Mathematical Challenges

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1. Let $\mathcal{D}_N := \left\{ x^n \dfrac{d}{dx}, \,|\, \mathbb{Z} \ni n \geq N \right\}$ be a set of linear operators on smooth real functions. For which values of $N \in \mathbb{Z} \cup \{\pm\infty\}$ do they generate a real Lie algebra, and are there isomorphic ones among them? Note that any linear combination of basis vectors has only finitely many nonzero coefficients.

   Reason: Infinite-Dimensional Lie Algebras.

   Solution: Let $\mathfrak{D}_N$ be the Lie algebra generated by $\mathcal{D}_N$. Then

$$\left[x^n \frac{d}{dx}, x^m \frac{d}{dx} \right] = x^n \frac{d}{dx} \circ x^m \frac{d}{dx} - x^m \frac{d}{dx} \circ x^n \frac{d}{dx} = (m-n) x^{n+m-1} \frac{d}{dx}$$

   defines a closed Lie structure for the values $N \in \{-\infty, 0, 1, \ldots, +\infty\}$. Now

$$\mathfrak{D}_N / [\mathfrak{D}_N, \mathfrak{D}_N] = \operatorname{span} \left\{ x^n \frac{d}{dx} \,|\, N \leq n \leq 2N-1 \right\}$$

   are of different dimensions N for different values of $N \geq 1$. Hence none of them are isomorphic. Since $[\mathfrak{D}_0, \mathfrak{D}_0] = \mathfrak{D}_0$ this is true for $N \geq 0$. Moreover $\mathfrak{D}_{+\infty} = \{0\}$ and $[\mathfrak{D}_{-\infty}, \mathfrak{D}_{-\infty}] = \mathfrak{D}_{-\infty}$.

   Since any linear combinations of the basis vectors contains only finitely many nonzero coefficients, $\mathfrak{D}_{-\infty} \not\cong \mathfrak{D}_0$:

   Let $D_n := x^n \dfrac{d}{dx}$. If $X = \sum_{k \geq 0} x_k D_k \in \mathfrak{D}_0$ with $m = \max\{k \,|\, x_k \neq 0\}$ then

$$\begin{aligned}
(\operatorname{ad} D_0)^{m+1}(X) &= \sum_{k=1}^{m} (\operatorname{ad} D_0)^m x_k k D_{k-1} \\
&= \sum_{k=2}^{m} (\operatorname{ad} D_0)^m x_k k(k-1) D_{k-2} \\
&\vdots \\
&= \sum_{k=m}^{m} (\operatorname{ad} D_0) x_k \frac{k!}{(k-m)!} D_{k-m} \\
&= x_m m! [D_0, D_0] = 0
\end{aligned}$$

   Assume there is an isomorphism $\varphi : \mathfrak{D}_0 \longrightarrow \mathfrak{D}_{-\infty}$. Then

$$(\operatorname{ad} \varphi(D_0))^k (\varphi(X)) = \varphi((\operatorname{ad} D_0)^k (X))$$

Hence there is an element $Y := \varphi(D_0) \in \mathfrak{D}_{-\infty}$ such that for any vector $\varphi(X) \in \mathfrak{D}_{-\infty}$ there is an $n \in \mathbb{N}$ such that $(\operatorname{ad} Y)^n(\varphi(X)) = 0$. Since φ is an isomorphism, this is especially true for all $\varphi(X) = D_n$ $(n \in \mathbb{Z})$. Let $V_m := \operatorname{span}\{\ldots, D_{m-1}, D_m\}$. Say

$$Y = D_m + R \text{ with } R \in V_{m-1}$$

$$\begin{aligned}
[Y, D_n] &= [D_m, D_n] + [R, D_n] \equiv (n-m)D_{n+m-1} \mod V_{n+m-2} \\
[Y, [Y, D_n]] &= (n-m)(n-1)D_{n+2m-2} \mod V_{n+2m-3} \\
[Y, [Y, [Y, D_n]]] &= (n-m)(n-1)(n+m-2)D_{n+3m-3} \mod V_{n+3m-4} \\
&\vdots
\end{aligned}$$

Thus $\operatorname{ad} Y$ acts nilpotent on D_n only if the leading coefficient becomes zero, i.e. if

$$\begin{aligned}
0 &= (n-m)(n-1)(n+m-2)(n+2m-3)(n+3m-4)\cdots \\
&= (n-m)(n-m+1\cdot(m-1)) + (n-m+2\cdot(m-1))\cdots \\
&\Longleftrightarrow \\
0 &= (n-m) + k(m-1) = n-k+m(k-1) \text{ for some } k \in \mathbb{N}_0 \\
&\Longleftrightarrow \\
k(m-1) &= m-n \text{ for some } k \in \mathbb{N}_0
\end{aligned}$$

If $m \notin \{0,1,2\}$ then we choose $n = 2$ so $k = \dfrac{m-2}{m-1} \notin \mathbb{N}_0$, and $(\operatorname{ad} Y)^k(D_2) \neq 0$ for all $k \in \mathbb{N}_0$. If $m = 2$ we choose $n = 3$, and again we get a $k = -1 \notin \mathbb{N}_0$. For $m = 1$ we can also choose $n = 2$ without a solution for $k \in \mathbb{N}_0$. It remains to consider the case $m = 0$ in which case we choose $n = -1$ so that there is no solution for $k \in \mathbb{N}_0$.

2. Let $c \in (0,1)$. Show that the function $f : [0,c] \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} -\dfrac{1}{\log x} & \text{if } 0 < x \leq c \\ 0 & \text{if } x = 0 \end{cases}$$

is uniformly continuous, but not Hölder continuous.

Reason: Hölder Continuity.

Solution: Set $\delta := e^{-1/\varepsilon}$ for an $\varepsilon > 0$. Then

$$|f(x) - f(0)| = |-\frac{1}{\log x} - 0| = -\frac{1}{\log x} < -\frac{1}{\log \delta} = \varepsilon$$

for all $|x| < \delta$ which shows that $f(x)$ is continuous at 0, hence continuous on $[0, c]$.

For the sake of completion we will show that every continuous function on a closed interval $[a, b]$ is automatically uniformly continuous (Theorem of Heine).

Assume $f(x)$ was not uniformly continuous. Then there is an $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ exist points $y_n, z_n \in [a, b]$ with

$$|y_n - z_n| < \frac{1}{n} \text{ and } |f(y_n) - f(z_n)| \geq \varepsilon$$

We choose a convergent subsequence $(y_{n_k}) \subseteq (y_n)$ by the theorem of Bolzano-Weierstraß. Say $p := \lim_{n\to\infty} y_{n_k} \in [a, b]$. Since $|y_{n_k} - zn_k| < n_k^{-1}$ we have $p := \lim_{n\to\infty} z_{n_k}$, too. By continuity of $f(x)$ we thus get

$$\left| \lim_{n\to\infty} (f(y_{n_k}) - f(z_{n_k})) \right| = f(p) - f(p) = 0$$

which contradicts $|f(y_{n_k}) - f(z_{n_k})| \geq \varepsilon$ for all $k \in \mathbb{N}$.

It remains to show that $f(x)$ is not Hölder continuous. Assume it is and there are $C > 0$ and $\alpha \in (0, 1]$ such that

$$|f(x) - f(0)| = f(x) \leq Cx^\alpha \ \forall\ x \in (0, c]$$

This means by the rule of L'Hôpital

$$C \geq \lim_{x\downarrow 0} \frac{f(x)}{x^\alpha} = \lim_{x\downarrow 0} \frac{(-x^{-\alpha})'}{(\log x)'} = \lim_{x\downarrow 0} \alpha x^{-\alpha} = \lim_{x\downarrow 0} \frac{\alpha}{x^\alpha} = \infty$$

which is obviously a contradiction.

3. Consider the equation $pV - C(A - B\sqrt{p} + T) = 0$ where A, B, C are constant parameters, $p = p(T, V)$ vapor pressure, $V = V(T, p)$ molar volume, and $T = T(p, V)$ absolute temperature. Prove by three *different* methods that

$$\left(\frac{\partial V}{\partial T}\right)_p \cdot \left(\frac{\partial T}{\partial p}\right)_V \cdot \left(\frac{\partial p}{\partial V}\right)_T = -1$$

Reason: Antoine Equation.

Solution: Let $f(p, V, T) = pV - C(A - B\sqrt{p} + T) = 0$.

(a) Implicit Function Theorem.

We get from the implicit function theorem applied to resp.

$$f(T,V(T))=0 \Longrightarrow \frac{\partial f}{\partial T}+\frac{\partial f}{\partial V}\cdot\frac{\partial V}{\partial T}=0 \Longrightarrow \left(\frac{\partial V}{\partial T}\right)_p = \left(-\frac{\left(\frac{\partial f}{\partial T}\right)}{\left(\frac{\partial f}{\partial V}\right)}\right)$$

$$f(p,T(p))=0 \Longrightarrow \frac{\partial f}{\partial p}+\frac{\partial f}{\partial T}\cdot\frac{\partial T}{\partial p}=0 \Longrightarrow \left(\frac{\partial T}{\partial p}\right)_V = \left(-\frac{\left(\frac{\partial f}{\partial p}\right)}{\left(\frac{\partial f}{\partial T}\right)}\right)$$

$$f(V,p(V))=0 \Longrightarrow \frac{\partial f}{\partial V}+\frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial V}=0 \Longrightarrow \left(\frac{\partial p}{\partial V}\right)_T = \left(-\frac{\left(\frac{\partial f}{\partial V}\right)}{\left(\frac{\partial f}{\partial p}\right)}\right)$$

Multiplication and reducing the quotients yield the result.

