} + \sqrt{1 - \left(\frac{y}{x}\right)^2} = z + \sqrt{1 - z^2}$$

with

$$z' = \frac{y'x - y}{x^2} = \frac{\sqrt{x^2 - y^2}}{x^2} = \frac{1}{x} \sqrt{1 - z^2} = \frac{1}{x} \left[\underbrace{z + \sqrt{1 - z^2}}_{:=g(z)=y'} - z \right]$$

Now we have

$$\int \frac{dz}{g(z) - z} = \int \frac{dz}{\sqrt{1 - z^2}} = \arcsin(z) + C = \int \frac{dx}{x} = \log|x| + C'$$

hence

$$y = x \cdot \sin(\log|x| + C)$$

We cannot rule out that $g(z_0) = z_0$ for some value z_0 . This means we have $\sqrt{1 - z_0^2} = 0$ or $z_0 = \pm 1$, i.e. $y = \pm x$ are also solutions.

9. Let $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}_{\geq 0}$ be two sequences of nonnegative numbers, where not all sequence elements vanish, and be $p, q \in \mathbb{R}$ with $1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1$. Prove

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n b_m}{n+m} < \frac{\pi}{\sin(\pi/p)} \cdot \left(\sum_{n=1}^{\infty} a_n^p \right)^{\frac{1}{p}} \cdot \left(\sum_{m=1}^{\infty} b_m^q \right)^{\frac{1}{q}}$$

Reason: Hilbert's Inequality.

Solution: $f(x) = \frac{1}{(1+x)x^\alpha}$ is for $0 < \alpha < 1$ strictly monotone decreasing, hence

$$\sum_{m=1}^{\infty} \frac{1}{(1 + \frac{m}{n}) \cdot (\frac{m}{n})^\alpha} \cdot \frac{1}{n} \stackrel{\text{Riemann sum}}{<} \int_0^\infty \frac{dx}{(1+x)x^\alpha} \stackrel{(*)}{=} \frac{\pi}{\sin \pi \alpha}$$

Proof of (*):

$$\begin{aligned} \int_0^\infty \frac{dx}{(1+x)x^\alpha} &= \int_0^\infty x^{-\alpha} \int_0^\infty e^{-(1+x)t} dt dx \\ &= \int_0^\infty \int_0^\infty x^{-\alpha} e^{-t} e^{-xt} dt dx \\ &\stackrel{u=tx}{=} \int_0^\infty \int_0^\infty \left(\frac{u}{t}\right)^{-\alpha} e^{-u} e^{-t} t^{-1} du dx \\ &= \int_0^\infty u^{-\alpha} e^{-u} du \int_0^\infty t^{\alpha-1} e^{-t} dt \\ &= \Gamma(1-\alpha)\Gamma(\alpha) \stackrel{\alpha \notin \mathbb{Z}}{=} \frac{\pi}{\sin \pi \alpha} \end{aligned}$$

Now we have

$$\begin{aligned}
 \sum_{n,m=1}^{\infty} \frac{a_n b_m}{n+m} &= \sum_{n,m=1}^{\infty} \frac{a_n}{(n+m)^{1/p} \left(\frac{m}{n}\right)^{1/(pq)}} \cdot \frac{b_m}{(n+m)^{1/q} \left(\frac{n}{m}\right)^{1/(pq)}} \\
 &\stackrel{\text{Hölder}}{\leq} \left(\sum_{n,m=1}^{\infty} \frac{a_n^p}{(n+m) \left(\frac{m}{n}\right)^{\frac{1}{q}}} \right)^{\frac{1}{p}} \cdot \left(\sum_{n,m=1}^{\infty} \frac{b_m^q}{(n+m) \left(\frac{n}{m}\right)^{\frac{1}{p}}} \right)^{\frac{1}{q}} \\
 &= \left(\sum_{n=1}^{\infty} a_n^p \cdot \sum_{m=1}^{\infty} \frac{1}{\left(1 + \frac{m}{n}\right) \left(\frac{m}{n}\right)^{\frac{1}{q}}} \cdot \frac{1}{n} \right)^{\frac{1}{p}} \\
 &\quad \cdot \left(\sum_{m=1}^{\infty} b_m^q \cdot \sum_{n=1}^{\infty} \frac{1}{\left(\frac{n}{m} + 1\right) \left(\frac{n}{m}\right)^{\frac{1}{p}}} \cdot \frac{1}{m} \right)^{\frac{1}{q}} \\
 &< \left(\sum_{n=1}^{\infty} a_n^p \right)^{\frac{1}{p}} \cdot \left(\sum_{m=1}^{\infty} b_m^q \right)^{\frac{1}{q}} \cdot \left(\frac{\pi}{\sin\left(\frac{\pi}{q}\right)} \right)^{\frac{1}{p}} \cdot \left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \right)^{\frac{1}{q}} \\
 &\stackrel{(**)}{=} \left(\sum_{n=1}^{\infty} a_n^p \right)^{\frac{1}{p}} \cdot \left(\sum_{m=1}^{\infty} b_m^q \right)^{\frac{1}{q}} \cdot \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}
 \end{aligned}$$

Proof of (**):

$$\sin \frac{\pi}{q} = \sin \left(\pi \left(1 - \frac{1}{p} \right) \right) = \sin \pi \cos \left(\frac{\pi}{p} \right) - \cos \pi \sin \left(\frac{\pi}{p} \right) = \sin \frac{\pi}{p}$$

10. Let $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ be an integrable function and $p > 1$. Prove

$$\int_0^{\infty} \left(\frac{1}{x} \int_0^x f(t) dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^{\infty} (f(x))^p dx$$

Hint: Substitute $t = xu^{p/r}$ and at the end $r = p - 1$.

Reason: Hardy's Inequality for Integrals.

Solution: Let $F(x) := \int_0^x f(t) dt$ which becomes by the substitution $t = xu^{p/r}$, $dt = x^{\frac{p}{r}} u^{-1+p/r} du$, $u = t^{r/p} x^{-r/p}$, $du = \frac{r}{p} t^{-1+r/p} x^{-r/p} dt$

$$F(x)^p = \left(\int_0^x f(t) dt \right)^p = x^p \left(\frac{p}{r} \right)^p \left(\int_0^1 f \left(xu^{\frac{p}{r}} \right) u^{\frac{p}{r}-1} du \right)^p$$

Since $u \mapsto u^p$ is convex, we can apply Jensen's theorem for convex functions (see October 2019 / 4b) and get

$$\begin{aligned} \left(\int_0^1 f\left(xu^{\frac{p}{r}}\right) u^{\frac{p}{r}-1} du \right)^p &\leq \int_0^1 \left[f\left(xu^{\frac{p}{r}}\right) \right]^p \cdot u^{p(\frac{p}{r}-1)} du \\ &= \int_0^x f(t)^p t^{p-r} x^{r-p} \left(\frac{r}{p}\right) t^{\frac{r}{p}-1} x^{-\frac{r}{p}} dt \\ &= \frac{x^{r-\frac{r}{p}}}{x^p} \cdot \frac{r}{p} \cdot \int_0^x f(t)^p t^{p-r+\frac{r}{p}-1} dt \end{aligned}$$

Hence

$$\begin{aligned} F(x)^p &\leq x^{r-\frac{r}{p}} \left(\frac{p}{r}\right)^{p-1} \int_0^x f(t)^p t^{p-r+\frac{r}{p}-1} dt \\ \int_0^\infty F(x)^p x^{-r-1} dx &\leq \int_0^\infty x^{r-\frac{r}{p}} \left(\frac{p}{r}\right)^{p-1} \int_0^x f(t)^p t^{p-r+\frac{r}{p}-1} dt x^{-r-1} dx \\ &= \left(\frac{p}{r}\right)^{p-1} \int_0^\infty \int_0^x f(t)^p t^{p-r+\frac{r}{p}-1} x^{-\frac{r}{p}-1} dx dt \\ &= \left(\frac{p}{r}\right)^{p-1} \int_0^\infty f(t)^p t^{p-r+\frac{r}{p}-1} \int_t^\infty x^{-\frac{r}{p}-1} dx dt \\ &= \left(\frac{p}{r}\right)^p \int_0^\infty f(t)^p t^{p-r-1} dt \end{aligned}$$

With $r = p - 1$ we get

$$\int_0^\infty F(x)^p x^{-p} dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f(t)^p dt$$

which had to be shown.

11. (HS-1) Choose any odd prime, square it and subtract one. Show that the result is always divisible by twenty-four except for three. What can be said, if we take the prime up to the power four, and subtract one?

Reason: Divisibility.

Solution: Let p be the chosen prime. Then the number we get is $n(p) = p^2 - 1 = (p-1)(p+1)$. $n(3) = 8$ is obviously not divisible by 24, but e.g. $n(5) = 24$, $n(7) = 48$ are. Now let us assume $p > 3$. Since p is odd, $p \pm 1$ are both even, hence $4 = 2 \cdot 2 \mid n(p)$. But from two consecutive even numbers, one has to be divisible by 4, which means $8 = 2 \cdot 4 \mid n(p)$. Finally we have three consecutive numbers $p-1, p, p+1$ and one of them has to be divisible by three. As it cannot be p by assumption,

$3 \mid n(p)$. Because 3 and 8 are coprime, we even get $3 \cdot 8 = 24 \mid n(p)$.

In this case we have $m(p) = p^4 - 1 = (p^2 - 1)(p^2 + 1) = (p - 1)(p + 1)(p^2 + 1)$. For small primes we get $m(3) = 80$, $m(5) = 624$, $m(7) = 2401$, $m(11) = 14,460$. Let us assume $p > 5$. As before we get $24 \mid m(p)$. Since $p^2 + 1$ is always even for odd values of p , we have another factor 2 and $48 \mid m(p)$. Now primes greater than 5 can only have one of the digits $\{1, 3, 7, 9\}$ as their last one.

If $p \equiv 1 \pmod{10}$ then $5 \mid (p - 1)$.

If $p \equiv 3, 7 \pmod{10}$ then $5 \mid (p^2 + 1)$ since $3^2 + 1 = 10$, $7^2 + 1 = 50$ and for $p = a \cdot 10 + r > 11$ we get for

$$p^2 + 1 = (a \cdot 10 + r)^2 = 100a^2 + 20ar + r^2 \equiv r^2 \pmod{10}$$

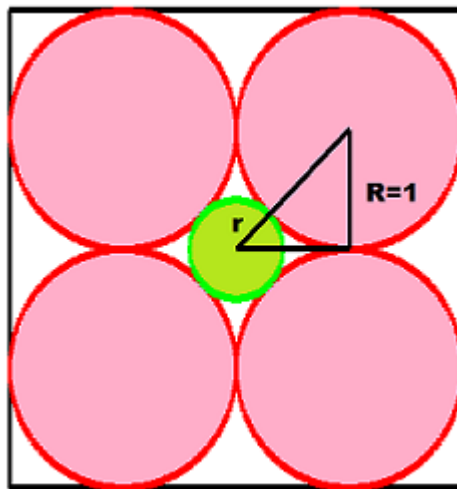
and again a zero at the end for $r = 3, 7$.

If $p \equiv 9 \pmod{10}$ then $5 \mid (p + 1)$.

Thus we have in total $5 \cdot 48 = 240 \mid m(p)$ again because 5 and 48 are coprime.

12. (HS-2) In a square of side length 4, there is a circle of radius 1 in each corner. In the center of the square is another circle that touches the other four. Analogously, in the three-dimensional case, in the center of a cube of edge length 4, there would be a sphere which would touch eight spheres of radius 1 placed in the corners of the cube. In which dimension does the central hypersphere become so large that it touches all sides of the hypercube?

Reason: Abstract Geometry.



Solution: Pythagoras gives us $(r + R)^2 = R^2 + R^2$ and for $R = 1$ we have $r = \sqrt{2} - 1$. Pythagoras applied once more for the next dimension results in

$$(r + R)^2 = (R^2 + R^2) + R^2 = 3R^2 \implies r = \sqrt{3} - 1$$

which continues with every new dimension. So the radius of the sphere inside equals $r = \sqrt{n} - 1$ in dimension n . In dimension 4 the inner sphere is as big as the outer ones. In order to touch the boundary of the hypercube, we need $r \geq 2$, i.e. $n = 9$.

The nine-dimensional central hypersphere touches all 18 bounding sides, eight-dimensional hypercubes, of the nine-dimensional hypercube. In the ten-dimensional space, parts of the central hypersphere are even outside the ten-dimensional hypercube. The higher the dimensions get, the more this effect intensifies.

13. (HS-3) There is only one rule at Christmas at the world's richest family: The gifts have to be expensive, heavy and glamorous. So they all present statues of pure gold. It may be large figure, a tiger sculpture or an opulent candlestick. The eldest son who doesn't live at home anymore receives gifts of nine tons total, but none of which is heavier than a ton. He wants to bring home all of them, but only could rent trucks which can load three tons maximal. How many trucks are needed to at least be able to transport all gifts of gold at the same time?