(b) Implicit Differentiation.

$$\frac{\partial f}{\partial T}=0=p\frac{\partial V}{\partial T}-C \Longrightarrow \left(\frac{\partial V}{\partial T}\right)_p=\frac{C}{p}$$

$$\frac{\partial f}{\partial p}=0=V+\frac{CB}{2\sqrt{p}}-C\frac{\partial T}{\partial p} \Longrightarrow \left(\frac{\partial T}{\partial p}\right)_V=\frac{V}{C}+\frac{B}{2\sqrt{p}}$$

$$\begin{aligned}\frac{\partial f}{\partial V}&=0=V\cdot\frac{\partial p(V)}{\partial V}+p+BC\frac{\partial\sqrt{p(V)}}{\partial V}\\ &=V\left(\frac{\partial p}{\partial V}\right)_T+p+\frac{BC}{2\sqrt{p}}\cdot\left(\frac{\partial p}{\partial V}\right)_T\\ &\Longrightarrow \left(\frac{\partial p}{\partial V}\right)_T=-\frac{p}{V+\frac{BC}{2\sqrt{p}}}\end{aligned}$$

Hence $\left(\dfrac{\partial V}{\partial T}\right)_p \cdot \left(\dfrac{\partial T}{\partial p}\right)_V \cdot \left(\dfrac{\partial p}{\partial V}\right)_T$ is equal to

$$\begin{aligned}
&\frac{C}{p} \cdot \left(\frac{V}{C} + \frac{B}{2\sqrt{p}}\right) \cdot \left(-\frac{2p\sqrt{p}}{2V\sqrt{p}+BC}\right) \\
&\quad = -\left(\frac{2V\sqrt{p}}{2p\sqrt{p}} + \frac{BC}{2p\sqrt{p}}\right) \cdot \left(-\frac{2p\sqrt{p}}{2V\sqrt{p}+BC}\right) = -1
\end{aligned}$$

(c) Solving for the Functions.

$$f(p,V,T) = 0 \Longrightarrow V = \frac{C}{p} \cdot (A - B\sqrt{p} + T) \Longrightarrow \left(\frac{\partial V}{\partial T}\right)_p = \frac{C}{p}$$

$$f(p,V,T) = 0 \Longrightarrow T = \frac{pV}{C} - A + B\sqrt{p} \Longrightarrow \left(\frac{\partial T}{\partial p}\right)_V = \frac{V}{C} + \frac{B}{2\sqrt{p}}$$

To solve $f(p,V,T) = 0$ for p, we have to consider a quadratic equation in $\sqrt{p} > 0$.

$$\begin{aligned}
0 &= p + \frac{CB}{V}\sqrt{p} - \frac{CA + CT}{V} \\
\sqrt{p} &= -\frac{CB}{2V} + \sqrt{\frac{C^2B^2}{4V^2} + \frac{CA+CT}{V}} \\
p &= \left(\frac{1}{2V}\left(-CB + \sqrt{C^2B^2 + 4VC(A+T)}\right)\right)^2
\end{aligned}$$

$$\begin{aligned}
\left(\frac{\partial p}{\partial V}\right)_T &= \frac{\partial p}{\partial V} = \frac{\partial(\sqrt{p}^2)}{\partial V}) = 2\sqrt{p}\cdot\frac{\partial(\sqrt{p})}{\partial V}\\
&= 2\sqrt{p}\cdot\frac{\partial}{\partial V}\left\{\frac{1}{2V}\left(-CB+\sqrt{C^2B^2+4VC(A+T)}\right)\right\}\\
&= 2\sqrt{p}\cdot\left\{-\frac{1}{2V^2}\left(-CB+\sqrt{C^2B^2+4VC(A+T)}\right)\right\}\\
&+ 2\sqrt{p}\cdot\frac{1}{2V}\cdot\frac{1}{2}\cdot\frac{4C(A+T)}{\sqrt{C^2B^2+4VC(A+T)}}\\
&= -\frac{\sqrt{p}}{V^2}\left(-CB+CB+2V\sqrt{p}\right)+\frac{2\sqrt{p}C(A+T)}{V\cdot\left(CB+2V\sqrt{p}\right)}\\
&= -\frac{2p}{V}+\frac{2\sqrt{p}\left(pV+CB\sqrt{p}\right)}{V\cdot\left(CB+2V\sqrt{p}\right)}\\
&= \frac{-2pCB-4pV\sqrt{p}+2\sqrt{p}pV+2CBp}{V\cdot\left(CB+2V\sqrt{p}\right)}\\
&= -\frac{2p\sqrt{p}}{CB+2V\sqrt{p}} = \frac{-p}{V+\dfrac{CB}{2\sqrt{p}}}
\end{aligned}$$

and thus

$$\begin{aligned}
\left(\frac{\partial V}{\partial T}\right)_p\cdot\left(\frac{\partial T}{\partial p}\right)_V\cdot\left(\frac{\partial p}{\partial V}\right)_T &= \frac{C}{p}\cdot\left(\frac{V}{C}+\frac{B}{2\sqrt{p}}\right)\cdot\frac{-p}{V+\dfrac{CB}{2\sqrt{p}}}\\
&= \left(\frac{V}{p}+\frac{CB}{2p\sqrt{p}}\right)\cdot\frac{-2p\sqrt{p}}{2V\sqrt{p}+CB}\\
&= \frac{-2V\sqrt{p}}{2V\sqrt{p}-CB}-\frac{CB}{2V\sqrt{p}+CB} = -1
\end{aligned}$$

4. Calculate

$$\left(\frac{\partial V}{\partial T}\right)_p \text{ and } \left(\frac{\partial V}{\partial p}\right)_T$$

for $V = V(T,p)$ from the equation of state

$$\left(p+\frac{a}{V^2}\right)(V-b) = R\cdot T\,;\; a,b,R>0$$

Reason: Van der Waals Equation.

Solution:

(a) If we differentiate $\left(p+\dfrac{a}{V^2}\right)(V-b)=R\cdot T$ along T with constant p we get

$$\begin{aligned}
R &= (V-b)\left(\frac{\partial}{\partial T}\left(p+\frac{a}{V^2}\right)\right)_p+\left(p+\frac{a}{V^2}\right)\left(\frac{\partial}{\partial T}(V-b)\right)_p\\
&= \frac{2(b-V)a}{V^3}\left(\frac{\partial V}{\partial T}\right)_p+\left(p+\frac{a}{V^2}\right)\left(\frac{\partial V}{\partial T}\right)_p\\
&= \frac{1}{V^3}\left(2ab-aV+pV^3\right)\left(\frac{\partial V}{\partial T}\right)_p\\
\left(\frac{\partial V}{\partial T}\right)_p &= \frac{RV^3}{pV^3-aV+2ab}
\end{aligned}$$

(b) If we differentiate $\left(p+\dfrac{a}{V^2}\right)(V-b)=R\cdot T$ along p with constant T we get

$$\begin{aligned}
0 &= (V-b)\left(\frac{\partial}{\partial p}\left(p+\frac{a}{V^2}\right)\right)_T+\left(p+\frac{a}{V^2}\right)\left(\frac{\partial}{\partial p}(V-b)\right)_T\\
&= (V-b)\left(1-\frac{2a}{V^3}\left(\frac{\partial V}{\partial p}\right)_T\right)+\left(p+\frac{a}{V^2}\right)\left(\frac{\partial V}{\partial p}\right)_T\\
&= (V-b)+\left(\frac{2a(b-V)}{V^3}+\frac{pV^3}{V^3}+\frac{aV}{V^3}\right)\left(\frac{\partial V}{\partial p}\right)_T\\
\left(\frac{\partial V}{\partial p}\right)_T &= \frac{V^3(b-V)}{pV^3-aV+2ab}
\end{aligned}$$

5. Let $\sigma\in\operatorname{Aut}(S_n)$ be an automorphism of the symmetric group S_n $(n\geq 4)$ such that σ sends transpositions to transpositions, then prove that σ is an inner automorphism. Determine the inner automorphism groups of the symmetric and the alternating groups for $n\geq 4$.

Reason: Inner Automorphisms of Permutation Groups.

Solution: Suppose that $\sigma(1,r)=(a_r,b_r)$ for each $r\in\{1,2,\ldots,n\,|\,n>3\}$. Then for $r\geq 3$

$$\sigma(1,r,2)=\sigma((1,2)(1,r))=\sigma(1,2)\sigma(1,r)=(a_2,b_2)(a_r,b_r)$$

is an element of order 3, hence either $a_r\in\{a_2,b_2\}$ or $b_r\in\{a_2,b_2\}$. By symmetry reasons we may assume that $a_r\in\{a_2,b_2\}$ for all $r\geq 3$.

We claim that either $a_r = a_2$ for all r or $a_r = b_2$ for all r. Assume that instead there are $r \neq s$ such that $a_r = a_2$ and $a_s = b_2$. Note that $(1,r,2)(1,s,2) = (1,s)(2,r)$ is of order 2. However,

$$\begin{aligned}\sigma((1,r,2)(1,s,2)) &= (a_2,b_2)(a_r,b_r)(a_2,b_2)(a_s,b_s)\\ &= (a_2,b_2)(a_2,b_r)(a_2,b_2)(b_2,b_s) = (b_2,b_s,b_r)\end{aligned}$$

is of order 3 if $2 \neq r \neq s \neq 2$. This is a contradiction. Thus we either have $a_2 = a_r$ or $b_2 = b_r$ for all $r \geq 3$. W.l.o.g. let $a_2 = a_r$, i.e. $\sigma(1,r) = (a_2, b_r)$ for all $r \geq 3$. Since σ is an isomorphism, we have $b_r \neq b_s$ if $r \neq s$ because $\sigma(1,r) \neq \sigma(1,s)$. Let π be a permutation for which $\pi(1) = a_2$ and $\pi(r) = b_r$ for all $r \geq 3$. This uniquely determines π, because we determined $n-1$ values, and bijectivity determines the last value. Now

$$\sigma(1,r) = (a_r,b_r) = (a_2,b_r) = \pi \circ (1,r) \circ \pi^{-1}$$

and $\sigma = \operatorname{Inn}(\pi) \in \operatorname{Inn}(S_n)$.

Consider $G \stackrel{\pi}{\longrightarrow} \operatorname{Inn}(G)$ defined by $g \longmapsto (x \mapsto g^{-1}xg)$. Then $\ker \pi = Z(G)$ and $\operatorname{Inn}(G) \cong G/Z(G)$. Since the alternating groups A_n are simple for $n > 4$, we have $\operatorname{Inn}(A_n) \cong A_n$. The symmetric groups S_n for $n > 4$ have only A_n as nontrivial normal subgroup, i.e. $S_n \cong A_n \rtimes \mathbb{Z}_2$. Furthermore the centers of S_n are trivial, i.e. $\operatorname{Inn}(S_n) \cong S_n$. In case of $n = 4$ we also have the Klein four-group

$$V_4 = [A_4, A_4] \triangleleft A_4 \triangleleft S_4.$$

Since V_4 is not abelian, it cannot be the center of either permutation group, i.e. $\operatorname{Inn}(A_4) \cong A_4$ and $\operatorname{Inn}(S_4) \cong S_4$.

6. Consider a code $C \subseteq \mathbb{F}_q^n$ with minimal Hamming distance $d > n \cdot \dfrac{q-1}{q}$. Prove that the number of possible code words is restricted by

$$c := \#C \leq \frac{d}{d - n \cdot \dfrac{q-1}{q}}$$

Reason: Plotkin Bound.

Solution: Let $s := \sum_{(x,y)\in C\times C} d(x,y)$ be the sum of all Hamming

distances in C. Since $d(x,y) \geq d$ for different code words x, y we immediately have
$$s \geq c \cdot (c-1) \cdot d.$$
We want to show that
$$s \leq c^2 \cdot n \cdot \left(1 - \frac{1}{q}\right) = c^2 \cdot n \cdot \frac{q-1}{q}$$
from which we get
$$c(c-1)d \leq c^2 n \frac{q-1}{q} \Longleftrightarrow cd - cn\frac{q-1}{q} \leq d \Longleftrightarrow c \leq \frac{d}{d - n\dfrac{q-1}{q}}$$
which we will have to show. Let $C = \{x^{(1)}, \ldots, x^{(c)}\}$. We define the number of code words that have an $a \in \mathbb{F}_q$ at $k-$th position by
$$t_k(a) := \#\{1 \leq j \leq c \,|\, x_k^{(j)} = a\}.$$
Obviously $\sum_{a \in \mathbb{F}_q} t_k(a) = c$. The number of pairs $(x,y) \in C \times C$ which are different at position k is $\sum_{a \in \mathbb{F}_q} t_k(a)(c - t_k(a))$. Therefore the sum of all Hamming distances equals
$$s = \sum_{k=1}^{n} \sum_{a \in \mathbb{F}_q} t_k(a)(c - t_k(a)) = \sum_{k=1}^{n} \left(c^2 - \sum_{a \in \mathbb{F}_q} t_k(a)^2\right)$$
According to the Cauchy-Schwarz inequality we have
$$\left(\sum_{a \in \mathbb{F}_q} t_k(a)\right)^2 \leq \left(\sum_{a \in \mathbb{F}_q} t_k(a)^2\right) \cdot \left(\sum_{a \in \mathbb{F}_q} 1^2\right) = q \cdot \sum_{a \in \mathbb{F}_q} t_k(a)^2$$
$$-\sum_{a \in \mathbb{F}_q} t_k(a)^2 \leq -\frac{1}{q}\left(\sum_{a \in \mathbb{F}_q} t_k(a)\right)^2$$
$$\begin{aligned} s = \sum_{k=1}^{n} \left(c^2 - \sum_{a \in \mathbb{F}_q} t_k(a)^2\right) &\leq \sum_{k=1}^{n} \left(c^2 - \frac{1}{q}\left(\sum_{a \in \mathbb{F}_q} t_k(a)\right)^2\right) \\ &= \sum_{k=1}^{n} \left(c^2 - \frac{1}{q}c^2\right) = nc^2\left(1 - \frac{1}{q}\right) \end{aligned}$$

7. Prove that the Cantor dust on the real line contains uncountable infinitely many points, and that it is a fractal by calculating its Hausdorff-Besicovitch dimension.

   Reason: Cantor Set.

   Solution: Cantor dust is constructed by cutting out the interval $(1/3, 2/3)$ from $[0, 1]$, then cutting out the middle third from the remaining intervals and so on.

   We can ask at any step of the construction for any remaining point, whether a point is placed on the left or on the right from the nearest removed interval. Let's write a "0" for left and a "1" for right. The result is an infinite, binary sequence that determines the given point of the Cantor dust, and vice versa: each such sequence determines a point in the Cantor dust. The Cantor dust is therefore equivalent to $[0, 1]^{\mathbb{N}}$, which is equivalent to the real numbers. Since the real numbers are uncountable infinitely many, the Cantor dust is, too.

   To determine the fractal Hausdorff dimension D_H, we cover the object with the least number $N(\varepsilon)$ of circles of diameter ε and define

   $$D_H = -\lim_{\varepsilon\to 0} \log_\varepsilon N(\varepsilon).$$

   This means in our case

ε	1	1/3	1/9	1/27	1/81	...
$N(\varepsilon)$	1	2	4	8	16	...

   i.e. $N(3^{-n}) = 2^n$ and $N(\varepsilon) = 2^{-\log_3 \varepsilon} = \varepsilon^{-\frac{\log 2}{\log 3}}$, i.e. $D_H = \dfrac{\log 2}{\log 3} \approx 0.631$. The Cantor dust is a fractal, since $D_H > D_T = 0$, the topological dimension D_T.

8. Define the harmonic number $H(p) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{p-1} = \frac{a}{b}$. Show that $p^2 \,|\, a$ for primes $p > 3$.

   Reason: Theorem of Wolstenholme.

   Solution: Consider the polynomial

$$f(x) = \prod_{k=1}^{p-1}(x-k) = x^{p-1} - a_{p-2}(p)x^{p-2} \pm \ldots - a_1(p)x + a_0(p)$$

   where $a_0(p) = (p-1)!$ and $a_1(p) = (p-1)!H(p)$. We want to show that $p^2 \,|\, a_1(p)$. We therefore pass to the finite field $\mathbb{F}_p$, so that $\mathbb{F}_p[x] \ni f(x) \stackrel{(*)}{=} x^{p-1} - 1$. Hence $p \,|\, a_k(p)$ for all $1 \leq k \leq p-2$ and $a_0(p) = (p-1)! \equiv -1 \mod p$. This is known as Wilson's theorem.

   $(*)$ The polynomial $x^p - x = x(x^{p-1} - 1)$ has only simple zeros, because $(x^p - x)' = px^{p-1} - 1 \not\equiv 0 \mod p$. But there are at most p numbers in $\mathbb{F}_p$ for the p many zeros, hence $\mathbb{F}_p[x] \ni x(x-1)(x-2)\cdots(x-(p-1)) = x^p - x = x \cdot f(x)$.

$$\begin{aligned} f(p) = (p-1)! &= p^{p-1} - a_{p-2}(p)p^{p-2} \pm \ldots - a_1(p)p + a_0(p) \\ &= p^{p-1} - a_{p-2}(p)p^{p-2} \pm \ldots - a_1(p)p + (p-1)! \\ a_1(p) &= p^{p-2} - a_{p-2}(p)p^{p-3} \pm \ldots + a_2(p)p \end{aligned}$$

   From $p > 3$ and $p \,|\, a_k$ follows that $p^2 \,|\, a_1(p) = (p-1)!H(p)$ and thus $p^2 \,|\, H(p)$ because p is prime.

9. An ideal coin is thrown three times in a row and then an ideal dice is thrown twice in a row. Each time you toss a coin you get one point if the coin shows "tails" and two points if the coin shows "heads". If you add the total of the two dice rolls to this number of points, you get the total number of points. Furthermore, let A be the event "the total number of points achieved is odd", B be the event "the total of the two dice rolls is divisible by 5", and C the event "the number of points achieved in the three coin tosses is at least 5". Investigate whether A, B, C are pairwise stochastically independent. Also investigate whether A, B, C are stochastically independent.

   Reason: Stochastic.

   Solution: The phase space $\Omega = \{1,2\}^3 \times \{1,2,\ldots,6\}^2$ and $p(\omega) := \frac{1}{|\Omega|} = \frac{1}{2^3 \cdot 6^2} = \frac{1}{288}$ for all $\omega \in \Omega$, such that (Ω, p) is a Laplace experiment. With $\vec{\omega}=(\omega_1,\ldots,\omega_5)\in\Omega$ let ω_i for $i\in\{1,2,3\}$ be the points achieved in the $i-$th coin flip, ω_4 the points of the first die roll, ω_5 the points of the second. Then

$$A=\left\{(\omega_1,\ldots,\omega_5)\in\Omega\,|\,\sum_{i=1}^{5}\omega_i=2n-1 \text{ for some } n\in\mathbb{N}\right\},$$

$$B=\{(\omega_1,\ldots,\omega_5)\in\Omega\,|\,\omega_4+\omega_5\in\{5,10\}\},$$

$$C=\left\{(\omega_1,\ldots,\omega_5)\in\Omega\,|\,\sum_{i=1}^{3}\omega_i=\in\{5,6\}\right\}.$$

We can choose $\omega_1,\ldots,\omega_4$ arbitrarily and have 3 possibilities left to an odd total number of points. There are 4 possibilities to achieve 5 by rolling the dice, and 3 to get 10. To end up with 5, resp. 6 points in the coin flips, there are 3, resp. 1 chances. Hence

$$P(A)=\frac{2^3\cdot 6\cdot 3}{288}=\frac{1}{2},P(B)=\frac{2^3\cdot(4+3)}{288}=\frac{7}{36},P(C)=\frac{(3+1)\cdot 6^2}{288}=\frac{1}{2}$$

Moreover

$$\begin{aligned}\vec{\omega}\in A\cap B \Longleftrightarrow &\left(\sum_{k=1}^{3}\omega_k\in\{4,6\}\wedge\omega_4+\omega_5=5\right)\\ &\vee\left(\sum_{k=1}^{3}\omega_k\in\{3,5\}\wedge\omega_4+\omega_5=10\right)\end{aligned}$$

$$P(A\cap B)=\frac{4\cdot 4+4\cdot 3}{288}=\frac{7}{72}=\frac{1}{2}\cdot\frac{7}{36}=P(A)\cdot P(B)$$

$$\begin{aligned}\vec{\omega}\in A\cap C \Longleftrightarrow &\left(\sum_{k=1}^{3}\omega_k=5\wedge\omega_4+\omega_5\in\{2,4,6,8,10,12\}\right)\\ &\vee\left(\sum_{k=1}^{3}\omega_k=6\wedge\omega_4+\omega_5\in\{3,5,7,9,11\}\right)\end{aligned}$$

$$P(A\cap C)=\frac{3\cdot 18+1\cdot 18}{288}=\frac{72}{288}=\frac{1}{4}=\frac{1}{2}\cdot\frac{1}{2}=P(A)\cdot P(C)$$

$$\vec{\omega}\in B\cap C\Longleftrightarrow\sum_{k=1}^{3}\omega_k\in\{5,6\}\wedge\omega_4+\omega_5\in\{5,10\}$$

$$P(B\cap C)=\frac{4\cdot 7}{288}=\frac{7}{72}=\frac{1}{2}\cdot\frac{7}{36}=P(B)\cdot P(C)$$

which proves that the events are pairwise independent. Finally we have

$$\begin{aligned}\vec{\omega} \in A \cap B \cap C \Longleftrightarrow &\left(\sum_{k=1}^{3} \omega_k = 6 \wedge \omega_4 + \omega_5 = 5\right)\\ &\vee \left(\sum_{k=1}^{3} \omega_k = 5 \wedge \omega_4 + \omega_5 = 10\right)\end{aligned}$$

$$P(A \cap B \cap C) = \frac{1 \cdot 4 + 3 \cdot 3}{288} = \frac{13}{288} \neq \frac{14}{288} = \frac{1}{2} \cdot \frac{7}{36} \cdot \frac{1}{2} = P(A) \cdot P(B) \cdot P(C)$$

which means that they are not independent as a whole.