Reason: Gold Transport.

Solution: If the gifts were ten statues of 900 kg each, then three trucks wouldn't be sufficient. Now we load the first truck until it carries at least two tons, which is possible, since all gifts weigh less than a ton. We do the same for truck number two and three. Hence we are left with less than three tons and a fourth truck will be sufficient.

14. (HS-4) Prove $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$ for $a, b, c > 0$

Reason: Nesbitt's Inequality.

Solution: As it is so important we recall the order of various mean

values: Let $x_1, \dots, x_n \in \mathbb{R}$. Then

$\bar{x}_{min} = \min\{x_1, \dots, x_n\}$	minimum
$\bar{x}_{harm} = \frac{n}{\frac{1}{x_1} + \dots + \frac{1}{x_n}}$	harmonic mean
$\bar{x}_{geom} = \sqrt[n]{x_1 \cdots x_n}, x_k > 0$	geometric mean
$\bar{x}_{arithm} = \frac{x_1 + \dots + x_n}{n}$	arithmetic mean
$\bar{x}_{quadr} = \sqrt{\frac{1}{n}(x_1^2 + \dots + x_n^2)}$	quadratic
$\bar{x}_{cubic} = \sqrt[3]{\frac{1}{n}(x_1^3 + \dots + x_n^3)}$	cubic
$\bar{x}_{max} = \max\{x_1, \dots, x_n\}$	maximum

$$\bar{x}_{min} \leq \bar{x}_{harm} \leq \bar{x}_{geom} \leq \bar{x}_{arithm} \leq \bar{x}_{quadr} \leq \bar{x}_{cubic} \leq \bar{x}_{max}$$

As a mnemonic we can think of $x_1 = 3, x_2 = 5$ where we have

$$3 < \frac{15}{4} = 3.75 < \sqrt{15} \approx 3.87 < 4 < \sqrt{17} \approx 4.12 < \sqrt[3]{76} \approx 4.24 < 5$$

This means in our situation

$$\begin{aligned} \frac{(a+b) + (b+c) + (c+a)}{3} &\geq \frac{3}{\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}} \\ 2(a+b+c) \cdot \left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \right) &\geq 9 \\ 1 + \frac{c}{a+b} + 1 + \frac{a}{b+c} + 1 + \frac{b}{c+a} &\geq \frac{9}{2} \\ \frac{c}{a+b} + \frac{a}{b+c} + \frac{b}{c+a} &\geq \frac{3}{2} \end{aligned}$$

15. (HS-5) Let $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n)$ be tuples of positive numbers. Prove

$$\prod_{k=1}^n (x_k + y_k)^{1/n} \geq \prod_{k=1}^n x_k^{1/n} + \prod_{k=1}^n y_k^{1/n}$$

Reason: Mahler's Inequality.

Solution: By the arithmetic-geometric mean inequality we have

$$\prod_{k=1}^n \left(\frac{x_k}{x_k + y_k} \right)^{1/n} \leq \frac{1}{n} \sum_{k=1}^n \frac{x_k}{x_k + y_k}, \quad \prod_{k=1}^n \left(\frac{y_k}{x_k + y_k} \right)^{1/n} \leq \frac{1}{n} \sum_{k=1}^n \frac{y_k}{x_k + y_k}$$

Hence

$$\prod_{k=1}^n \left(\frac{x_k}{x_k + y_k} \right)^{1/n} + \prod_{k=1}^n \left(\frac{y_k}{x_k + y_k} \right)^{1/n} \leq \frac{1}{n} \sum_{k=1}^n \frac{x_k + y_k}{x_k + y_k} = 1$$

and multiplying with the denominator

$$\prod_{k=1}^n x_k^{1/n} + \prod_{k=1}^n y_k^{1/n} \leq \prod_{k=1}^n (x_k + y_k)^{1/n}$$

2 November 2019

1. If f has the real Fourier representation

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

prove

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2)$$

Reason: Parseval Equation.

Solution: We can write f as complex Fourier series

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \text{ with } 2c_k = \begin{cases} a_k - ib_k & , k > 0 \\ a_0 & , k = 0 \\ a_k + ib_k & , k < 0 \end{cases}$$

Then $|f(x)|^2 = f(x) \overline{f(x)} = \sum_{k,l=-\infty}^{\infty} c_k \overline{c_l} e^{i(k-l)x}$ and

$$\begin{aligned} \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx &= \frac{1}{\pi} \sum_{k,l=-\infty}^{\infty} c_k \overline{c_l} \underbrace{\int_{-\pi}^{\pi} e^{i(k-l)x} dx}_{=2\pi\delta_{k,l}} \\ &= 2 \sum_{k=-\infty}^{\infty} |c_k|^2 = \frac{1}{2} \sum_{k=-\infty}^{\infty} |2c_k|^2 \\ &= \frac{1}{2} \left(\sum_{k=-\infty}^{-1} |a_k + ib_k|^2 + |a_0|^2 + \sum_{k=1}^{\infty} |a_k - ib_k|^2 \right) \\ &= \frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \end{aligned}$$

2. We define the weighted Hölder-mean as

$$M_w^p := \left(\sum_{k=1}^n w_k x_k^p \right)^{\frac{1}{p}}, \quad M_w^0 := \lim_{p \rightarrow 0} M_w^p = \prod_{k=1}^n x_k^{w_k}$$

for positive, real numbers $x_1, \dots, x_n > 0$ and a weight $w = (w_1, \dots, w_n)$ with $w_1 + \dots + w_n = 1$, $w_k > 0$ and a $p \in \mathbb{R} - \{0\}$.

Prove $M_w^r \leq M_w^s$ whenever $r < s$.

Hint: Use Jensen's theorem for convex functions (see October 2019 / 4a).

Reason: Inequality of Weighted Hölder Means.

Solution: By the rule of L'Hôpital we get

$$\begin{aligned} \log \prod_{k=1}^n x_k^{w_k} &= \sum_{k=1}^n w_k \log x_k \\ &= \lim_{p \rightarrow 0} \frac{\sum_{k=1}^n w_k x_k^p \log x_k}{\sum_{k=1}^n w_k x_k^p} \\ &= \lim_{p \rightarrow 0} \frac{(\log \sum_{k=1}^n w_k x_k^p)'}{(p)'} \\ &= \lim_{p \rightarrow 0} \frac{\log \sum_{k=1}^n w_k x_k^p}{p} \\ &= \lim_{p \rightarrow 0} \log (M_w^p) \end{aligned}$$

We now apply Jensen's inequality (see see October 2019 / 4a) for functions $x \mapsto x^q$ which are convex for $q \geq 1$, $x > 0$.

(a) $0 < r < s$.

In this case $q = \frac{s}{r} > 1$ and

$$\left(\sum_{k=1}^n w_k x_k^r \right)^{\frac{s}{r}} \leq \sum_{k=1}^n w_k (x_k^r)^{\frac{s}{r}} \implies \left(\sum_{k=1}^n w_k x_k^r \right)^{\frac{1}{r}} \leq \left(\sum_{k=1}^n w_k x_k^s \right)^{\frac{1}{s}}$$

(b) $r < s < 0$.

In this case $0 < -s < -r$ and by the previous case we have

$$\left(\sum_{k=1}^n w_k x_k^{-s} \right)^{\frac{1}{-s}} \leq \left(\sum_{k=1}^n w_k x_k^{-r} \right)^{\frac{1}{-r}} \implies \left(\sum_{k=1}^n w_k x_k^{-r} \right)^{\frac{1}{r}} \leq \left(\sum_{k=1}^n w_k x_k^{-s} \right)^{\frac{1}{s}}$$

As we have proven it for any $x_k > 0$ we have proven it for $\frac{1}{x_k}$ as well, which is what had to be shown.

(c) $r = 0$ or $s = 0$.

Since $\lim_{r \rightarrow 0} M_w^r = M_w^0 = \lim_{s \rightarrow 0} M_w^s$ the inequality $M_w^r \leq M_w^s$ holds true for $0 \leq r < s$ and $r < s \leq 0$, too.

(d) $r < 0 < s$.

This case follows from the transitive ordering $M_w^r \leq M_w^0 \leq M_w^s$.

3. (HS-1) Mr. Smith on a full up flight with 50 passengers on a CRJ100 had lost his boarding pass. The flight attendant tells him to sit anywhere. All other passengers sit on their booked seats, unless it is already occupied, in which case they randomly choose another seat just like Mr. Smith did. What are the chances that the last passenger gets the seat printed on his boarding pass?

Reason: Combinatorics.

Solution: Let's say passengers in the boarding queue and seats are numbered $1, \dots, 50$ and Mr. Smith is passenger 1. He could choose seat number one and seat number fifty with the same probability. Either case determines whether passenger 50 gets his correct seat or not. Now if he chooses, say seat number 25, passengers $1, \dots, 24$ can seat correctly and passenger 25 is now in the same situation Mr. Smith had been at the beginning, i.e. seat with the same probability on seat 1 or seat 50 which again determines the last passenger's fate by the same probability. If passenger 25 chooses another seat, then our situation loops.

These considerations show that ultimately only occupancies of places 1 and 50 are important. Once a passenger has chosen one of these two places at random, the outcome of the story is decided. If it's number one, passenger 50 will sit right. If it is number 50, it will not work anymore. How often passengers sit on seats during boarding that are not theirs, does not matter - as long as neither number 1 nor number 50 is affected. So the probability is exactly 0.5.

4. (HS-2) On the first flight day of a little island hopper there was no wind during the return flight. How does the total flight duration from outward and return flight change if, instead, a strong headwind blows on the way to the neighboring island - and on the way back, an equally strong tailwind?

Reason: Wind and Flight Duration.

Solution: Let us assume the flight path is of length 1 one way, at speed v and wind w . Then we need a time of $F_0 = \frac{2}{v}$ without wind.

With wind, we need a time $F_1 = \frac{1}{v-w} + \frac{1}{v+w} = \frac{2v}{v^2 - w^2}$. Hence

$$v^2 - w^2 < v^2 \iff F_0 = \frac{2}{v} < \frac{2v}{v^2 - w^2} = F_1$$

and the complete flight is longer with wind.

3 October 2019

1. Let $A = \sum_{k=0}^{\infty} a_k$, $B = \sum_{k=0}^{\infty} b_k$ be two convergent series one of which absolutely. The Cauchy-product $C = \sum_{k=0}^{\infty} c_k$ with $c_k = \sum_{j=0}^k a_j b_{k-j}$ converges then to AB . Give an example that absolute convergence of one factor is necessary.

Reason: Mertens' Theorem.

Solution: W.l.o.g. we assume that A converges absolutely. We note the partial sums $A_n = \sum_{k=0}^n a_k$, $B_n = \sum_{k=0}^n b_k$.

$$\begin{aligned} AB &= (A - A_n)B + \sum_{k=0}^n a_k B \\ S_n &= \sum_{k=0}^n c_k = \sum_{k=0}^n \sum_{j=0}^k a_j b_{k-j} = \sum_{k=0}^n a_k B_{n-k} \\ AB - S_n &= (A - A_n)B + \sum_{k=0}^n a_k (B - B_{n-k}) \end{aligned}$$

The first term converges to 0 and with $N := \lfloor \frac{n}{2} \rfloor$ we can write the second term

$$\sum_{k=0}^N (B - B_{n-k}) = \underbrace{\sum_{k=0}^N a_k (B - B_{n-k})}_{=P_n} + \underbrace{\sum_{k=N+1}^n a_k (B - B_{n-k})}_{=Q_n}$$

For P_n we have

$$|P_n| \leq \sum_{k=0}^N |a_k| \cdot |B - B_{n-k}| \leq \max_{N \leq k \leq n} |B - B_k| \cdot \sum_{k=0}^N |a_k| \longrightarrow 0$$

because A converges absolutely and $(B - B_k)_k$ is a bounded sequence converging to 0, i.e. there is a constant c such that $|B - B_k| < c$ for all $k \in \mathbb{N}_0$. Therefore we get

$$|Q_n| \leq \sum_{k=N+1}^n |a_k| \cdot |B - B_{n-k}| \leq c \sum_{k=N+1}^n |a_k| \longrightarrow 0$$

by the Cauchy criterion. Hence $AB - S_n \longrightarrow 0$ or $S_n \longrightarrow AB$.

An example for the necessity of absolute convergence for at least one

factor is $A = B = \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{k+1}}$ where $A = B = -(\sqrt{2} - 1)\zeta\left(\frac{1}{2}\right) \approx 0.605$. For a proof that the factors actually converge and the Cauchy product diverges, see the problems from September.

2. Prove that a T_0 topological group (Kolmogorov space) is already T_2 (Hausdorff space). Show that an infinite linear algebraic group with the Zariski topology is always T_0 but never T_2 . Why the discrepancy?