10. Show

$$C_n := \binom{2n}{n} - \binom{2n}{n+1} = \prod_{k=1}^{n} \frac{4k-2}{k+1}$$

and determine all primes in $\{C_n\}$.

Reason: Catalan Numbers.

Solution:

$$\begin{aligned}C_n = \binom{2n}{n} - \binom{2n}{n+1} &= \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n+1)!(n-1)!}\\ &= \frac{(2n)!(n+1) - (2n)!n}{n!(n+1)!} = \frac{(2n)!}{n!(n+1)!}\end{aligned}$$

$$\begin{aligned}\frac{C_{n+1}}{C_n} &= \frac{(2n+2)!}{(n+1)!(n+2)!} \cdot \frac{n!(n+1)!}{(2n)!}\\ &= \frac{(2n+2)(2n+1)}{(n+2)(n+1)} = \frac{2(2n+1)}{n+2} = \frac{4n+2}{n+2}\end{aligned}$$

Since $C_1 = 1$ we get

$$C_n = \frac{4n-2}{n+1} \cdot C_{n-1} = \frac{4n-2}{n+1} \cdot \frac{4n-6}{n} \cdot C_{n-2} = \ldots = \prod_{k=1}^{n} \frac{4k-2}{k+1}$$

$$\begin{aligned}C_0 = C_1 = 1\,,\ C_2 = 2\,,\ C_3 = 5\,,\ C_4 = 14\,,\ C_5 = 42\\ C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{2n(2n-1) \cdot \ldots \cdot (n+2)}{n!}\\ = 2 \cdot \frac{2n-1}{n-1} \cdot \ldots \cdot \frac{n+2}{2} > 2^{n-1} \geq 2n \text{ for } n \geq 4\end{aligned}$$

All prime factors of

$$C_n = 2^n \cdot \frac{(2n-1)\cdot(2n-2)\cdot\ldots\cdot 1}{(n+1)\cdot n\cdot\ldots\cdot 1}$$

are less than $2n$. Thus it is impossible that C_n itself is prime if $n \geq 4$.

11. (HS-1) Check whether there is a natural number $n \in \mathbb{N}$ such that $\sqrt{n}+\sqrt{n+4} \in \mathbb{Q}$. Note that zero is no natural number.

    Reason: Irrational Numbers.

    Solution: Assume $\sqrt{n}+\sqrt{n+4} \in \mathbb{Q}$. Then

    $$\mathbb{Q} \ni \left(\sqrt{n}+\sqrt{n+4}\right)^2 = 2n+4+2\underbrace{\sqrt{n(n+4)}}_{\in\mathbb{Q}}$$

    If $\sqrt{n(n+4)} = \frac{a}{b}$, then $a^2 = b^2\cdot n\cdot(n+4)$ and $n(n+4)$ is a square number (e.g. by the fundamental theorem of arithmetic).

    $$\begin{aligned}&n^2+2n+2 \leq n^2+4n\\ &\quad(n+1)^2 = n^2+2n+1 < n^2+4n < n^2+4n+4 = (n+2)^2\\ &\quad n(n+4) = n^2+4n \in ((n+1)^2,(n+2)^2)\end{aligned}$$

    so $n(n+4)$ cannot be a square number contradicting our assumption.

12. (HS-2) Assume that $n \in \mathbb{N}$ is odd, and $\{a_1, a_2, \ldots a_n\} = \{1, 2, \ldots, n\}$. Prove that

    $$(a_1-1)\cdot(a_2-2)\cdot\ldots\cdot(a_{n-1}-(n-1))\cdot(a_n-n)$$

    is always even.

    Reason: Pigeon Hole Principle.

    Solution: $n = 2m+1$ for some $m \in \mathbb{N}$. Among the numbers $\{a_1, \ldots, a_n\} = \{1, \ldots, n\}$ are therefore at most m numbers even, namely $\{2, 4, \ldots, 2m\}$. The set $\{a_1, a_3, \ldots, a_n\}$ of numbers with an odd index contains $m+1 = n-m$ many numbers, so at least one of them has to be odd. Thus at least one of the factors $(a_1-1), (a_3-3), \ldots, (a_n-n)$ is an even number, i.e. the product $\prod_{k=1}^{n}(a_k-k)$ is even, too.

13. (HS-3) Show that for every natural number $n \in \mathbb{N}$ there is a $c = c(n) \in \mathbb{R}$ such that for all real numbers $a > 0$

    $$a + a^2 + a^3 + \ldots + a^{2n-1} + a^{2n} \leq c(n)\cdot\left(1+a^{2n+1}\right).$$

Show that there is a smallest solution among all possible values $c(n)$ and determine it.

Reason: Calculus.

Solution: Let $a > 0$ be a real number. Then we have

$$0 \leq 1 - (n+1)a^{2n+1} - a^{2n+2} + (n+1)a^{2n+3} \text{ for all } n \in \mathbb{N}_0$$

For $n = 0$ it is $1 - a - a^2 + a^3 = (1-a)(1-a^2) = (1-a)^2(1+a) \geq 0$ and for $n = 1$ we have $1 - 2a^3 - a^4 + 2a^5 = (a-1)^2(a+1)(2a^2+a+1) \geq 0$.

$$\begin{aligned}
&1 - (n+2)a^{2n+3} - a^{2n+4} + (n+2)a^{2n+5} \\
&= a^2(1 - (n+1)a^{2n+1} - a^{2n+2} + (n+1)a^{2n+3}) + 1 - a^2 - a^{2n+3} + a^{2n+5} \\
&\geq 1 - a^2 - a^{2n+3} + a^{2n+5} \\
&= (1-a^2)(1-a^{2n+3}) = (1-a)^2(1+a) \cdot \frac{1-a^{2n+3}}{1-a} \\
&= (1-a)^2(1+a)(1+a+a^2+\ldots+a^{2n+2}) \geq 0
\end{aligned}$$

We now prove again by induction (case $n = 1$ see above) that

$$a + a^2 + a^3 + \ldots + a^{2n-1} + a^{2n} \leq n \cdot \left(1 + a^{2n+1}\right).$$

$$\begin{aligned}
&0 \leq -a - a^2 - a^3 + \ldots - a^{2n-1} - a^{2n} + n + na^{2n+1} \text{ and} \\
&0 \leq 1 - (n+1)a^{2n+1} - a^{2n+2} + (n+1)a^{2n+3} \text{ hence} \\
&0 \leq 1 - (n+1)a^{2n+1} - a^{2n+2} + (n+1)a^{2n+3} + n + na^{2n+1} - a - \ldots - a^{2n} \\
&0 \leq 1 + n - a - \ldots - a^{2n+1} - a^{2n+2} + (n+1)a^{2n+3} \\
&\ a + a^2 + a^3 + \ldots + a^{2n} + a^{2n+1} + a^{2n+2} \leq (n+1)(1 + a^{2n+3})
\end{aligned}$$

Therefore $c = c(n) = n$ is a possible solution. It remains to show that it is already the minimal solution, i.e. for any real number $c < n$ we must find a real number $a > 0$ such that

$$a + a^2 + a^3 + \ldots + a^{2n-1} + a^{2n} > c(n) \cdot \left(1 + a^{2n+1}\right).$$

However, with $c < n$ we have $2c < 2n$ which is exactly our requirement if we choose $a = 1$.

14. (HS-4) Given an integer k, determine all pairs $(x, y) \in \mathbb{Z}^2$ such that

$$x^2 + k \cdot y^2 = 4 \text{ and } k \cdot x^2 - y^2 = 2$$

Reason: Conic Sections.

Solution: Assume (x,y) is an integer solution. Then

$$\begin{aligned} x^2+k(kx^2-2)=x^2(1+k^2)-2k=4 \\ 0<x^2(1+k^2)=2(k+2) \end{aligned}$$

We can therefore exclude all $k\leq -3$.

$$\begin{aligned}
k=-2 &: 0<2=-2x^2-y^2\leq 0 \lightning \\
k=-1 &: 0<2=-x^2-y^2\leq 0 \lightning \\
k=0 &: 0<2=-y^2\leq 0 \lightning \\
k=1 &: 2x^2=2\cdot 3=6\Longrightarrow x^2=3\Longrightarrow x\notin\mathbb{Z} \lightning \\
k=2 &: 5x^2=2\cdot 4=8\Longrightarrow x^2=\frac{8}{5}\Longrightarrow x\notin\mathbb{Z} \lightning \\
k=3 &: 10x^2=10\Longrightarrow x=\pm 1\Longrightarrow y=\pm 1 \\
k>3 &\wedge y^2=0 \Longrightarrow 2=k\cdot x^2=4\cdot k>12 \lightning \\
k>3 &\wedge x^2=0 \Longrightarrow y^2=-2 \lightning \\
k\geq 4 &\wedge x^2\geq 1\wedge y^2\geq 1 \Longrightarrow 4=x^2+ky^2\geq 1+4\cdot 1=5 \lightning
\end{aligned}$$

Our equation system is thus not solvable, except $k=3$, in which case all four pairs $\{(x,y)\in\mathbb{Z}^2\,|\,x=\pm1,y=\pm1\}$ are the only solution. It is easy to check, that these pairs are indeed solutions.

15. (HS-5) Prove for every natural number $n\in\mathbb{N}$

$$\frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\ldots\cdot 2n}<\frac{1}{\sqrt{2n+1}}$$

Reason: Inequality.

Solution: For $k=1,\ldots,n$

$$\begin{aligned}
(2k-1)\cdot(2k+1)=4k^2-1<4k^2 &\Longrightarrow \frac{2k-1}{2k}<\frac{2k}{2k+1} \\
&\Longrightarrow \prod_{k=1}^{n}\frac{2k-1}{2k}<\prod_{k=1}^{n}\frac{2k}{2k+1} \\
&\Longrightarrow \left(\prod_{k=1}^{n}\frac{2k-1}{2k}\right)^2<\left(\prod_{k=1}^{n}\frac{2k-1}{2k}\right)\cdot\left(\prod_{k=1}^{n}\frac{2k}{2k+1}\right)=\frac{1}{2n+1} \\
&\Longrightarrow \prod_{k=1}^{n}\frac{2k-1}{2k}<\frac{1}{\sqrt{2n+1}}
\end{aligned}$$

2 May 2021

1. Integrate
$$\int_0^\infty \int_0^\infty e^{-\left(x+y+\frac{\lambda^3}{xy}\right)} x^{-\frac{2}{3}} y^{-\frac{1}{3}}\,dx\,dy$$

Reason: Liouville's Formula.

Solution: Set $R(\lambda)=\int_0^\infty\int_0^\infty e^{-\left(x+y+\frac{\lambda^3}{xy}\right)} x^{-\frac{2}{3}} y^{-\frac{1}{3}}\,dx\,dy$ and $z:=\dfrac{\lambda^3}{xy}$. Then

$$\frac{dz}{dx}=-\frac{\lambda^3}{x^2y}=-\frac{\lambda^3}{y}\left(\frac{yz}{\lambda^3}\right)^2=-yz^2\Rightarrow dx=-\frac{\lambda^3}{yz^2}\,dz$$

$$\begin{aligned}R'(\lambda)&=-\int_0^\infty\int_0^\infty \frac{3\lambda^2}{xy}\cdot e^{-\left(x+y+\frac{\lambda^3}{xy}\right)}\cdot x^{-\frac{2}{3}}\cdot y^{-\frac{1}{3}}\,dx\,dy\\&=3\int_0^\infty\int_\infty^0 \frac{z}{\lambda}\cdot e^{-\left(\frac{\lambda^3}{yz}+y+z\right)}\frac{\lambda^3}{yz^2}\cdot\frac{1}{\lambda^2}\cdot y^{\frac{2}{3}}\cdot z^{\frac{2}{3}}\cdot y^{-\frac{1}{3}}\,dz\,dy\\&=-3\int_0^\infty\int_0^\infty e^{-\left(\frac{\lambda^3}{yz}+y+z\right)}\cdot y^{-\frac{2}{3}}\cdot z^{-\frac{1}{3}}\,dz\,dy\\&=-3R(\lambda)\end{aligned}$$

Hence $R(\lambda)=R(0)e^{-3\lambda}$ and

$$\begin{aligned}R(0)&=\int_0^\infty\int_0^\infty e^{-x-y}\cdot x^{-\frac{2}{3}}\cdot y^{-\frac{1}{3}}\,dx\,dy=\int_0^\infty x^{\frac{1}{3}-1}e^{-x}\,dx\cdot\int_0^\infty y^{\frac{2}{3}-1}e^{-y}\,dy\\&=\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right)=\frac{\pi}{\sin(\pi/3)}=\frac{2\pi}{\sqrt{3}}\text{ and }R(\lambda)=\frac{2\pi}{\sqrt{3}}\cdot e^{-3\lambda}\end{aligned}$$

2. Let F_n be the free group of rank n with generators $\{w_1,\ldots,w_n\}$. Then

$$\prod_{i=1}^m w_{a_i}^{b_i}\in[F_n,F_n]\Longleftrightarrow\forall_{k=1}^m\sum_{a_i=k}b_i=0$$

Reason: Abstract Algebra.

Solution: Denote the right-hand side property by (P). Then

(1) if two elements of F_n satisfy (P), then so does their product,
(2) each element of the form $[x,y]$ (i.e. each generator of $[F_n,F_n]$) satisfies (P).

which proves the direction from left to right.

Now assume (P). We proceed by the length of the word m. The cases $m = 0$ and $m = 1$ are obviously true.

$$w_k^{b_1} x w_k^{b_2} y = [w_k^{-b_1}, x^{-1}] x w_k^{b_1+b_2} y$$

If $w_k^{b_1} x w_k^{b_2} y$ satisfies (P), so does $x w_k^{b_1+b_2} y$. But now the length of the latter is one less than the length of the former and the induction hypothesis applies.

Another way to see the statement is as follows: For an arbitrary group G, we have $g \in [G, G]$ if and only if $\bar{g} = g[G, G] = \bar{1}$ in the abelianization $G/[G, G]$. For the free group $G = F_n$ we have $F_n/[F_n, F_n] = \mathbb{Z}^n$ and the stated property follows immediately.

3. Calculate
$$\int_0^\pi \int_0^\pi \int_0^\pi \frac{1}{1 - \cos x \, \cos y \, \cos z} \, dx \, dy \, dz$$

Reason: Watson Integral.

Solution: We start with the Weierstraß substitution $t = \tan(x/2)$.

$$\cos x = \frac{1 - t^2}{1 + t^2}$$
$$\frac{dt}{dx} = \frac{d}{dx} \tan\left(\frac{x}{2}\right) = \frac{1}{2}\left(1 + \tan^2\left(\frac{x}{2}\right)\right) = \frac{1 + t^2}{2} \Longrightarrow dx = \frac{2\, dt}{1 + t^2}$$

$$x = r \sin\theta \cos\varphi \,, \; dx = r \cos\theta \cos\varphi \, d\theta$$
$$y = r \sin\theta \sin\varphi \,, \; dy = r \sin\theta \cos\varphi \, d\varphi$$
$$z = r \cos\theta \,, \; dz = \cos\theta \, dr$$

and rewrite

$$\begin{aligned}
I &:= \int_0^\pi \int_0^\pi \int_0^\pi \frac{1}{1-\cos x\,\cos y\,\cos z}\,dx\,dy\,dz\\
&= 8\int_0^\infty \int_0^\infty \int_0^\infty \frac{\frac{1}{1+x^2}\cdot\frac{1}{1+y^2}\cdot\frac{1}{1+z^2}}{1-\frac{1-x^2}{1+x^2}\cdot\frac{1-y^2}{1+y^2}\cdot\frac{1-z^2}{1+z^2}}\,dx\,dy\,dz\\
&= 8\int_0^\infty \int_0^\infty \int_0^\infty \frac{dx\,dy\,dz}{(1+x^2)(1+y^2)(1+z^2)-(1-x^2)(1-y^2)(1-z^2)}\\
&= 4\int_0^\infty \int_0^\infty \int_0^\infty \frac{dx\,dy\,dz}{x^2+y^2+z^2+x^2y^2z^2}\\
&= 4\int_0^{\pi/2} \int_0^{\pi/2} \int_0^\infty \frac{r^2\sin\theta\,dr\,d\theta\,d\varphi}{r^2+r^2\sin^2\theta\cos^2\varphi\, r^2\sin^2\theta\sin^2\varphi\, r^2\cos^2\theta}\\
&= 4\int_0^{\pi/2} \int_0^{\pi/2} \int_0^\infty \frac{dr}{1+\underbrace{\left(r\sin\theta\sqrt{\cos\theta}\sqrt{\sin\varphi\cos\varphi}\right)}_{=:s}^4}\sin\theta\,d\theta\,d\varphi\\
&= 4\int_0^\infty \frac{ds}{1+s^4}\cdot\int_0^{\pi/2}\frac{d\theta}{\sqrt{\cos\theta}}\cdot\int_0^{\pi/2}\frac{d\varphi}{\sqrt{\sin\varphi\cos\varphi}}\\
&= 4\cdot\frac{\pi}{2\sqrt{2}}\cdot\frac{\Gamma\left(\frac{1}{4}\right)^2}{2\sqrt{2\pi}}\cdot\frac{\Gamma\left(\frac{1}{4}\right)^2}{2\sqrt{\pi}}\\
&= \frac{1}{4}\Gamma\left(\frac{1}{4}\right)^4 = 2\pi\varpi^2 = 2G^2\pi^3 \approx 43.198
\end{aligned}$$

with the Gauß constant $G = \dfrac{2}{\pi}\displaystyle\int_0^1 \frac{ds}{\sqrt{1-s^4}}$.

4. Let G be a finite group, $\mathbb{K}$ a field such that $\operatorname{char}(\mathbb{K}) \nmid |G|$, and (ρ, V) and (τ, W) linear representations of G over $\mathbb{K}$. The $\mathbb{K}-$linear mapping

$$\begin{aligned}
\operatorname{Sym} \,:\, \operatorname{Hom}_{\mathbb{K}}(V,W) &\longrightarrow \operatorname{Hom}_{\mathbb{K}}(V,W)\\
\varphi &\longmapsto \operatorname{Sym}(\varphi) = \frac{1}{|G|}\sum_{g\in G}\tau(g)\circ\varphi\circ\rho(g^{-1})
\end{aligned}$$

is a projection onto the subspace

$$\operatorname{Hom}_{\mathbb{K}}((\rho,V),(\tau,W)) = \{\vartheta : V \longrightarrow W \,|\, \forall_{g\in G}\,:\, \tau(g)\circ\vartheta\circ\rho(g^{-1}) = \vartheta\}$$

of $\operatorname{Hom}_{\mathbb{K}}(V,W)$. Prove (mention) all five claims.

Reason: Group Representations.

Solution:

(a) **Well-definition.** The element $|G|=1+\ldots+1\in\mathbb{K}$ has a multiplicative inverse since $\operatorname{char}(\mathbb{K})\nmid |G|$, hence Sym is well-defined.

(b) **Homomorphism of representations.** Let $h\in G,\varphi\in \operatorname{Hom}_{\mathbb{K}}(V,W)$.

$$\begin{aligned}\tau(h)\circ\operatorname{Sym}(\varphi)&=\frac{1}{|G|}\sum_{g\in G}\tau(h)\circ\tau(g)\circ\varphi\circ\rho(g^{-1})\\&=\frac{1}{|G|}\sum_{g\in G}\tau(hg)\circ\varphi\circ\rho(g^{-1}h^{-1}h)\\&\stackrel{u=hg}{=}\frac{1}{|G|}\sum_{u\in G}\tau(u)\circ\varphi\circ\rho(u^{-1})\circ\rho(h)\\&=\operatorname{Sym}(\varphi)\circ\rho(h)\end{aligned}$$

(c) **Linearity.**

$$\operatorname{Sym}(\alpha\varphi+\beta\vartheta)=\alpha\operatorname{Sym}(\varphi)+\beta\operatorname{Sym}(\vartheta)$$

is a direct consequence of the definition of Sym .