Reason: Topological Groups.

Solution: Let G be a T_0 topological group. We first show that the singleton $\{e\}$ is a closed subset, where $e \in G$ is the neutral element.

Given any element $x \in G$, there is either an open neighborhood U_x of x with $e \notin U_x$ or an open neighborhood V of e with $x \notin V$. In the latter case, we may assume that $V = V^{-1}$. If not then we replace V with $V \cap V^{-1}$. The homeomorphism $f : G \rightarrow G, y \mapsto xy$ maps $e \mapsto x$. Let $U_x := f(V)$, which is an open neighborhood of $x = f(e)$ with $e \notin U_x$ since $e \in U_x$ would imply $e = f(y) = xy$ and $y = x^{-1} \in V$ contradicting $x \notin V = V^{-1}$. In any case we find for all $x \neq e$ and open neighborhood that doesn't contain e . We now take the union

$$U := \bigcup_{x \neq e} U_x$$

over all these neighborhoods. By construction, this is an open set with $U = G - \{e\}$, so $\{e\}$ is indeed closed.

The map $G \times G \rightarrow G, (g, h) \mapsto gh^{-1}$ is continuous as G is a topological group. Its preimage of the closed subset $\{e\}$ is the diagonal $\Delta G = \{(g, g) \mid g \in G\}$ which is therefore closed. Any topological space X is T_2 if and only if the diagonal ΔX is a closed subset of $X \times X$.

The statements about linear algebraic groups follow from general properties of the Zariski topology: points of affine varieties are closed because they correspond to maximal ideals; hence varieties are T_0 (and even T_1 but note that schemes have more points and are only T_0 in general). Since any two non-empty open subsets of an irreducible variety meet, varieties are never T_2 . Infinity is crucial here.

Explanation of the discrepancy: The above shows that while linear algebraic groups are groups with a topology, they are not topological groups! The reason is that the topology on $G \times G$ for a variety is not the product topology!

3. Let $\mathcal{O}(n)$ be the group of orthogonal real $n \times n$ matrices. For $f \in L^p = L^p(\mathbb{R}^n)$ we set

$$A.f(x) = f(A^{-1}x)$$

Show that $\mathcal{O}_f = \{ A.f \mid A \in \mathcal{O}(n) \} \subseteq L^p(\mathbb{R}^n)$ is compact.

Reason: Orthogonal Groups and Hilbert Spaces.

Solution: We use the theorem of Fréchet-Riesz-Kolmogorov. The set \mathcal{O}_f is closed and bounded:

$$\|A.f\|_p^p = \int_{\mathbb{R}^n} |A.f(x)|^p dx = \int_{\mathbb{R}^n} |f(A^{-1}x)|^p dx = \int_{\mathbb{R}^n} |f(x)|^p dx = \|f\|_p^p$$

due to the transformation theorem of integration ($|\det A| = 1$).

Let $(A_n.f)$ be an L^p convergent sequence with limit $h \in L^p$. Since $\mathcal{O}(n)$ is compact, the sequence of A_n has a convergent subsequence with limit $A \in \mathcal{O}(n)$.

$$\|A_n.f - A.f\|_p^p = \int_{\mathbb{R}^n} |f(A_n^{-1}Ax) - f(x)|^p dx$$

and this converges to 0 for $n \rightarrow \infty$. Hence $h = A.f$ and the sequence $(A_n.f)$ converges to $A.f$, i.e. $\mathcal{O}_f \subseteq L^p$ is closed.

To obtain compactness by Fréchet-Riesz-Kolmogorov, we have to check two conditions.

(a) to be shown:

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^n} |A.f(x+t) - A.f(x)|^p dx \xrightarrow{\text{uniformly in } A} 0$$

For $\varepsilon > 0$ there is a $\delta > 0$ with

$$\|t\| < \delta \implies \int_{\mathbb{R}^n} |f(x+t) - f(x)|^p dx < \varepsilon$$

and

$$\int_{\mathbb{R}^n} |A.f(x+t) - A.f(x)|^p dx = \int_{\mathbb{R}^n} |f(x+A^{-1}t) - f(x)|^p dx$$

and $\|A^{-1}t\| = \|t\|$. Hence we obtain $\int_{\mathbb{R}^n} |A.f(x+t) - A.f(x)|^p dx < \varepsilon$ for $\|t\| < \delta$ so the first condition is fulfilled.

(b) to be shown: For every $\varepsilon > 0$ there is an $M > 0$ such that

$$\int_{\mathbb{R}^n - B_M(0)} |A.f(x)|^p dx < \varepsilon \quad \forall A \in \mathcal{O}(n)$$

where $B_M(0)$ is the closed ball of radius M and center 0 .

Since $f \in L^p$, we have for every $\varepsilon > 0$ an $M > 0$ with

$$\int_{\mathbb{R}^n - B_M(0)} |f(x)|^p dx < \varepsilon$$

But the ball $B_M(0)$ is invariant under $A \in \mathcal{O}(n)$, so we get $A(B_M(0)) = B_M(0)$ and with the transformation theorem of integration

$$\int_{\mathbb{R}^n - B_M(0)} |A.f(x)|^p dx = \int_{\mathbb{R}^n - B_M(0)} |f(x)|^p dx < \varepsilon$$

and the second condition of the theorem is fulfilled, i.e. $\mathcal{O}_f \subseteq L^p$ is compact.

4. (a) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a convex function and $\lambda_1, \dots, \lambda_n$ positive weights, i.e. $\sum_{i=1}^n \lambda_i = 1$. Show that

$$f\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i f(x_i)$$

- (b) Let $g : [0, 1] \rightarrow \mathbb{R}$ be an integrable function such that the continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ is convex on the image of g . Prove

$$f\left(\frac{1}{b-a} \int_a^b g(x) dx\right) \leq \frac{1}{b-a} \int_a^b f(g(x)) dx$$

- (c) Prove without differentiation that the cylinder with the least surface area among the ones with given volume V is the cylinder whose height equals the diameter of its base.
- (d) Prove that for any sequence $a_n \geq \dots \geq a_1 > 0$ of positive real numbers

$$\frac{1}{\frac{1}{a_1} + \frac{2}{\frac{1}{a_1} + \frac{1}{a_2}} + \dots + \frac{n}{\frac{1}{a_1} + \dots + \frac{1}{a_n}}} < 2(a_1 + \dots + a_n)$$

Reason: Jensen's Inequality.

Solution:

(a) The definition of convexity is

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

which is our induction base. The step then is

$$\begin{aligned} f\left(\sum_{i=1}^n \lambda_i x_i\right) &= f\left(\sum_{i=1}^{n-1} \lambda_i x_i + \lambda_n x_n\right) \\ &= f\left((1 - \lambda_n) \underbrace{\sum_{i=1}^{n-1} \frac{\lambda_i}{1 - \lambda_n} x_i}_{=:y} + \lambda_n x_n\right) \\ &\leq (1 - \lambda_n)f(y) + \lambda_n f(x_n) \\ &= (1 - \lambda_n)f\left(\sum_{i=1}^{n-1} \frac{\lambda_i}{1 - \lambda_n} x_i\right) + \lambda_n f(x_n) \\ &\leq \sum_{i=1}^{n-1} \lambda_i f(x_i) + \lambda_n f(x_n) \\ &= \sum_{i=1}^n \lambda_i f(x_i) \end{aligned}$$

(b) By the previous part we have for an integrable function $\varphi : [0, 1] \rightarrow \mathbb{R}$ such that f is convex on its image, $\lambda_k = \frac{1}{n}$ and $x_k = \varphi\left(\frac{k}{n}\right)$

$$f\left(\sum_{k=1}^n \varphi\left(\frac{k}{n}\right) \cdot \frac{1}{n}\right) \leq \sum_{k=1}^n f\left(\varphi\left(\frac{k}{n}\right)\right) \cdot \frac{1}{n}$$

which becomes by the limit $n \rightarrow \infty$

$$f\left(\int_0^1 \varphi(u) du\right) \leq \int_0^1 f(\varphi(u)) du$$

Now we substitute $u = \frac{x - a}{b - a}$, $du = \frac{dx}{b - a}$, hence

$$f\left(\int_a^b \varphi\left(\frac{x - a}{b - a}\right) \frac{dx}{b - a}\right) \leq \int_0^1 f\left(\varphi\left(\frac{x - a}{b - a}\right)\right) \frac{dx}{b - a}$$

where we set $g(x) = \varphi\left(\frac{x-a}{b-a}\right)$ and get

$$f\left(\frac{1}{b-a} \int_a^b g(x) dx\right) \leq \frac{1}{b-a} \int_a^b f(g(x)) dx$$

(c) Let r, h, A, V be radius, height, surface and volume of the cylinder, resp. Then

$$\frac{A}{3\pi} = \frac{2r^2 + rh + rh}{3} \stackrel{AM \geq GM}{\geq} \sqrt[3]{2r^2 \cdot rh \cdot rh} = \sqrt[3]{\frac{2V^2}{\pi^2}} =: \text{const.} > 0$$

and equality holds for $h = 2r$.

(d) From $a_n \geq \dots \geq a_1 > 0$ we get

$$\frac{1}{a_1} + \dots + \frac{1}{a_n} \geq \frac{n}{a_n} \implies \frac{n}{\frac{1}{a_1} + \dots + \frac{1}{a_n}} \leq a_n < 2a_n$$

The inequality of our statement is clearly true for $n = 1$. By induction we have

$$\begin{aligned} \sum_{k=1}^n \frac{k}{\frac{1}{a_1} + \dots + \frac{1}{a_k}} &= \frac{n}{\frac{1}{a_1} + \dots + \frac{1}{a_n}} + \sum_{k=1}^{n-1} \frac{k}{\frac{1}{a_1} + \dots + \frac{1}{a_k}} \\ &< \frac{n}{\frac{1}{a_1} + \dots + \frac{1}{a_n}} + 2(a_1 + \dots + a_{n-1}) \\ &< 2a_n + 2(a_1 + \dots + a_{n-1}) \\ &= 2(a_1 + \dots + a_n) \end{aligned}$$

5. Let $p(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ be a nonlinear polynomial with $a_n = 1$ and suppose $(x-1)^{k+1} \mid p(x)$ for some positive integer k . Prove that

$$\sum_{j=0}^{n-1} |a_j| > 1 + \frac{2k^2}{n}$$

Hint: At some stage of the proof you will need Chebyshev polynomials.

Reason: Tricky polynomial inequality.

Solution: We first prove the following statement:

For any polynomial $q(y)$ with degree at most k , we have

$$\sum_{j=0}^n a_j q(j) = 0 \quad (*)$$

We define for $0 \leq \nu \leq k$ the polynomials

$$\varphi_0(x) = 1, \varphi_\nu(x) = x(x-1)(x-2) \cdots (x-\nu+1)$$

and prove

$$\sum_{j=0}^n a_j \varphi_\nu(j) = p^{(\nu)}(1)$$

by induction on ν . For $\nu = 0$ we have $a_0 + \dots + a_n = p(1)$ and for $\nu = 1$ it's

$$\begin{aligned} a_0 \varphi_1(0) + a_1 \varphi_1(1) + a_2 \varphi_1(2) + \dots + a_{n-1} \varphi_1(n-1) + a_n \varphi_1(n) \\ = a_0 \cdot 0 + a_1 \cdot 1 + a_2 \cdot 2 + \dots + a_{n-1} \cdot (n-1) + 1 \cdot n \\ = p'(1) \end{aligned}$$

$$p^{(\nu)}(x) = \sum_{j=0}^n a_j (x^j)^{(\nu)} = \sum_{j=\nu}^n a_j (x^j)^{(\nu)} = \sum_{j=\nu}^n a_j \varphi_\nu(j) x^{j-\nu}$$

The induction step is now

$$\begin{aligned} p^{(\nu+1)}(1) &= \left(\sum_{j=\nu}^n a_j \varphi_\nu(j) x^{j-\nu} \right)' (1) = \left(\sum_{j=\nu+1}^n a_j \varphi_\nu(j) (j-\nu) x^{j-\nu-1} \right) (1) \\ &= \left(\sum_{j=\nu+1}^n a_j \varphi_{\nu+1}(j) x^{j-\nu-1} \right) (1) = \sum_{j=\nu+1}^n a_j \varphi_{\nu+1}(j) = \sum_{j=0}^n a_j \varphi_{\nu+1}(j) \end{aligned}$$

Since $\{\varphi_0, \varphi_1, \dots, \varphi_k\}$ is a basis of the vector space of all polynomials up to degree k we may write $q(x) = \sum_{\nu=0}^k q_\nu \varphi_\nu(x)$ which gives us

$$\sum_{j=0}^n a_j q(j) = \sum_{j=0}^n a_j \sum_{\nu=0}^k q_\nu \varphi_\nu(j) = \sum_{\nu=0}^k q_\nu \sum_{j=0}^n a_j \varphi_\nu(j) = \sum_{\nu=0}^k q_\nu p^{(\nu)}(1) = 0$$

as $(x-1)^{k+1} \mid p(x)$, so $(*)$ is proven.