(d) **Image is a subspace of** $\operatorname{Hom}_{\mathbb{K}}(V,W)$. We just have proven that Sym is a $\mathbb{K}-$linear homomorphism of representations, i.e. especially spans a subspace of all $\mathbb{K}-$linear homomorphisms $V\longrightarrow W$.

(e) Sym **is a projection onto** $\operatorname{Hom}_{\mathbb{K}}((\rho,V),(\tau,W))$. Let ϑ be a homomorphism of representations, i.e. $\tau(g)\circ\vartheta\circ\rho(g^{-1})=\vartheta$. Thus

$$\operatorname{Sym}(\vartheta)=\frac{1}{|G|}\sum_{g\in G}\tau(g)\circ\vartheta\circ\rho(g^{-1})=\frac{1}{|G|}\sum_{g\in G}\vartheta=\frac{1}{|G|}\cdot|G|\cdot\vartheta=\vartheta$$

and especially $\operatorname{Sym}\circ\operatorname{Sym}=\operatorname{Sym}$ and

$$\operatorname{Sym}(\operatorname{Hom}_{\mathbb{K}}(V,W))\subseteq\operatorname{Hom}_{\mathbb{K}}((\rho,V),(\tau,W))$$

5. Let $f(x)=x^3-\dfrac{49}{6}x^2+\dfrac{39}{2}x-\dfrac{31}{3}$. Prove that there are at least one a,b such that $f^2(a)=a\,,\,f(a)\neq a$ and $f^4(b)=b\,,\,f^k(b)\neq b\ (k<4)$ where $f^n:=f\circ f^{n-1},f^1=f$.

Is this true for every even power?

Reason: Theorem of Sharkovskii.

Solution: We observe that $f(1) = 2, f(2) = 4, f(4) = 1$, which means $f^3(1) = f(f(f(1))) = 1$. This means that $x = 1$ is a periodic point of order 3 of the continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$. A periodic point p of order m is a point such that $f^m(p) = p$ and $f^k(p) \neq p$ for some $m \in \mathbb{N}$ and all $0 < k < m$. The claim now follows from the theorem of Sharkovskii:

Consider the total (Sharkovskii) order "$\preccurlyeq_S$"

$$3, 5, 7, 9, \ldots, 2\cdot 3, 2\cdot 5, 2\cdot 7, \ldots, 2^2\cdot 3, 2^2\cdot 5, \ldots, 2^3\cdot 3, \ldots, \ldots, 2^4, 2^3, 2^2, 2, 1$$

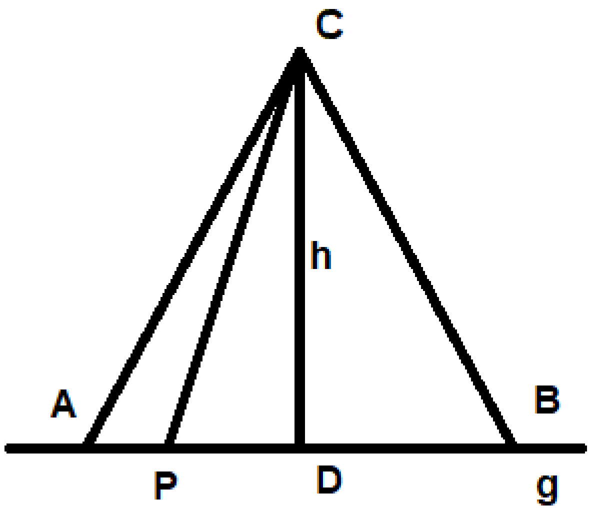
of the natural numbers. If the continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ has a periodic point of order m, and $m \preccurlyeq_S n$, then there is at least one periodic point of order n.

Since we have a periodic point of order 3, we have at least one periodic point of any order.

6. Prove the equivalence of the theorem of Pythagoras with the following transversal theorem about isosceles triangles:

   Given an isosceles triangle $\triangle ABC$ with baseline $\overline{AB} \subseteq g$, peak C, i.e. $|AC| = |BC|$, and g the straight along the baseline. Moreover let $P \in g$ be an arbitrary point. Then

$$|CP|^2 = |CA|^2 + |PA| \cdot |PB| \text{ if } P \notin \overline{AB}$$
$$|CP|^2 = |CA|^2 - |PA| \cdot |PB| \text{ if } P \in \overline{AB}$$



Reason: Geometry.

Solution: The height h divides the baseline $\overline{AB}$ into equal halves and intersects g at point D. We thus get two right triangles $\triangle PDC$ and $\triangle BDC$ on which we can apply the theorem of Pythagoras.

Case I: $P \notin \overline{AB}$

$$\begin{aligned}|CP|^2 &= |CD|^2 + \left(\frac{|AB|}{2} + |PA|\right)^2 \\ |CB|^2 &= |CD|^2 + \left(\frac{|AB|}{2}\right)^2 \\ |CP|^2 &= |CB|^2 - \left(\frac{|AB|}{2}\right)^2 + \left(\frac{|AB|}{2} + |PA|\right)^2 \\ &= |CB|^2 + |PA|^2 + 2 \cdot |PA| \cdot |AB| \\ &= |CB|^2 + |PA| \cdot |PB| = |CA|^2 + |PA| \cdot |PB|\end{aligned}$$

Case II: $P \in \overline{AB}$

$$\begin{aligned}
|CP|^2 &= |CD|^2 + \left(\frac{|AB|}{2} - |PA|\right)^2 \\
|CB|^2 &= |CD|^2 + \left(\frac{|AB|}{2}\right)^2 \\
|CP|^2 &= |CB|^2 - \left(\frac{|AB|}{2}\right)^2 + \left(\frac{|AB|}{2} - |PA|\right)^2 \\
&= |CB|^2 + |PA|^2 - 2 \cdot |PA| \cdot |AB| \\
&= |CB|^2 + |PA| \cdot |PB| = |CA|^2 - |PA| \cdot |PB|
\end{aligned}$$

Conversely we consider the special case $P = D$. Then

$$|CD|^2 = |CP|^2 = |CA|^2 - |PA| \cdot |PB| = |CA|^2 - |DA|^2$$

is the transversal theorem for the isosceles triangle $\triangle ABC$, which is the theorem of Pythagoras for the right triangle $\triangle ADC$. However, any given right triangle can be mirrored at one of its legs to make it an isosceles triangle for which the theorem of Pythagoras is a consequence of the transversal theorem.

7. Let α be an algebraic number of degree $n \geq 1$. Then there is a real number $c > 0$ such that for all $\mathbb{Q} \ni \dfrac{p}{q} \neq \alpha$

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{c}{q^n}$$

Reason: Liouville's Approximation Theorem.

Solution: Let $f(x) = a_n x^n + \ldots + a_1 x + a_0 \in \mathbb{Z}[x]$ be the minimal polynomial of α with $a_n \neq 0$. This means we can factorize $f(x) = (x - \alpha) \cdot g(x)$ in $\mathbb{C}[x]$. The function $\mathbb{R} \longrightarrow \mathbb{C}\,,\, x \longmapsto g(x)$ is continuous, i.e. there are real numbers $c_1, c_2 > 0$ such that $|g(x)| \leq c_1$ whenever $|\alpha - x| < c_2$. Since f has only finitely many zeros, we may assume w.l.o.g. that no other zero lies in the neighborhood of α, i.e. that $f(x) \neq 0$ for all $|\alpha - x| < c_2$ and $x \neq \alpha$. Set $c := \min\{c_2, c_1^{-1}\}$.

Assume that there are $p, q \in \mathbb{Z}, q \geq 1$ such that

$$\left| \alpha - \frac{p}{q} \right| < \frac{c}{q^n} < c \leq c_2$$

hence $|g(p/q)| \leq c_1$ and

$$\left|f\left(\frac{p}{q}\right)\right| = \left|\frac{p}{q}-\alpha\right| \cdot g\left(\frac{p}{q}\right) < \frac{c}{q^n}\cdot c_1 \leq \frac{1}{q^n} \Longrightarrow \left|q^n f\left(\frac{p}{q}\right)\right| < 1$$

But $q^n f\left(\dfrac{p}{q}\right) = a_np^n + qa_{n-1}p^{n-1} + \ldots + q^{n-1}a_1p + q^na_0 \in \mathbb{Z}$ which implies that $f\left(\dfrac{p}{q}\right) = 0$ which can only happen if $\dfrac{p}{q} = \alpha$ in the chosen neighborhood of α.

Another way to prove Liouville's approximation theorem is the following. If $\alpha = a + ib \notin \mathbb{R}$ then

$$\frac{|b|}{q^n} \leq |b| \leq \sqrt{\left(a-\frac{p}{q}\right)^2 + b^2} = \left|\alpha - \frac{p}{q}\right|$$

and $c := |\mathfrak{Im}(\alpha)|$ proves the statement of the theorem. We may thus assume that $\alpha \in \mathbb{R}$. Let $r > 0$ and $M_r := \max\{f'(x) : |x-\alpha| \leq r\}$. Now we choose $c := \min\{r, M_r^{-1}\}$ and assume

$$\left|\alpha - \frac{p}{q}\right| \leq r.$$

There is a $\xi \in \left[\alpha, \dfrac{p}{q}\right]$ i.e. especially $|\xi - \alpha| \leq r$, such that

$$f\left(\frac{p}{q}\right) = f\left(\frac{p}{q}\right) - f(\alpha) = \left(\frac{p}{q} - \alpha\right)\cdot f'(\xi)$$

by the mean value theorem of differential calculus and $|f'(\xi)| \leq M_r$. Again we have $q^n f\left(\dfrac{p}{q}\right) \in \mathbb{Z}$. The polynomial f is irreducible over $\mathbb{Z}$ by its minimality and so irreducible over $\mathbb{Q}$ by Gauß's lemma for polynomials. Then either $\alpha = \dfrac{p}{q}$ or $f\left(\dfrac{p}{q}\right) \neq 0$. The former is impossible, so the latter must hold. Then

$$\begin{aligned}\left|q^n f\left(\frac{p}{q}\right)\right| \geq 1 \Longrightarrow \frac{1}{q^n} &\leq \left|f\left(\frac{p}{q}\right)\right| = \left|\frac{p}{q}-\alpha\right| \cdot |f'(\xi)| \\ \Longrightarrow \frac{1}{q^n} &\leq \left|\frac{p}{q}-\alpha\right| \cdot M_r \\ \Longrightarrow \left|\alpha - \frac{p}{q}\right| &\geq \frac{1}{M_rq^n} \geq \frac{c}{q^n}\end{aligned}$$

8. Let $a_{n+1} = 2 + \sqrt{4 + a_n}$, $a_0 \geq -4$, be a sequence of real numbers. Determine - if existent - its limit in dependence of the initial value a_0, and show that $a_n \in [2, 5]$ in cases where $a_0 \in [-4, 5]$, and $a_n \geq 5$ in cases where $a_0 \geq 5$ ($n \in \mathbb{N}$).