To prove the original statement now let

$$q(x) = T_k \left(\frac{2}{n-1} x - 1 \right)$$

with the k -th Chebyshev polynomial.

https://en.wikipedia.org/wiki/Chebyshev_polynomials

Then $q(0), \dots, q(n-1) \in T_k([-1, 1]) \subseteq [-1, 1]$ and

$$\begin{aligned} q(n) &= T_k\left(\frac{n+1}{n-1}\right) = \cosh\left(k \cdot \operatorname{arcosh}\left(\frac{n+1}{n-1}\right)\right) \\ &= \cosh\left(k \cdot \log\left(\frac{n+1}{n-1} + \sqrt{\left(\frac{n+1}{n-1}\right)^2 - 1}\right)\right) \\ &= \cosh\left(k \cdot \log\left(\frac{(\sqrt{n}+1)^2}{n-1}\right)\right) = \cosh\left(k \cdot \log\left(\frac{\sqrt{n}+1}{\sqrt{n}-1}\right)\right) \\ &= \cosh\left(k \cdot \log\left(\frac{1 + \frac{1}{\sqrt{n}}}{1 - \frac{1}{\sqrt{n}}}\right)\right) > \cosh\left(k \cdot \frac{2}{\sqrt{n}}\right) \end{aligned}$$

where we have used that $n > 1$, and that \cosh is strictly monotone increasing for positive arguments, and $\log\left(\frac{1+x}{1-x}\right) > 2x$ for $x < 1$.

$$\frac{1+x}{1-x} = 1 + 2 \sum_{n=1}^{\infty} x^n > 1 + 2x + 2 \cdot \frac{2}{2!} x^2 + 2 \cdot \frac{2^{3-1}}{3!} x^3 + 2 \cdot \frac{2^3}{4!} x^4 + \dots = e^{2x}$$

Note that by definition of $q(x)$ we have $q(0), \dots, q(n-1) \in [-1, 1]$ and we have shown

$$\sum_{j=0}^{n-1} |a_j| \geq \sum_{j=0}^{n-1} a_j(-q(j)) \stackrel{(*)}{=} a_n q(n) = q(n) > \cosh\left(k \cdot \frac{2}{\sqrt{n}}\right) > 1 + \frac{2k^2}{n}$$

6. Consider the triangle $A = (0, 0)$, $B = (2\sqrt{3}, 0)$, $C = (3 - \sqrt{3}, -3 + 3\sqrt{3})$. Now choose on each side a point, M_a, M_b, M_c , such that the new triangle built by those points is of minimal perimeter. What is the area of the $\triangle(M_a, M_b, M_c)$?

Reason: Fagnano Triangle.

Solution: The solution to the optimization problem is the Fagnano or orthic triangle, which is built by the base points of all heights. In our

triangle these are

$$\begin{aligned} M_a &= (\sqrt{3}, \sqrt{3}) & a &= \frac{1}{2} + \frac{1}{6}\sqrt{3} \\ M_b &= \left(\frac{1}{2}\sqrt{3}, \frac{3}{2}\right) & b &= \frac{1}{4} + \frac{1}{4}\sqrt{3} \\ M_c &= (3 - \sqrt{3}, 0) & c &= -\frac{1}{2} + \frac{1}{2}\sqrt{3} \end{aligned}$$

We take $M_a M_c$ as baseline, which results in the straight line, height and base point equations

$$\begin{aligned} g : \vec{x}_g &= (3 - 2\sqrt{3}, -\sqrt{3})^\tau \cdot g + (\sqrt{3}, \sqrt{3})^\tau & 0 \leq g \leq 1 \\ h_g : \vec{x} &= (-\sqrt{3}, -3 + 2\sqrt{3})^\tau \cdot h_g + \left(\frac{1}{2}\sqrt{3}, \frac{3}{2}\right)^\tau & 0 \geq h_g \geq -\frac{1}{4}\sqrt{3} \\ H : \left(\frac{3}{4} + \frac{1}{2}\sqrt{3}, \frac{3}{4}\sqrt{3}\right) & & g = \frac{1}{4} \end{aligned}$$

For the area of $\triangle(M_a, M_b, M_c)$ we get

$$\begin{aligned} A &= \frac{1}{2} \cdot |g| \cdot |h_g| \\ &= \frac{1}{2} \cdot \|(3 - 2\sqrt{3}, -\sqrt{3})\| \cdot 1 \cdot \|(-\sqrt{3}, -3 + 2\sqrt{3})\| \cdot \left| -\frac{1}{4}\sqrt{3} \right| \\ &= \frac{1}{8}\sqrt{3} \cdot \sqrt{3 + (3 - 2\sqrt{3})^2} \cdot \sqrt{(-3 + 2\sqrt{3})^2 + 3} \\ &= -\frac{9}{2} + 3\sqrt{3} \approx 0.696 \end{aligned}$$

The three straights of the triangle and their heights are

$$\begin{aligned} a : \vec{x}_a &= (3 - 3\sqrt{3}, -3 + 3\sqrt{3})^\tau \cdot a + (2\sqrt{3}, 0)^\tau = \vec{x}_a \cdot a + \vec{s} & 0 \leq a \leq 1 \\ b : \vec{x}_b &= (3 - \sqrt{3}, -3 + 3\sqrt{3})^\tau \cdot b = \vec{x}_b \cdot b & 0 \leq b \leq 1 \\ c : \vec{x}_c &= (2\sqrt{3}, 0)^\tau \cdot c = \vec{x}_c \cdot c & 0 \leq c \leq 1 \\ h_a : \vec{x} &= (-3 + 3\sqrt{3}, -3 + 3\sqrt{3})^\tau \cdot h_a & 0 \leq h_a \leq \frac{1}{2} + \frac{1}{6}\sqrt{3} \\ h_b : \vec{x} &= (-3 + 3\sqrt{3}, -3 + \sqrt{3})^\tau \cdot h_b + (2\sqrt{3}, 0)^\tau & 0 \geq h_b \geq -\frac{3}{4} - \frac{1}{4}\sqrt{3} \\ h_c : \vec{x} &= (0, 2\sqrt{3})^\tau \cdot h_c + (3 - \sqrt{3}, -3 + 3\sqrt{3})^\tau & 0 \geq h_c \geq -\frac{3}{2} + \frac{1}{2}\sqrt{3} \end{aligned}$$

Solution

The orthic triangle, with vertices at the base points of the

altitudes of the given triangle, has the smallest perimeter of all triangles inscribed into an acute triangle, hence it is the solution of Fagnano's problem. Fagnano's original proof used calculus methods and an intermediate result given by his father Giulio Carlo de'Toschi di Fagnano. Later however several geometric proofs were discovered as well, amongst others by Hermann Schwarz and Lipót Fejér. These proofs use the geometrical properties of reflections to determine some minimal path representing the perimeter.

Physical principles

A solution from physics is found by imagining putting a rubber band that follows Hooke's Law around the three sides of a triangular frame ABC , such that it could slide around smoothly. Then the rubber band would end up in a position that minimizes its elastic energy, and therefore minimize its total length. This position gives the minimal perimeter triangle. The tension inside the rubber band is the same everywhere in the rubber band, so in its resting position, we have, by Lami's theorem, $\angle bcA = \angle acB$, $\angle caB = \angle baC$, $\angle abC = \angle cbA$. Therefore, this minimal triangle is the orthic triangle.

https://en.wikipedia.org/wiki/Fagnano%27s_problem

Proof by geometry:

https://azimpremjiuniversity.edu.in/SitePages/pdf/05-shailesh_fagnanosproblemaddendum_classroom.pdf

Proof by analytical geometry:

<http://forumgeom.fau.edu/FG2007volume7/FG200728.pdf>

7. (HS-1) Calculate the masses of Sun, Earth and Jupiter. You may assume circular orbits. We further calculate with the following data:

the gravitational constant $G = 6.67 \cdot 10^{-11} \frac{m^3}{kg \cdot s^2}$

the Kepler constant for our solar system $C = \frac{T^2}{R^3} = 0.29 \cdot 10^{-18} \frac{s^2}{m^3}$

the acceleration by gravity on earth $\gamma = 9.81 \frac{m}{s^2}$

the earth's radius $R = 6,370 \text{ km}$

Io's orbital radius $R_I = 4.22 \cdot 10^8 \text{ m}$

Io's orbital period $T_I = 1.77 \text{ d}$

Reason: Some Basic Astronomy.

Solution:

- (a) Sun. The gravitational force of the sun on a planet is given by $F_G = G \frac{m_s \cdot m_p}{R^2}$ which is the radial force of the planet, i.e. $F_R = \frac{4\pi^2 R m_p}{T^2}$. Both forces are equal so we get

$$m_s = \frac{4\pi^2}{G} \cdot \frac{R^3}{T^2} = \frac{4\pi^2}{G \cdot C} \approx \frac{4\pi^2}{1.9343} \cdot 10^{29} \cdot \frac{\text{kg} \cdot \text{s}^2}{\text{m}^3} \cdot \frac{\text{m}^3}{\text{s}^2} \approx 2.041 \cdot 10^{30} \text{ kg}$$

- (b) Earth. In case of our own planet, we have again $F_G = G \frac{m_e \cdot m}{R^2}$ as gravitational force of a mass m on the planet. It equals its weight $F_w = m \cdot \gamma$, hence we have

$$\begin{aligned} m_e &= \frac{\gamma \cdot R^2}{G} \approx \frac{9.81 \cdot 6,370,000^2}{6.67 \cdot 10^{-11}} \cdot \frac{\text{m} \cdot \text{m}^2 \cdot \text{kg} \cdot \text{s}^2}{\text{s}^2 \cdot \text{m}^2} \\ &\approx \frac{9.81 \cdot 6.37^2}{6.67} \cdot 10^{23} \text{ kg} \approx 5.968 \cdot 10^{24} \text{ kg} \end{aligned}$$

- (c) Jupiter. The calculations for Jupiter (and Io) is analog to that of the Sun (and Jupiter). Hence we get

$$\begin{aligned} m_j &= \frac{4\pi^2}{G} \cdot \frac{R_I^3}{T_I^2} \\ &\approx \frac{4\pi^2}{6.67} \cdot 10^{11} \cdot \frac{4.22^3}{1.77^2} \cdot 10^{24} \cdot \frac{1}{8.64^2} \cdot 10^{-8} \cdot \frac{\text{kg} \cdot \text{s}^2 \cdot \text{m}^3}{\text{m}^3 \cdot \text{s}^2} \\ &\approx 1.9 \cdot 10^{27} \text{ kg} \end{aligned}$$

8. (HS-2) A car drives at 72 km/h directly past a resting observer when the driver presses its horn. By what interval does the pitch of the horn change as the car passes the observer? (Speed of sound $s = 340 \text{ m/s}$.)

Reason: Doppler Effect.

Solution: The car's speed is 72 km/h = 20 m/s. Let the horn's pitch be ν . If the car approaches the observer, he will hear a frequency

$$\nu' = \frac{\nu}{1 - \frac{v}{s}}$$

If the car evicts the observer, he will hear a frequency

$$\nu'' = \frac{\nu}{1 + \frac{v}{s}}$$

For the interval we get

$$\frac{\nu'}{\nu''} = \frac{s+v}{s-v} = \frac{360 \text{ m/s}}{320 \text{ m/s}} = \frac{9}{8}$$

which is a full musical tone.

9. (HS-3) Consider the sphere $\mathbb{S}^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = r^2\}$ and a point $P \in \mathbb{S}^2$. Determine the set of all center points of all chords starting in P .

Reason: Geometry.

Vector calculations are easier than coordinate calculations.

Solution: The variable endpoint X of the chord is on the sphere, so for its position vector we have $\vec{x}^2 = r^2$. The position vector of the center of the chord \overline{PX} is thus

$$\vec{c} = \frac{\vec{p} + \vec{x}}{2} \iff \vec{x} = 2\vec{c} - \vec{p}$$

hence $r^2 = (2\vec{c} - \vec{p})^2$ or $\left(\vec{c} - \frac{\vec{p}}{2}\right)^2 = \frac{r^2}{4}$. So the set of points we were looking for are all on a sphere with center $\overline{OP}/2 = \vec{p}/2$ and radius $r/2$. All points of this sphere are on the other hand a center of some chord of the original sphere with endpoint P , since we can go back. The point P itself is the center of the chord \overline{PP} .

10. (HS-4) At the monthly meeting of former mathematics students, six members choose a real number a , which has to be guessed by a seventh mathematician who had left the room before. He gets the following information after he returned:

- (1) a is rational.
- (2) a is an integer divisible by 14.
- (3) a is real and its square equals 13.
- (4) a is an integer divisible by 7.

- (5) a is real and the inequality $0 < a^3 + a < 8,000$ holds.
- (6) a is even.

He is told, that all pairs $(1, 2), (3, 4), (5, 6)$ always consist of a true and a false statement. What is a ?

Reason: Puzzle.

Solution: Assume $a \notin \mathbb{Z}$. Then (2),(4),(6) are false and thus (1),(3),(5) true, which cannot be since $\sqrt{13} \notin \mathbb{Q}$. This means that $a \in \mathbb{Z}$ and statement (4) is true. As $\mathbb{Z} \subseteq \mathbb{Q}$ statement (2) is false and a is not divisible by 14, hence odd. So we have additionally that $0 < a^3 + a = a(a^2 + 1) < 8,000$. This implies $a > 0$. On the other end it implies $a < 20$. But only $a = 7$ is odd and divisible by 7 in this range. So (1),(4),(5) are true and (2),(3),(6) are false.

4 September 2019

1. We all know that the geometric mean is less than the arithmetic mean. I memorize it with $3 \cdot 5 < 4 \cdot 4$. Now we consider the arithmetic-geometric mean $M(a, b)$ between the two others. Let a, b be two nonnegative real numbers. We set $a_0 = a$, $b_0 = b$ and define the sequences (a_k) , (b_k) by

$$a_{k+1} := \frac{a_k + b_k}{2}, \quad b_{k+1} = \sqrt{a_k b_k} \quad k = 0, 1, \dots$$

Then the arithmetic-geometric mean $M(a, b)$ is the common limit

$$\lim_{n \rightarrow \infty} a_n = M(a, b) = \lim_{n \rightarrow \infty} b_n$$

It is not hard to show that both sequences converge and that their limit is the same by using the known inequality and the monotony of the sequences.

Prove that for positive $a, b \in \mathbb{R}$ holds

$$T(a, b) := \frac{2}{\pi} \int_0^{\pi/2} \frac{d\varphi}{\sqrt{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi}} = \frac{1}{M(a, b)}$$

Reason: The arithmetic-geometric mean.

Solution: We show $T(a, b) = T\left(\frac{1}{2}(a+b), \sqrt{ab}\right)$, the Laden transformation. By repetition and the limiting process we get

$$T(a, b) = T(M(a, b), M(a, b)) = \frac{2}{\pi} \int_0^{\pi/2} \frac{d\varphi}{M(a, b)} = \frac{1}{M(a, b)}$$

With $t := b \tan \varphi$ we get

$$\cos^2 \varphi = \frac{b^2}{b^2 + t^2}, \quad \sin^2 \varphi = \frac{t^2}{b^2 + t^2}, \quad d\varphi = \frac{b}{b^2 + t^2} dt$$

and

$$\begin{aligned} T(a, b) &= \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{dt}{\sqrt{(a^2 + t^2)(b^2 + t^2)}} \\ &= \frac{1}{\pi} \int_{-\infty}^0 \frac{dt}{\sqrt{(a^2 + t^2)(b^2 + t^2)}} + \frac{1}{\pi} \int_0^{+\infty} \frac{dt}{\sqrt{(a^2 + t^2)(b^2 + t^2)}} \end{aligned}$$

In the first integral we substitute $t = x - C(x)$, in the second $t = x + C(x)$ where we set $C(x) = \sqrt{ab + x^2}$ for short. Then

$$\begin{aligned}
 T(a, b) &= \frac{1}{\pi} \int_{-\infty}^{+\infty} \left\{ \frac{1 - x/C(x)}{\sqrt{[a^2 + (x - C(x))^2] \cdot [b^2 + (x - C(x))^2]}} \right. \\
 &\quad \left. + \frac{1 + x/C(x)}{\sqrt{[a^2 + (x + C(x))^2] \cdot [b^2 + (x + C(x))^2]}} \right\} dx \\
 &= \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{dx}{C(x)} \left\{ \frac{C(x) - x}{\sqrt{[a^2 + (C(x) - x)^2] \cdot [b^2 + (C(x) - x)^2]}} \right. \\
 &\quad \left. + \frac{C(x) + x}{\sqrt{[a^2 + (C(x) + x)^2] \cdot [b^2 + (C(x) + x)^2]}} \right\} \\
 &= \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{dx}{C(x)} \left\{ \frac{C(x)^2 - x^2}{\sqrt{(C(x) + x)^2 \cdot [a^2 + (C(x) - x)^2] \cdot [b^2 + (C(x) - x)^2]}} \right. \\
 &\quad \left. + \frac{C(x)^2 - x^2}{\sqrt{(C(x) - x)^2 \cdot [a^2 + (C(x) + x)^2] \cdot [b^2 + (C(x) + x)^2]}} \right\} \\
 &= \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{dx}{C(x)} \left\{ \frac{ab}{\sqrt{(C(x) + x)^2 \cdot [a^2 b^2 + (C(x) - x)^4 + (a^2 + b^2) \cdot (C(x) - x)^2]}} \right. \\
 &\quad \left. + \frac{ab}{\sqrt{(C(x) - x)^2 \cdot [a^2 b^2 + (C(x) + x)^4 + (a^2 + b^2) \cdot (C(x) + x)^2]}} \right\} \\
 &= \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{dx}{C(x)} \left\{ \frac{1}{\sqrt{(C(x) + x)^2 + (C(x) - x)^2 + (a^2 + b^2)}} \right. \\
 &\quad \left. + \frac{1}{\sqrt{(C(x) - x)^2 + (C(x) + x)^2 + (a^2 + b^2)}} \right\} \\
 &= \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{2}{C(x)} \cdot \frac{dx}{\sqrt{2C(x)^2 + 2x^2 + a^2 + b^2}} \\
 &= \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{2 dx}{\sqrt{2(ab + x^2)^2 + 2x^2(ab + x^2) + (a^2 + b^2)(ab + x^2)}} \\
 &= \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{[(a + b)/2]^2 + x^2} \cdot [\sqrt{ab^2} + x^2]} \\
 &= T\left(\frac{a+b}{2}, \sqrt{ab}\right)
 \end{aligned}$$

2. If A, B, C, D are four points in the plane, show that

$$\det \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & |AB|^2 & |AC|^2 & |AD|^2 \\ 1 & |AB|^2 & 0 & |BC|^2 & |BD|^2 \\ 1 & |AC|^2 & |BC|^2 & 0 & |CD|^2 \\ 1 & |AD|^2 & |BD|^2 & |CD|^2 & 0 \end{bmatrix} = 0$$

Reason: Cayley Menger Determinant.

Solution: If we view A, B, C, D as vectors in \mathbb{R}^2 , then we have the usual cosine rule $|AB|^2 = |A|^2 + |B|^2 - 2A \cdot B$, and similarly for all the other distances. The matrix can then be written as $M + M^T - 2G$, where

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & |A|^2 & |B|^2 & |C|^2 & |D|^2 \\ 1 & |A|^2 & |B|^2 & |C|^2 & |D|^2 \\ 1 & |A|^2 & |B|^2 & |C|^2 & |D|^2 \\ 1 & |A|^2 & |B|^2 & |C|^2 & |D|^2 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & A \cdot A & A \cdot B & A \cdot C & A \cdot D \\ 0 & B \cdot A & B \cdot B & B \cdot C & B \cdot D \\ 0 & C \cdot A & C \cdot B & C \cdot C & C \cdot D \\ 0 & D \cdot A & D \cdot B & D \cdot C & D \cdot D \end{bmatrix}$$

$\text{rk } M = \text{rk } M^T = 1$ and $G = S \cdot S^T$ with the 5×2 matrix with rows $0, A, B, C, D$, i.e. $\text{rk } G \leq 2$. Hence $\text{rk } (M + M^T - 2G) \leq 1 + 1 + 2 = 4 < 5$ and its determinant vanishes.

3. Let $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ a linear, continuous (= bounded) operator on Hilbert spaces. Prove that the following are equivalent:

- (a) T is invertible.
- (b) There exists a constant $\alpha > 0$, such that $T^*T \geq \alpha I_{\mathcal{H}_1}$ and $TT^* \geq \alpha I_{\mathcal{H}_2}$. $A \geq B$ means $\langle (A - B)\xi, \xi \rangle \geq 0$ for all ξ .

Reason: Linear Operators.

Solution:

(a) \implies (b) If T is invertible so is $T^* : \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ with inverse $(T^*)^{-1} = (T^{-1})^*$. Define $\alpha := \|T^{-1}\|^{-2} = \|(T^*)^{-1}\|^{-2}$. Note that for $\xi \in \mathcal{H}_1$ we have

$$\|\xi\| = \|T^{-1}(T(\xi))\| \leq \|T^{-1}\| \cdot \|T(\xi)\|$$

and therefore

$$\langle T^*T\xi, \xi \rangle_{\mathcal{H}_1} = \langle T\xi, T\xi \rangle_{\mathcal{H}_2} = \|T\xi\|^2 \geq \|T^{-1}\|^{-2} \|\xi\|^2 = \alpha \langle \xi, \xi \rangle_{\mathcal{H}_1}$$

This shows $\langle (T^*T - \alpha I_{\mathcal{H}_1})\xi, \xi \rangle \geq 0$ for all $\xi \in \mathcal{H}_1$, so $T^*T - \alpha I_{\mathcal{H}_1} \in \mathcal{B}(\mathcal{H}_1)$ is positive. The positivity of $TT^* - \alpha I_{\mathcal{H}_2} \in \mathcal{B}(\mathcal{H}_2)$ follows accordingly.

(b) \implies (a) Assume there is an $\alpha > 0$ such that $T^*T \geq \alpha I_{\mathcal{H}_1}$ and $TT^* \geq \alpha I_{\mathcal{H}_2}$. Thus

$$\|T\xi\|^2 = \langle T\xi, T\xi \rangle = \langle T^*T\xi, \xi \rangle \geq \alpha \langle \xi, \xi \rangle = \|\xi\|^2$$

and we get

$$\|T\xi\| \geq \sqrt{\alpha}\|\xi\| \quad \forall \xi \in \mathcal{H}_1$$

On the one hand this shows that T is injective, and similar that T^* is injective, too. Therefore

$$\overline{R(T)} = (\ker(T^*))^\perp = \{0_{\mathcal{H}_2}\}^\perp = \mathcal{H}_2$$

and we only have to show that $R(T)$ is closed, hence T is also surjective. Now since T is bounded, we have

$$C \cdot \|\xi\| \geq \|T\| \cdot \|\xi\| \geq \|T\xi\| \geq \sqrt{\alpha} \cdot \|\xi\|$$

which makes the norm on \mathcal{H}_1 equivalent to the by \mathcal{H}_2 induced norm on $R(T)$. Thus $R(T)$ is again a Banach space and therefore closed.

4. Let $a, b \in \mathbb{F}$ be non-zero elements in a field of characteristic not two. Let A be the four dimensional \mathbb{F} -space with basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and the bilinear and associative multiplication defined by the conditions that 1 is a unity element and

$$\mathbf{i}^2 = a, \mathbf{j}^2 = b, \mathbf{ij} = -\mathbf{ji} = \mathbf{k}.$$

Then $A = \left(\frac{a, b}{\mathbb{F}}\right)$ is called a (generalized) quaternion algebra over \mathbb{F} . Show that A is a simple algebra whose center is \mathbb{F} .

Reason: Associative Algebras.

Solution: For convenience we use the Lie bracket for $[x, y] = xy - yx$. If $x = c_0 + c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ then

$$\begin{aligned} [\mathbf{i}, x] &= (2ac_3)\mathbf{j} + (2c_2)\mathbf{k} \\ [\mathbf{j}, x] &= (-2bc_3)\mathbf{i} + (-2c_1)\mathbf{k} \\ [\mathbf{k}, x] &= (2bc_2)\mathbf{i} + (-2ac_1)\mathbf{j} \end{aligned}$$

In case $x \in Z(A)$ we get $c_3 = c_2 = c_1 = 0$ and vice versa, hence $Z(A) = \mathbb{F}$. Now let $\{0\} \neq I \subseteq A$ be a non-zero ideal of A and $0 \neq x \in I$. As a two sided ideal we have

$$\begin{aligned} [\mathbf{j}, [\mathbf{i}, x]] &= (-4bc_2)\mathbf{i} \in I \\ [\mathbf{k}, [\mathbf{j}, x]] &= (4abc_3)\mathbf{j} \in I \\ [\mathbf{i}, [\mathbf{k}, x]] &= (-4ac_1)\mathbf{k} \in I \end{aligned}$$

If one of the coefficients c_1, c_2, c_3 is unequal zero, then I contains a unit of A and thus $I = A$. If $c_1 = c_2 = c_3 = 0$ then $x \neq 0$ implies $c_0 \neq 0$ and I again contains a unit. In all cases we have $A = I$.