   Reason: Recursion.

   Solution: By the monotony of the root function we get for $a_0 \in [-4, 5]$ that $a_1 \in [2, 5]$ and by induction $a_n \in [2+\sqrt{6}, 5] \subseteq [2, 5]$. In case $a_0 \geq 5$ we have again by monotony and induction $a_n \geq 5$.

$$\begin{aligned} a_{n+1} - a_n = 2 + \sqrt{4 + a_n} - a_n &= \frac{4 + a_n - (a_n - 2)^2}{\sqrt{4 + a_n} + a_n - 2} \\ &= \frac{a_n(5 - a_n)}{\sqrt{4 + a_n} + a_n - 2} = \begin{cases} \geq 0 & \text{if } a_n \in [2, 5] \\ \leq 0 & \text{if } a_n \geq 5 \end{cases} \end{aligned}$$

   The sequence is thus monotone increasing for $a_0 \in [2, 5]$ and monotone decreasing for $a_0 \geq 5$. This implies convergence in both cases (a_1 is an upper bound in the latter case). Now consider the fixed point equation

$$a = 2 + \sqrt{4 + a} \Longrightarrow (a - 2)^2 = 4 + a \Longrightarrow a(a - 5) = 0$$

   Testing both solutions gives us $a = 5$ as unique possible fixed point, i.e. $\lim_{n\to\infty} a_n = 5$ for any initial value $a_0 \geq -4$.

9. Calculate center, foci, semi-axis, and area of the maximal inscribed ellipse of the triangle $(1, 1), (5, 2), (3, 6)$.

   Reason: Geometry.

   Solution: Set

$$\begin{aligned} \mathbb{C}[x] \ni p(z) &= (z - 1 - i)(z - 5 - 2i)(z - 3 - 6i) \\ &= z^3 - (9 + 9i)z^2 + (3 + 52i)z + (33 - 39i) \end{aligned}$$

   The maximal inscribed ellipse of the triangle of zeros of $p(x)$ is thus the **Steiner inellipse**, where the sides of the triangle are tangents at their midpoints. The foci are the zeros of $p'(z)$ and the center the zero of $p''(x)$ by Marden's theorem.

$$\begin{aligned} 0 = p'(z) &= 3z^2 - (18 + 18i)z + (3 + 52i) \\ &= 3\left(z - (3 + 3i) - \sqrt{-1 + \frac{2i}{3}}\right)\left(z - (3 + 3i) + \sqrt{-1 + \frac{2i}{3}}\right) \\ &\approx 3\,(z - 3.32 - 4.05i)\,(z - 2.68 - 1.95i) \\ 0 = p''(x) &= 6\,(z - (3 + 3i)) \end{aligned}$$

Hence the center of the ellipse is $(3,3)$ and the foci $(2.68,1.95),(3.32,4.05)$. The area of an ellipse is given as $A=\pi ab$, so we have to compute the semi-axis of a Steiner inellipse within $\triangle ABC$ with center S.

$$A=(1,1)\,,\;B=(5,2)\,,\;C=(3,6)\,,\;S=(3,3)\,,$$
$$M:=\frac{1}{4}\left(\overline{SC}^2+\frac{1}{3}\overline{AB}^2\right)\,,\;N:=\frac{1}{4\sqrt{3}}\cdot\left|\det\left(\vec{SC},\vec{AB}\right)\right|$$
$$a=\frac{1}{2}\left(\sqrt{M+2N}+\sqrt{M-2N}\right)\,,\;b=\frac{1}{2}\left(\sqrt{M+2N}-\sqrt{M-2N}\right)$$

$$M=\frac{1}{4}\left(9+\frac{1}{3}\cdot 17\right)=\frac{11}{3}$$
$$N=\frac{1}{4\sqrt{3}}\left|\det\left(\begin{bmatrix}0&4\\3&1\end{bmatrix}\right)\right|=\sqrt{3}$$
$$a=\frac{1}{2\sqrt{3}}\left(\sqrt{11+6\sqrt{3}}+\sqrt{11-6\sqrt{3}}\right)$$
$$b=\frac{1}{2\sqrt{3}}\left(\sqrt{11+6\sqrt{3}}-\sqrt{11-6\sqrt{3}}\right)$$
$$ab=\frac{1}{12}\left(\left(11+6\sqrt{3}\right)-\left(11-6\sqrt{3}\right)\right)=\sqrt{3}$$
$$A=\pi\sqrt{3}\approx 5.44\quad=\frac{\pi}{3\sqrt{3}}\cdot A_{\triangle}$$

10. Let $A,B\in\mathbb{M}(n,\mathbb{F})$ be two square $n\times n$ matrices over a field $\mathbb{F}$. Show that the minimal polynomials of AB and BA are the same in case A is regular. Is it true as well, if A is singular?

Reason: Matrices and Minimal Polynomials.

Solution: For any polynomial $p(x)\in\mathbb{F}[x]$ we have

$$p(AB)A=Ap(BA)\text{ i.e. }0=p(AB)=Ap(BA)A^{-1}\Longleftrightarrow p(BA)=0$$

This means that the minimal polynomials of AB and BA divide each other, and are therefore equal. Now set

$$A:=\begin{bmatrix}1&0\\0&0\end{bmatrix}\,,\;B:=\begin{bmatrix}0&1\\0&0\end{bmatrix}$$
$$AB=\begin{bmatrix}0&1\\0&0\end{bmatrix}\,,\;BA=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

so $m_{AB}(x)=x^2\neq x=m_{BA}(x)$.

11. (HS-1) For which positive real numbers $\mathbb{R} \ni a, b > 0$ does

$$f(a,b) = \frac{a^4}{b^4} + \frac{b^4}{a^4} - \frac{a^2}{b^2} - \frac{b^2}{a^2} + \frac{a}{b} + \frac{b}{a}$$

assume a minimal value, and which one?

Reason: Real Numbers.

Solution:

$$\begin{aligned} f(a,b) &= \left(\frac{a^2}{b^2} + \frac{b^2}{a^2}\right)^2 - 2 - \left(\frac{a^2}{b^2} + \frac{b^2}{a^2}\right) + \left(\frac{a}{b} + \frac{b}{a}\right) \\ &= \left(\frac{a^2}{b^2} + \frac{b^2}{a^2} - \frac{1}{2}\right)^2 - \frac{9}{4} + \left(\frac{a}{b} + \frac{b}{a}\right) \\ &= \left(\left(\frac{a}{b} - \frac{b}{a}\right)^2 + \frac{3}{2}\right)^2 - \frac{9}{4} + \left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^2 + 2 \\ &\geq \frac{9}{4} - \frac{9}{4} + 2 = 2 \end{aligned}$$

where equality holds if and only if $\frac{a}{b} = \frac{b}{a}$ which is equivalent to $a = b$ since both are positive. Hence $f(a,b)$ assumes its minimal value if and only if $a = b$ in which case $f(a,a) = 2$.

12. (HS-2) Find all pairs (x, y) of integers such that

$$y^2 = x \cdot (x+1) \cdot (x+7) \cdot (x+8)$$

Reason: Integers.

Solution: Assume (x, y) is a solution, $u := x + 4$ and $t = u^2 - (25/2)$. Then

$$y^2 = x(x+1)(x+7)(x+8) = (u-4)(u-3)(u+3)(u+4) = (u^2-9)(u^2-16)$$
$$y^2 = \left(t + \frac{7}{2}\right)\left(t - \frac{7}{2}\right) = t^2 - \frac{49}{4}$$
$$49 = 4t^2 - 4y^2 = (2t - 2y)(2t + 2y) \text{ with } 2t \in \mathbb{Z}$$

Therefore $\{2t-2y, 2t+2y\} = \{2x^2+16x+7 \pm 2y\} = \{\pm 1, \pm 7, \pm 49\}$.

$2t+2y$	$2t-2y$	t	y	u	x	$x(x+1)(x+7)(x+8)$
49	1	25/2	12	5	1	144
				−5	−9	144
				0	−4	144
7	7	7/2	0	3	−1	0
				−3	−7	0
				4	0	0
				−4	−8	0
1	49	25/2	−12	5	1	144
				−5	−9	144
				0	−4	144
−1	−49	−25/2	12	5	1	144
				−5	−9	144
				0	−4	144
−7	−7	−7/2	0	3	−1	0
				−3	−7	0
				4	0	0
				−4	−8	0
−49	−1	−25/2	−12	5	1	144
				−5	−9	144
				0	−4	144

Possible pairs (x,y) are thus $(-9,-12),(-9,12),(-8,0),(-7,0)$, $(-4,-12),(-4,12),(-1,0),(0,0),(1,-12),(1,12)$.

13. (HS-3) Show that

$$\underbrace{\left| x - \frac{\sin(x)(14+\cos(x))}{9+6\cos(x)} \right|}_{=:f(x)} \leq 10^{-4} \text{ for } x \in \left[0, \frac{\pi}{4}\right]$$

You may use $\pi = 3.14159 + \delta$, $\sqrt{2} = 1.41421 + \varepsilon$ with $\delta, \varepsilon \in (0, 10^{-5})$.

Reason: Error Margins.

Solution: To shorten notation we set $c := \cos(x)$ and $s := \sin(x)$.

Then

$$\begin{aligned}
f'(x) &= 1 - \frac{(14c+c^2-s^2)(9+6c)-(14s+cs)(-6s)}{(9+6c)^2}\\
&= \frac{1}{(9+6c)^2}\left[(9+6c)^2-(9+6c)(14c+c^2-s^2)-6s(14s+cs)\right]\\
&= \frac{1}{(9+6c)^2}\left[81-18c-57c^2-6c^3-75s^2\right]\\
&= \frac{1}{(9+6c)^2}\left(24-18c-6c^3-18(1-c^2)\right)\\
&= \frac{6-18c+18c^2-6c^3}{(9+6c)^2} = \frac{2(1-c)^3}{3(3+2c)^2}
\end{aligned}$$

So $f'(x)>0$ for $x\in(0,\pi/4]$ and $f'(0)=0$. This means that $f(x)$ is strictly monotone increasing on the interval $[0,\pi/4]$. Hence

$$0\leq f(x)\leq f(\pi/4)=\frac{\pi}{4}-\frac{1}{\sqrt{2}}\cdot\frac{14+\dfrac{1}{\sqrt{2}}}{9+6\dfrac{1}{\sqrt{2}}}=\frac{\pi}{4}-\frac{41\sqrt{2}-25}{42}$$

Now $\pi/4=0.7853975+\dfrac{\delta}{4}<0.7854$ and $\dfrac{41\sqrt{2}-25}{42}=\dfrac{41\cdot 1.41421-25+41\varepsilon}{42}>\dfrac{32.98261}{42}>0.7853$, i.e. $0\leq f(x)<0.7854-0.7853=10^{-4}$.