5. Prove that the quaternion algebra $\left(\frac{a, 1}{\mathbb{F}}\right) \cong \mathbb{M}(2, \mathbb{F})$ is isomorphic to the matrix algebra of 2×2 matrices for every $a \in \mathbb{F} - \{0\}$.

Reason: Quaternions.

Solution: Direct calculation of the multiplication tables by setting

$$\begin{aligned} e_{11} &= \frac{1}{2}(1 - \mathbf{j}) \\ e_{22} &= \frac{1}{2}(1 + \mathbf{j}) \\ e_{12} &= \frac{1}{2a}(\mathbf{i} - \mathbf{k}) \\ e_{21} &= \frac{1}{2}(\mathbf{i} + \mathbf{k}) \end{aligned}$$

6. Show that there are infinitely many primes of the form $4k + 3$, $k \in \mathbb{N}_0$.

Reason: Number Theory.

Solution: Assume there are only finitely many primes of the form $4k + 3$: p_1, \dots, p_n . Set $z := 4p_1 \cdot \dots \cdot p_n - 1$. Then $z = 4k + 3$ with $k = p_1 \cdot \dots \cdot p_n - 1 \in \mathbb{N}_0$. Let $z = q_1 \cdot \dots \cdot q_m$ be the prime decomposition of z . Since z is odd, $q_i \neq 2$ for all $i \in \{1, \dots, m\}$. This all q_i are either of the form $q_i = 4r_i + 1$ or of the form $q_i = 4s_i + 3$ with $r_i, s_i \in \mathbb{N}_0$. Assume all q_i were of the form $4r_i + 1$, then $3 \equiv z \equiv 1 \pmod{4}$ which is impossible. Therefore at least one q_i has the form $4s_i + 3$. But for this prime we have $q_i \in \{p_1, \dots, p_n\}$, say $q_i = p_j$. Now

$$p_j = q_i \mid q_1 \cdot \dots \cdot q_m = z = 4p_1 \cdot \dots \cdot p_n - 1$$

which is impossible. Hence there are infinitely many primes of the form $4k + 3$.

7. Do $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ and $\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} \right)^2$ converge or diverge?

Reason: Product of converging series can diverge.

Solution: $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ converges according to the Leibniz criterion,

because $a_n := \frac{1}{\sqrt{n+1}}$ is strictly monotone decreasing. For the Cauchy

product $\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n \right)^2 = \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} \right)^2$ we get

$$\begin{aligned} c_n &= \sum_{k=0}^n a_{n-k} a_k \\ &= \sum_{k=0}^n \frac{(-1)^{n-k}}{\sqrt{n-k+1}} \cdot \frac{(-1)^k}{\sqrt{k+1}} \\ &= (-1)^n \sum_{k=0}^n \frac{1}{\sqrt{n-k+1} \cdot \sqrt{k+1}} \end{aligned}$$

From $0 \leq (\sqrt{a} - \sqrt{b})^2$ we get $\sqrt{a}\sqrt{b} \leq \frac{1}{2}(a+b)$ for $a, b > 0$ and so

$$\begin{aligned} |c_n| &= \sum_{k=0}^n \frac{1}{\sqrt{n-k+1} \cdot \sqrt{k+1}} \\ &\geq \sum_{k=0}^n \frac{1}{\frac{1}{2}(n-k+1+k+1)} \\ &= \sum_{k=0}^n \frac{2}{\sqrt{n+2}} \\ &= \frac{2(n+1)}{n+2} \\ &= \frac{2 + \frac{2}{n}}{1 + \frac{2}{n}} \rightarrow 2 \quad (n \rightarrow \infty) \end{aligned}$$

Hence (c_n) isn't a null sequence and $\sum_{n=0}^{\infty} c_n$ diverges.

8. Consider the curve $\gamma : \mathbb{R} \mapsto \mathbb{C}$, $\gamma(t) = \cos(\pi t) \cdot e^{\pi i t}$. Find the minimal period of γ (a), prove that $\gamma(\mathbb{R}) \equiv \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 - x = 0\}$ (b), show that $\gamma(\mathbb{R})$ is symmetric to the x -axis (c), and parameterize γ with respect to its arc length (d).

Reason: Differential Geometry.

Solution: We have a natural parameterization by

$$\gamma(t) = (\cos^2(\pi t), \sin(\pi t) \cos(\pi t))$$

and thus $\gamma(t+2) = \gamma(t)$. Since $\sin(\pi t) \cos(\pi t) = \frac{1}{2} \sin(2\pi t)$ we even get $\gamma(t) = \gamma(t+1)$, and thus a minimal period of 1.

Now let $(x, y) \in \gamma(\mathbb{R}) \subseteq \mathbb{R}^2$. Then

$$\begin{aligned} x^2 + y^2 - x &= \cos^4(\pi t) + \sin^2(\pi t) \cos^2(\pi t) - \cos^2(\pi t) \\ &= \cos^2(\pi t) \cdot (\cos^2(\pi t) + \sin^2(\pi t) - 1) \\ &= 0 \end{aligned}$$

If $(x, y) \in \mathbb{R}^2$ with $x^2 + y^2 - x = 0$, then we have to find a $t \in \mathbb{R}$ with $\gamma(t) = (x, y)$.

$$x^2 + y^2 - x = 0 \iff x = \frac{1}{2} \pm \sqrt{\frac{1}{4} - y^2}$$

Since (x, y) exists by assumption, $\frac{1}{4} - y^2 \geq 0$ and $x \in [0, 1]$ which allows us to choose $x = \cos^2(\pi t)$.

$$y^2 = x - x^2 = \cos^2(\pi t) - \cos^4(\pi t) = \cos^2(\pi t)(1 - \cos^2(\pi t)) = \cos^2(\pi t) \sin^2(\pi t)$$

For the positive solution we are done. If $y = -\cos(\pi t) \sin(\pi t)$ we observe that $y = -\cos(\pi t) \sin(\pi t) = \cos(-\pi t) \sin(-\pi t)$. As $x = \cos^2(\pi t) = \cos^2(-\pi t)$, we also have shown the existence of t . In combination we have $\gamma(\mathbb{R}) \equiv \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 - x = 0\}$. As $y, -y$ yield the same point, the symmetry with respect to the x -axis is obvious. By

$$x^2 + y^2 - x = 0 \iff \left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$$

we see, that γ is the circle with center $(\frac{1}{2}, 0)$ and radius $\frac{1}{2}$. The arc length is given by

$$\begin{aligned} s(t) &= \int_0^t \|\dot{\gamma}(t)\| dt \\ &= \int_0^t \sqrt{(-\pi \sin(\pi t)e^{\pi it} + i\pi \cos(\pi t)e^{\pi it}) \cdot (-\pi \sin(\pi t)e^{-\pi it} - i\pi \cos(\pi t)e^{-\pi it})} dt \\ &= \int_0^t \sqrt{\pi^2 \sin^2(\pi t) + \pi^2 \cos^2(\pi t)} dt \\ &= \int_0^t \pi dt \\ &= \pi t \end{aligned}$$

and the inverse is $\Phi(s) = \frac{s}{\pi}$. Then we have $p := \gamma \circ \Phi : \mathbb{R} \rightarrow \mathbb{C}$ with

$$p(s) = \gamma(\Phi(s)) = \gamma\left(\frac{s}{\pi}\right) = \cos(s) \cdot e^{is}$$

9. Let $\gamma : I \rightarrow \mathbb{R}^n$ be a regular curve with unit tangential vector $T = \frac{d}{dt}\gamma$. A (orthonormal) **frame** is a (smooth) C^∞ -transformation $F : I \rightarrow \text{SO}(n)$ with $F(t)e_1 = T(t)$ where $\{e_i\}$ is the standard basis of \mathbb{R}^n . The pair (γ, F) is called a **framed curve**, and the matrix A given by $\frac{d}{dt}F = F' = FA$ is called **derivation matrix** of F .

Let $F_0 : \mathbb{R} \rightarrow \text{SO}(n)$ be a frame of a regular curve $\gamma : \mathbb{R} \rightarrow \mathbb{R}^n$. Show that

- If $F : \mathbb{R} \rightarrow \text{SO}(n)$ is another frame of γ , then there exists a transformation $\Phi : \mathbb{R} \rightarrow \text{SO}(n)$ with $\Phi(t)e_1 = e_1$ for all $t \in \mathbb{R}$ and $F = F_0\Phi$.
- If on the other hand $\Phi : \mathbb{R} \rightarrow \text{SO}(n)$ is a smooth transformation with $\Phi(t)e_1 = e_1$, then $F := F_0 \cdot \Phi$ defines a new frame of γ .
- If A_0 is the derivation matrix of F_0 , and A the derivation matrix of the transformed frame $F := F_0\Phi$ with Φ as above, then

$$A = \Phi^{-1}A_0\Phi + \Phi^{-1}\Phi'$$

Reason: Gauge Transformation.

Solution:

- (a) We define Φ by $\Phi(t) := F_0(t)^{-1} \cdot F(t)$ which is a transformation $\mathbb{R} \rightarrow \text{SO}(n)$. Since both are frames of the same curve γ we get for all $t \in \mathbb{R}$

$$F(t)e_1 = F_0(t)e_1 = T(t) \implies \Phi(t)e_1 = F_0(t)^{-1} \cdot F(t)e_1 = e_1$$

- (b) F is obviously a transformation from \mathbb{R} to $\text{SO}(n)$ with

$$F(t)e_1 = F_0(t) \cdot \Phi(t)e_1 = F_0(t)e_1 = T(t)$$

and F is a frame of γ .

- (c) We calculate

$$A = F^{-1}F' = (F_0\Phi)^{-1}(F_0\Phi)' = \Phi^{-1}F_0^{-1}(F_0'\Phi + F_0\Phi') = \Phi^{-1}A_0\Phi + \Phi^{-1}\Phi'$$

10. (HS-1) Show that the number of ways to express a positive integer n as the sum of consecutive positive integers is equal to the number of odd factors of n .

Reason: Partitions.

Solution: From $n = r + (r+1) + \dots + (r+k) = \frac{1}{2}k(k+1) + (k+1)r$ we get $2n = (k+1)(2r+k)$. Now either $k+1$ or $2k+r$ is odd, so every odd factor of n results in a partition of n as sum of consecutive positive integers. If we have two decompositions $2n = (k+1)(2r+k) = (l+1)(2s+l)$ and $k+1 = l+1$ or $2r+k = 2s+l$ are the same odd numbers, then $k = l$ in both cases. If we have $k+1 = 2s+l$ then $l+1 = k+2-2s = \frac{2n}{2s+l} = 2r+k$ and $r = 1-s$ or $r, s \in \{0, 1\}$. For $(r, s) = (0, 1)$ we get $2n = (k+1)k = (l+1)(k+1)$ or $k = l+1$, and for $(r, s) = (1, 0)$ we have $2n = (l+1)l = l(2+k)$ or $l = k+1$ which is again the same odd factor as $2n = (l+2)(l+1) = (k+1)(k+2)$ is the same decomposition.

11. (HS-2) How many solutions in non-negative integers are there to the equation:

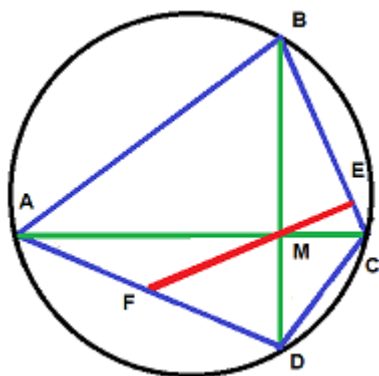
$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 32$$

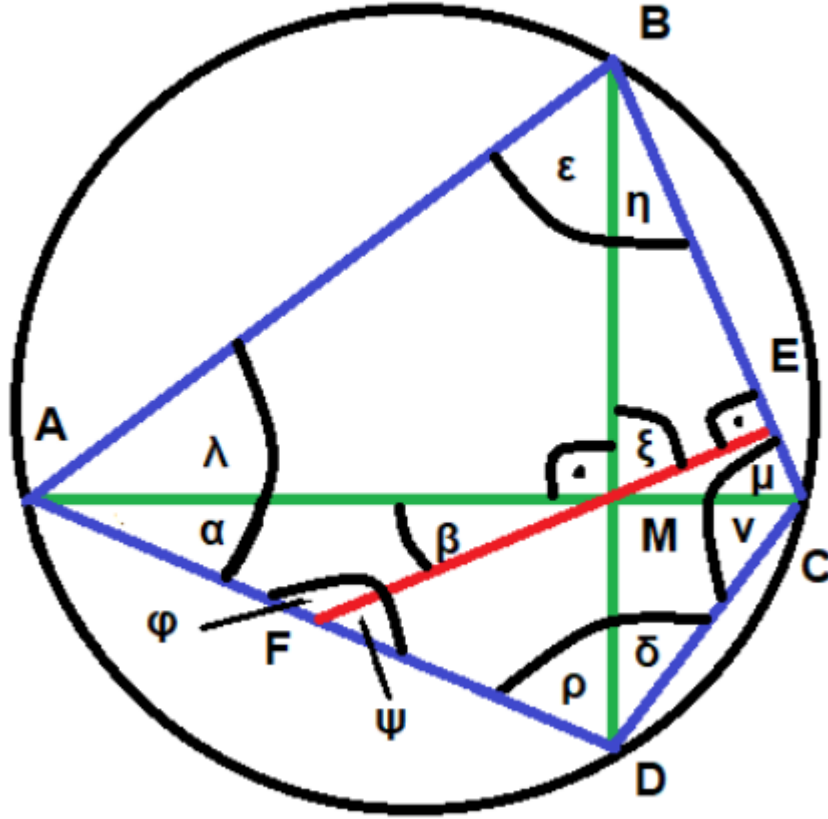
Reason: Partitions.

Solution: Assume we have 37 boxes in which we place 32 pebbles and 5 partition sticks. Then we have $\binom{37}{5} = 435,897$ possibilities, if we

allow zero (stick in the first box) as a summand.

12. (HS-3) Let A, B, C and D be four points on a circle such that the lines AC and BD are perpendicular. Denote the intersection of AC and BD by M . Drop the perpendicular from M to the line BC , calling the intersection E . Let F be the intersection of the line EM and the edge AD . Then F is the midpoint of AD .





Reason: Brahmagupta Theorem.

Solution: We need to prove that $AF = FD$. We will prove that both AF and FD are in fact equal to FM . By the first theorem of chords in an inscribed quadrilateral: $AM \cdot CM = BM \cdot DM$ we see that the slopes of AD and BC are reciprocal. Thus a rotary reflection maps BC on AD , i.e. the triangles $\triangle AMD$ and $\triangle BMC$ are similar. Hence $\alpha = \eta$, $\rho = \mu$. Since $\eta + \xi = \beta + \xi = \pi/2$ we get $\beta = \eta = \alpha$ and $\triangle AMF$ is an isosceles triangle, i.e. $AF = FM$.

$$\begin{aligned}\varphi &= \pi - \alpha - \beta \\ &= \pi - 2\beta \\ &= \psi + \varphi - 2\beta\end{aligned}$$

and $\psi = 2\beta$. Hence $\rho = \pi - 2\beta - \xi = \pi/2 - \beta = \xi$ so $\triangle FDM$ is also an isosceles triangle, i.e. $FM = FD$.

13. (HS-4) Prove that every non negative natural number $n \in \mathbb{N}_0$ can be written as

$$n = \frac{(x+y)^2 + 3x + y}{2}$$

with uniquely determined non negative natural numbers $x, y \in \mathbb{N}_0$.

Reason: Puzzle.

Solution: We set $s = x + y$, so $s \geq x \geq 0$ and for a given s we get as possible values for n the numbers

$$n = \frac{s^2 + s}{2} + x \in \left\{ \frac{s^2 + s}{2}, \frac{s^2 + s}{2} + 1, \dots, \frac{s^2 + s}{2} + s \right\} \subseteq \mathbb{N}_0$$

If we define $I_s := \left[\frac{s^2 + s}{2}, \frac{s^2 + s}{2} + s \right] \cap \mathbb{N}_0$ we observe that

$$\left(\frac{s^2 + s}{2} + s \right) + 1 = \frac{(s+1)^2 + (s+1)}{2}$$

and the I_s are a disjoint coverage of \mathbb{N}_0 . Thus all n belong to some I_s and it cannot belong to two.

14. (HS-5) Calculate

$$S = \int_{\frac{1}{2}}^3 \frac{1}{\sqrt{x^2 + 1}} \frac{\log(x)}{\sqrt{x}} dx + \int_{\frac{1}{3}}^2 \frac{1}{\sqrt{x^2 + 1}} \frac{\log(x)}{\sqrt{x}} dx$$

Reason: Multiplicative Integration Symmetry.

Solution: The functions are continuous in the area of integration. Assume that the anti-derivative is $F(x)$. Then we have to calculate $S = F(3) - F\left(\frac{1}{2}\right) + F(2) - F\left(\frac{1}{3}\right) = F(3) - F\left(\frac{1}{3}\right) + F(2) - F\left(\frac{1}{2}\right)$ and we can calculate the integrals

$$\mathcal{I}_n = \int_{\frac{1}{n}}^n \frac{1}{\sqrt{x^2 + 1}} \frac{\log(x)}{\sqrt{x}} dx = \int_{\frac{1}{n}}^n \frac{1}{\sqrt{x + \frac{1}{x}}} \frac{\log(x)}{x} dx$$

Set $y = \frac{1}{x}$. This means $\frac{dy}{dx} = -\frac{1}{x^2} = -y^2$. We also have to switch the

integration bounds $x = n$ to $y = \frac{1}{n}$ and $x = \frac{1}{n}$ to $y = n$. Thus

$$\begin{aligned}\mathcal{I}_n &= \int_n^{\frac{1}{n}} \frac{y}{\sqrt{\frac{1}{y} + y}} \log\left(\frac{1}{y}\right) \frac{-1}{y^2} dy \\&= \int_n^{\frac{1}{n}} \frac{1}{\sqrt{\frac{1}{y} + y}} \log(y) \frac{1}{y} dy \\&= - \int_{\frac{1}{n}}^n \frac{1}{\sqrt{\frac{1}{y} + y}} \frac{\log(y)}{y} dy \\&= - \int_{\frac{1}{n}}^n \frac{1}{\sqrt{y\left(\frac{1}{y} + y\right)}} \frac{\log(y)}{\sqrt{y}} dy \\&= - \int_{\frac{1}{n}}^n \frac{1}{\sqrt{1 + y^2}} \frac{\log(y)}{\sqrt{y}} dy \\&= - \int_{\frac{1}{n}}^n \frac{1}{\sqrt{x^2 + 1}} \frac{\log(x)}{\sqrt{x}} dx \\&= -\mathcal{I}_n\end{aligned}$$

If the integral equals its negative, then it has to be zero for any positive n . Hence $S = 0$.

5 August 2019

1. Three identical airplanes start at the same time at the vertices of an equilateral triangle with side length L . Let's say the origin of our coordinate system is the center of the triangle. The planes fly at a constant speed v above ground in the direction of the clockwise next airplane. How long will it take for the planes to reach the same point, and which are the flight paths?

Reason: Mechanics.

Solution: The side length of the triangle at $t = 0$ is $L(0) = L$. For the position $\vec{r}(t)$ of the first airplane we have $|\vec{r}(0)| = r(0) = \frac{2}{3}L \cos \frac{\pi}{6} = \frac{L}{\sqrt{3}}$. The distance between the airplanes are the same at any point in time, because of the symmetry, i.e. the airplanes will always mark the vertices of an equilateral triangle with its center at the origin. Thus the angle between the velocity $\vec{v}(t)$ and the position $\vec{r}(t)$ is always

$$\angle(\vec{v}(t), \vec{r}(t)) = \psi(t) = \psi(0) = \psi = \pi - \frac{\pi}{6}$$

Thus we have

$$\begin{aligned}\vec{v}(t) &= \dot{\vec{r}}(t) \\ \vec{r}(t) \cdot \vec{v}(t) &= \vec{r}(t) \cdot \dot{\vec{r}}(t) \\ r \cdot v \cdot \cos \psi &= \frac{1}{2} \frac{d}{dt} (\vec{r}(t) \cdot \vec{r}(t)) \\ r \cdot v \cdot \cos \psi &= \frac{1}{2} \frac{dr^2}{dt} \\ r \cdot v \cdot \cos \psi &= r \frac{dr}{dt} \\ \frac{dr}{dt} &= -v \frac{\sqrt{3}}{2} \\ r(t) &= \frac{L}{\sqrt{3}} - v \frac{\sqrt{3}}{2} t\end{aligned}$$

Hence $r(t_f) = 0$ implies $t_f = \frac{2L}{3v}$.

To get the flight path we decompose $\vec{v}(t)$ in components parallel and perpendicular to $\vec{r}(t)$. The perpendicular component is $|v_{\perp}| = v \cdot \sin \psi = \frac{v}{2}$ so we have the angular velocity $\dot{\omega}(t) = \frac{v_{\perp}(t)}{r(t)}$. We parameterize the

motion by cylindric coordinates $\vec{r}(t) = (r(t) \cos \varphi(t), -r(t) \sin \varphi(t), 0)^T$ and receive the momentary rotation angle by the integration

$$\begin{aligned}\varphi(t) &= \varphi(0) + \int_0^t \omega(t') dt' \\ &= \varphi(0) + \frac{v}{2} \int_0^t \frac{1}{r(t')} dt' \\ &= \varphi(0) + \frac{v}{2} \int_0^t \frac{1}{\frac{L}{\sqrt{3}} - v \frac{\sqrt{3}}{2} t'} dt' \\ &= \varphi(0) + \int_0^t \frac{1}{\frac{2L}{\sqrt{3}v} - \sqrt{3} t'} dt' \\ &= \varphi(0) + \frac{1}{\sqrt{3}} \int_0^t \frac{1}{\frac{2L}{3v} - t'} dt' \\ &= \varphi(0) + \frac{1}{\sqrt{3}} \log \left(\frac{\frac{2L}{3v}}{\frac{2L}{3v} - t} \right) \\ &= \varphi(0) + \frac{1}{\sqrt{3}} \log \left(\frac{r(0)}{r(t)} \right)\end{aligned}$$

so the flight path is the logarithmic spiral with

$$r(t) = r(0) \cdot e^{-\sqrt{3}(\varphi(t) - \varphi(0))}$$

The distance towards the center decreases by a factor of $e^{-2\pi\sqrt{3}} \approx 1.88 \cdot 10^{-5}$ with every complete turn.

2. The Schwarzian derivative of a holomorphic function f is given by

$$S_f(z) = \{f, z\} := \frac{d}{dz} \left(\frac{f''(z)}{f'(z)} \right) - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2 = \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left(\frac{f''(z)}{f'(z)} \right)^2$$

Prove a chain rule for the Schwarzian derivative and show that

$$\{f, z\} < 0 \wedge \{h, z\} < 0 \implies \{f \circ h, z\} < 0$$

Reason: Dynamical Systems.

Solution: The formula we want to prove is

$$S_{f \circ h}(z) = S_f(h(z)) \cdot (h'(z))^2 + S_h(z)$$

$$\begin{aligned}
 (f \circ h)'(z) &= f'(h(z))h'(z) \\
 (f \circ h)''(z) &= f''(h(z))(h'(z))^2 + f'(h(z))h''(z) \\
 (f \circ h)'''(z) &= f'''(h(z))(h'(z))^3 + 3f''(h(z))h'(z)h''(z) + f'(h(z))h'''(z) \\
 S_{fh}(z) &= \frac{(fh)'''(z)}{(fh)'(z)} - \frac{3}{2} \left(\frac{(fh)''(z)}{(fh)'(z)} \right)^2 \\
 &= \frac{f'''(h(z))(h'(z))^3 + 3f''(h(z))h'(z)h''(z) + f'(h(z))h'''(z)}{f'(h(z))h'(z)} \\
 &\quad - \frac{3}{2} \left(\frac{f''(h(z))(h'(z))^2 + f'(h(z))h''(z)}{f'(h(z))h'(z)} \right)^2 \\
 &= \frac{f'''(h(z))}{f'(h(z))} \cdot (h'(z))^2 + 3 \frac{f''(h(z))}{f'(h(z))} \cdot h''(z) + \frac{h'''(z)}{h'(z)} \\
 &\quad - \frac{3}{2} \left(\frac{f''(h(z))}{f'(h(z))} \cdot h'(z) + \frac{h''(z)}{h'(z)} \right)^2 \\
 &= S_f(h(z)) \cdot (h'(z))^2 + S_h(z) \\
 &\quad + 3 \frac{f''(h(z))}{f'(h(z))} \cdot h''(z) - \frac{3}{2} \cdot 2 \cdot \frac{f''(h(z))}{f'(h(z))} \cdot h''(z) \\
 &= S_f(h(z)) \cdot (h'(z))^2 + S_h(z)
 \end{aligned}$$

and from $S_f(z) < 0$ and $S_h(z) < 0$ we thus have $S_{fh}(z) < 0$.

Schwarzian derivatives are used in dynamical systems to investigate attractors, in flows of surfaces, or in the theory of Schwarz-Christoffel mappings.

3. (HS-1) David drives to work every working day by car. Outside towns he drives at an average speed of 180 km/h. On the 10 km in town, he drives at an average speed of 40 km/h. As a result, he is often too fast and gets a ticket. Meanwhile he has realized that things can not go on like this and he decides to reduce his average speed by 20 km/h in town as well as outside. How long is his way to work, if this reduces his average speed by 40 km/h on total?

Reason: Puzzle.

Solution: Let y be the length of his path in town, and x outside of town, each measured in km. We will later set $y = 10$. Originally he

needed $\frac{x}{180} + \frac{y}{40}$ hours, and now he needs $\frac{x}{160} + \frac{y}{20}$ hours. In order that the average speed decreases by exactly 40 km/h the following equation has to hold:

$$\frac{\frac{x+y}{\frac{x}{180} + \frac{y}{40}}}{\frac{x}{160} + \frac{y}{20}} = 40$$

$$40 = (x+y) \cdot \left(\frac{180 \cdot 40}{40x + 180y} - \frac{160 \cdot 20}{20x + 160y} \right)$$

$$1 = (x+y) \cdot \left(\frac{9}{2x+9y} - \frac{4}{x+8y} \right)$$

$$(2x+9y) \cdot (x+8y) = (x+y) \cdot (9x+72y-8x-36y)$$

$$2x^2 + 72y^2 + 25xy = x^2 + 36y^2 + 37xy$$

$$x^2 + 36y^2 - 12xy = 0$$

$$(x-6y)^2 = 0$$

So a necessary and sufficient condition is $x = 6y = 60$ km and his total way is 70 km long.

4. (HS-2) Show that $2x^6 + 3y^6 = z^3$ has no other rational solutions than $x = y = z = 0$.

Reason: Puzzle.

Solution: For an integer p the cube has only possible remainders $\{0, 1, 6\}$ from division by 7 so the remainder of p^6 will be either one or zero.

$$p = 2^n \cdot (2k+1) \implies p^3 = 8^n \cdot (2k+1)^3 \equiv r^3 \pmod{7}$$

where r is odd, i.e. $r^3 \in \{1^3, 3^3, 5^3, 7^3\} \equiv \{0, 1, 6\} \pmod{7}$. Hence for any integer solution

$$\{0, 1, 6\} \ni z^3 = 2x^6 + 3y^6 \in \{0, 2, 3, 5\} \pmod{7}$$

and z is divisible by 7 and then all are: $x, y, z \equiv 0 \pmod{7}$.

Let $q = \max\{p \in \mathbb{N} \mid 7^p \mid x \text{ and } 7^p \mid y\}$. Then $7^{6q} \mid z^3$, i.e. $7^{2q} \mid z$ and $\left(\frac{x}{7^q}, \frac{y}{7^q}, \frac{z}{7^{2q}}\right)$ is again an integer solution, so 7 divides all of them, which is impossible by maximality of q . This shows that $(0, 0, 0)$ is the only integer solution.

Now let (x, y, z) be with rationals and L the least common multiple of the denominators of x, y, z . Then (Lx, Ly, L^2z) is an integer solution, i.e. $x = y = z = 0$.

5. (HS-3) Let $x, y, z \in \mathbb{R} - \{0\}$ such that

$$x + \frac{y}{z} = 2, \quad y + \frac{z}{x} = 2, \quad z + \frac{x}{y} = 2$$

Show that $s := x + y + z$ can only have the values 3 or 7. You do not need to solve the equation system.

Reason: Puzzle.

Solution: The equations without denominators are

$$xz + y = 2z, \quad yx + z = 2x, \quad zy + x = 2y$$

hence $xz + yx + zy = 2(z + x + y) - (y + z + x) = x + y + z = s$. In the second step we multiply them

$$\begin{aligned} 1 &= \frac{y}{z} \cdot \frac{z}{x} \cdot \frac{x}{y} \\ &= (2 - x)(2 - y)(2 - z) \\ &= 8 - 4(x + y + z) + 2(xy + yz + zx) - xyz \\ &= 8 - 4s + 2s - xyz \\ &\implies xyz = 7 - 2s \end{aligned}$$

From the first step we also get

$$xzy + y^2 = 2yz, \quad xyz + z^2 = 2xz, \quad xyz + x^2 = 2xy$$

and thus

$$\begin{aligned} 3xyz + x^2 + y^2 + z^2 &= 2(xy + xz + yz) \\ 3xyz + (x + y + z)^2 &= 4(xy + xz + yz) \\ 3xyz + s^2 &= 4s \end{aligned}$$

Now we have $3(7 - 2s) + s^2 = 4s$ or $(s - 3)(s - 7) = 0$ and s can only have the values $s \in \{3, 7\}$.

There are actually only 4 solutions of the equation system:

$$(1, 1, 1), \left(\sqrt{7} \cot \frac{\pi}{7}, \sqrt{7} \cot \frac{2\pi}{7}, \sqrt{7} \cot \frac{4\pi}{7}\right)$$

and the cyclic permutations. They all solve $(t - 1)(t^3 - 7t^2 + 7t + 7) = 0$.

6 July 2019

- (a) Prove that every symmetric and positive definite matrix $A \in \mathbb{M}(n, \mathbb{R})$ can be uniquely written as $A = L \cdot L^\tau$, where L is a lower triangular matrix with positive diagonal elements.

(b) Calculate L for $A = \begin{bmatrix} 4 & 2 & 4 & 4 \\ 2 & 10 & 17 & 11 \\ 4 & 17 & 33 & 29 \\ 4 & 11 & 29 & 39 \end{bmatrix}$.

Reason: Cholesky Decomposition

Solution:

- (a) Induction over n . The statement is obviously true for $n = 1$. Let it be true for matrices in $\mathbb{M}(n-1, \mathbb{R})$. We will write

$$A = \begin{bmatrix} d & v^\tau \\ v & G \end{bmatrix}$$

Since A is positive definite, $x^\tau A x > 0$ for all $x \neq 0$. For $x = e_i := (0, \dots, 0, 1, 0, \dots, 0)^\tau$ we get $a_{ii} = e_i^\tau A e_i > 0$. Therefore we have $d > 0$.

With $H = G - \frac{vv^\tau}{d} \cdot I_{n-1}$ we get

$$A = \begin{bmatrix} d & v^\tau \\ v & G \end{bmatrix} = \begin{bmatrix} \sqrt{d} & 0 \\ \frac{v}{\sqrt{d}} & I_{n-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & H \end{bmatrix} \cdot \begin{bmatrix} \sqrt{d} & \frac{v^\tau}{\sqrt{d}} \\ 0 & I_{n-1} \end{bmatrix}$$

H is symmetric by definition and also positive definite:

$$0 < \begin{bmatrix} -\frac{x^\tau v}{d} & x^\tau \end{bmatrix} \cdot \begin{bmatrix} d & v^\tau \\ v & G \end{bmatrix} \cdot \begin{bmatrix} -\frac{x^\tau v}{d} \\ x \end{bmatrix} = x^\tau \left(G - \frac{vv^\tau}{d} \right) x = x^\tau H x$$

Thus we can write $H = L_H L_H^\tau$ per induction assumption with a lower triangular matrix with positive diagonal elements. Finally we get

$$\begin{aligned} A &= \begin{bmatrix} \sqrt{d} & 0 \\ \frac{v}{\sqrt{d}} & I_{n-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & L_H \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & L_H^\tau \end{bmatrix} \cdot \begin{bmatrix} \sqrt{d} & \frac{v^\tau}{\sqrt{d}} \\ 0 & I_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{d} & 0 \\ \frac{v}{\sqrt{d}} & L_H \end{bmatrix} \cdot \begin{bmatrix} \sqrt{d} & \frac{v^\tau}{\sqrt{d}} \\ 0 & L_H^\tau \end{bmatrix} \\ &= L L^\tau \end{aligned}$$

$$(b) \quad L = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 2 & 5 & 2 & 0 \\ 2 & 3 & 5 & 1 \end{bmatrix}$$

2. Let $L \subseteq H$ be a nonempty, closed, and convex set in a Hilbert space. Prove that there is an element of minimal norm in L . **Reason:** Completeness Properties.

Solution: Let $d := \inf\{\|f\| : f \in L\}$. Then there is a sequence $(f_n) \subseteq L$ such that $\lim_{n \rightarrow \infty} \|f_n\| = d$. By direct computation

$$\begin{aligned} \left\| \frac{f_n - f_m}{2} \right\|^2 &= 2 \left\| \frac{f_n}{2} \right\|^2 + 2 \left\| \frac{f_m}{2} \right\|^2 - \left\| \frac{f_n + f_m}{2} \right\|^2 \\ &\leq 2 \left\| \frac{f_n}{2} \right\|^2 + 2 \left\| \frac{f_m}{2} \right\|^2 - d^2 \end{aligned}$$

where the inequality follows from convexity. Therefore

$$\|f_n - f_m\|^2 \leq 2\|f_n\|^2 + 2\|f_m\|^2 - 4d^2$$

and so

$$\limsup_{n,m \rightarrow \infty} \|f_n - f_m\|^2 \leq 2d^2 + 2d^2 - 4d^2 = 0$$

which shows that (f_n) is a Cauchy sequence and as L is closed and therewith a complete subset in H , we conclude that there is $f \in L$ with $\lim_{n \rightarrow \infty} f_n = f$, which implies $\|f\| = \lim_{n \rightarrow \infty} \|f_n\| = d$.

3. (HS-1) Is $N := 21^{39} + 39^{21}$ divisible by 45? Why, why not?

Reason: Puzzle.

Solution: $45 = 9 \cdot 5$ and $9|N$, so it remains to show that $5|N$. The last digit of 21^n is 1 and the last digit of 39^{2n+1} is 9 for any natural number n . Hence $10|N$ and especially $5|N$.

4. (HS-2) Let $0 < u, v, w < 1$. Show that among the numbers $u(1-v)$, $v(1-w)$, $w(1-u)$ is at least one value not greater than $\frac{1}{4}$.

Reason: Puzzle.

Solution: Let us assume

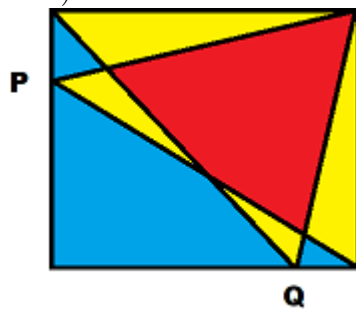
$$uvw(1-u)(1-v)(1-w) > \left(\frac{1}{4}\right)^3 = \frac{1}{64}$$

But $u(1-u) = \frac{1}{4} - (u - \frac{1}{2})^2 \leq \frac{1}{4}$ and likewise $v(1-v) \leq \frac{1}{4}$, and $w(1-w) \leq \frac{1}{4}$. Multiplying all three inequalities yields

$$u(1-u)v(1-v)w(1-w) \leq \frac{1}{64}$$

against our assumption.

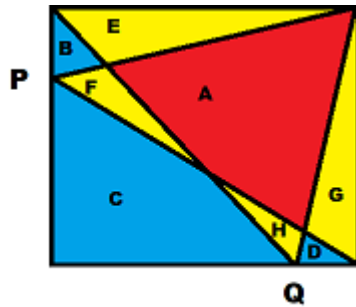
5. (HS-3) What is the ratio between the red and the blue area?



The points P and Q are anywhere on their edges.

Reason: Geometry.

Solution: Let's first label the areas and call the area of the square X .

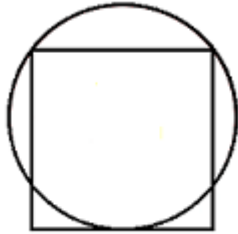


Since height and baseline of both triangles equal the side length of the square, their area is half of X :

$$A + E + H = \frac{1}{2}X = A + F + G = B + C + D + F + G = B + C + D + H + E$$

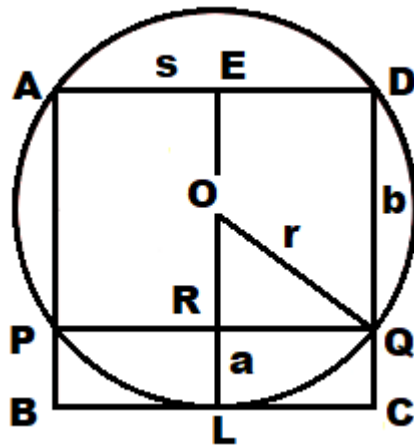
This means $\frac{A}{B + C + D} = \frac{A}{\frac{1}{2}X - F - G} = \frac{A}{A} = 1$

6. (HS-4) In what ratio does the circumference of the circle divide the left and right sides of the square?



Reason: Geometry.

Solution: Let's first label the graphic.



Pythagoras for $\triangle ORQ$ gives us $r^2 = \left(\frac{b}{2}\right)^2 + \left(\frac{s}{2}\right)^2$ and

$$s = \overline{AD} = \overline{AB} = \overline{EOL} = EO + OL = \frac{b}{2} + r$$

Thus $\left(s - \frac{b}{2}\right)^2 = \left(\frac{b}{2}\right)^2 + \left(\frac{s}{2}\right)^2$ or $bs = \frac{3}{4}s^2$ or $\frac{s}{b} = \frac{4}{3}$.

From $a + b = s$ we get $\frac{a}{b} = \frac{1}{3}$.