14. (HS-4) If $f(x)=a_nx^n+\ldots+a_1x+a_0\in\mathbb{R}[x]$ is a real polynomial of degree n which doesn't have real zeros, and $h\in\mathbb{R}$ a real number, then

$$F(x):=f(x)+h\cdot f'(x)+h^2\cdot f''(x)+\ldots+h^n\cdot f^{(n)}(x)$$

doesn't have real zeros either.

Reason: Polynomial Zeros.

Solution: $F(x)$ is a real polynomial of degree n, too, and n has to be even, so we may assume w.l.o.g. that $f(x)>0$ for all $x\in\mathbb{R}$, since we could otherwise work with $-f(x)$ instead. Since $F(x)$ and $f(x)$ are of the same degree with the same leading coefficient a_n, $F(x)$ has a global minimum, because $f(x)$ has, and their limits at $x\to\pm$ are the same. This means there is a point x_0 such that $F(x)\geq F(x_0)$ for all $x\in\mathbb{R}$

and it is sufficient to show that $F(x_0) > 0$.

$$\begin{aligned}
F(x) &= \sum_{k=0}^{n} h^k \frac{d^k}{dx^k} f(x) = \sum_{k=0}^{\infty} h^k \frac{d^k}{dx^k} f(x) = \left(1 - h\frac{d}{dx}\right)^{-1} f(x) \\
f(x) &= \left(1 - h\frac{d}{dx}\right) F(x) = F(x) - hF'(x) \\
F(x) &= f(x) + hF'(x) \\
F(x_0) &= f(x_0) + hF'(x_0) = f(x_0) > 0
\end{aligned}$$

15. (HS-5) Solve the following real equations system:

$$\begin{aligned}
x + y &= az \\
x - y &= bz \\
x^2 + y^2 &= cz
\end{aligned}$$

Reason: Calculus.

Solution: The first two equations can be rewritten as

$$x = \frac{a+b}{2}\, z \, , \; y = \frac{a-b}{a}\, z$$

so

$$0 = \left(\frac{a+b}{2}\right)^2 z^2 + \left(\frac{a-b}{2}\right)^2 z^2 - cz = z \cdot \left(\frac{a^2+b^2}{2}\, z - c\right)$$

If $z = 0$ then $x = y = 0$ which is a solution for any choice of $a, b, c \in \mathbb{R}$.
If $a = b = 0$ then $x = y = 0$ and $cz = 0$.

Now let $a^2 + b^2 \neq 0$ and $z = \dfrac{2c}{a^2+b^2}$, $x = \dfrac{c(a+b)}{a^2+b^2}$, $y = \dfrac{c(a-b)}{a^2+b^2}$.
These are necessary and sufficient conditions to solve the equations.
We have therefore the following solutions:

a	b	c	x	y	z
0	0	0	0	0	z
a	b	c	0	0	0
a	b	c	$\dfrac{a+b}{a^2+b^2}\, c$	$\dfrac{a-b}{a^2+b^2}\, c$	$\dfrac{2}{a^2+b^2}\, c$

3 April 2021

1. Let T be a planet's orbital period, a the length of the semi-major axis of its orbit. Then
$$T'(a) = \gamma \sqrt[3]{T(a)}\,,\, T(0) = 0$$
with a constant proportional factor $\gamma > 0$. Solve this equation for all $a \in \mathbb{R}$ and determine whether the solution is unique and why.

   Reason: Uniqueness in the theorem of Picard-Lindelöf.

   Solution: Division by $\gamma \sqrt[3]{T(a)}$ yields
$$\begin{aligned} \gamma^{-1} \int \frac{\frac{dT(a)}{da}}{\sqrt[3]{T(a)}}\, da &= \int 1\, da \\ \gamma^{-1} \int \frac{1}{\sqrt[3]{T(a)}}\, dT(a) &= a - C \\ \gamma^{-1} \frac{3}{2} \sqrt[3]{T(a)^2} &= a - C \\ T(a) &= \left(\frac{2\gamma}{3}(a-C)\right)^{3/2} = \gamma'(a-C)^{3/2} \end{aligned}$$
The global solutions are thus
$$T(a) = \begin{cases} 0 & , a \leq C \\ \gamma'(a-C)^{3/2} & , a > C \end{cases}$$
which are infinitely many, for any $C \in \mathbb{R}_0^+$ and $\gamma' = (2\gamma/3)^{3/2}$.

   All conditions for the theorem of Picard-Lindelöf hold, except the Lipschitz continuity of $f(a,T) = \gamma \sqrt[3]{T(a)}$ at $T(0) = 0$. The function $x \longmapsto \sqrt[3]{x}$ isn't Lipschitz continuous in any neighborhood of 0. This example shows that Lipschitz continuity is crucial for the uniqueness part in the (local and global version) of the theorem of Picard-Lindelöf.

2. Show that the Hadamard (elementwise) product of two positive definite complex matrices is again positive definite.

   Reason: Schur product theorem.

   Solution: A complex matrix A can be written as the sum of a Hermitian and a skew-Hermitian matrix:
$$A = \frac{1}{2}\left(A + A^\dagger\right) + \frac{1}{2}\left(A - A^\dagger\right)$$

Therefore, if A is positive definite, we have for $x \neq 0$

$$\begin{aligned}
0 < 2\langle Ax, x\rangle &= \langle \left(A + A^\dagger\right) x, x\rangle + \langle \left(A - A^\dagger\right) x, x\rangle \\
&= \langle x, \left(A + A^\dagger\right)^\dagger x\rangle + \langle x, \left(A - A^\dagger\right)^\dagger x\rangle \\
&= \langle x, \left(A + A^\dagger\right) x\rangle - \langle x, \left(A - A^\dagger\right) x\rangle \\
&= 2\langle x, A^\dagger x\rangle
\end{aligned}$$

and A is Hermitian. As such it can be unitary diagonalized, i.e. $A = UDU^\dagger$ for a real diagonal matrix D and a unitary matrix U. As all eigenvalues of the positive definite matrix A are all positive, we can draw the square root of $D = R^2$. If we define

$$\sqrt{A} := URU^\dagger$$

then $\sqrt{A} \cdot \sqrt{A} = (URU^\dagger)(URU^\dagger) = URU^{-1}URU^\dagger = UDU^\dagger = A$ and the square root is Hermitian again: $\sqrt{A}^\dagger = \sqrt{A}$.

The Hadamard product of any two matrices $A \odot B$ is defined by elementwise multiplication, i.e.

$$\begin{aligned}
\operatorname{tr}\left(A^\tau \operatorname{diag}(\bar{v}) B \operatorname{diag}(w)\right) &= \sum_{k=1}^{n} \left[\left(a_{ji}\bar{v}_j\right)\left(b_{ij}w_i\right)\right]_{kk} \\
&= \sum_{k=1}^{n} \left[\left(\sum_{l=1}^{n} a_{li}\bar{v}_l b_{lj} w_j\right)_{ij}\right]_{kk} \\
&= \sum_{k=1}^{n}\sum_{l=1}^{n} \left(a_{lk}\bar{v}_l b_{lk} w_k\right) \\
&= v^\dagger \left(A \odot B\right) w
\end{aligned}$$

Now let's assume that A, B are positive definite. Then

$$\begin{aligned}
\langle v, (A \odot B)v\rangle &= v^\dagger (A \odot B) v = \operatorname{tr}\left(A^\tau \operatorname{diag}(\bar{v}) B \operatorname{diag}(v)\right) \\
&= \operatorname{tr}\left(\sqrt{A}\sqrt{A} \operatorname{diag}(\bar{v}) \sqrt{B}\sqrt{B} \operatorname{diag}(v)\right) \\
&= \operatorname{tr}\left(\left(\sqrt{A} \operatorname{diag}(\bar{v}) \sqrt{B}\right)\left(\sqrt{B} \operatorname{diag}(v) \sqrt{A}\right)\right) \\
&= \operatorname{tr}\left(\underbrace{\left(\sqrt{B} \operatorname{diag}(v) \sqrt{A}\right)^\dagger}_{=:C^\dagger} \underbrace{\left(\sqrt{B} \operatorname{diag}(v) \sqrt{A}\right)}_{=:C}\right) \\
&= \operatorname{tr}\left(C^\dagger C\right) = \sum_{i,j} \bar{c}_{ij} c_{ij} > 0
\end{aligned}$$

for $v \neq 0$ and equal to zero if and only if $v = 0$.

3. A function $f : S^k \longrightarrow X$ is called antipodal if it is continuous and $f(-x) = -f(x)$ for all $x \in S^k$ and any topological space $X \subseteq \mathbb{R}^m$.

   Show that the following statements are equivalent:

   (a) For every antipodal map $f : S^n \longrightarrow \mathbb{R}^n$ there is a point $x \in S^n$ satisfying $f(x) = 0$.

   (b) There is no antipodal map $f : S^n \longrightarrow S^{n-1}$.

   (c) There is no continuous mapping $f : B^n \longrightarrow S^{n-1}$ that is antipodal on the boundary.

   Assume the conditions hold. Prove Brouwer's fixed point theorem:

   Any continuous map $f : B^n \longrightarrow B^n$ has a fixed point.

   Reason: Borsuk-Ulam Theorem.

   Solution: $(a) \Longrightarrow (b)$ Assume there is a antipodal map $f : S^n \longrightarrow S^{n-1}$. If we compose it with the inclusion $\iota : S^{n-1} \hookrightarrow \mathbb{R}^n$ then it remains antipodal and according to (a) there is a point $\iota(f(x)) = 0$ which can only occur if $f(x) = 0$. However, $f(S^n) \subseteq S^{n-1} \not\ni 0$, a contradiction.

   $(b) \Longrightarrow (a)$ Assume there is a antipodal map $f : S^n \longrightarrow \mathbb{R}^n$ such that $f(x) \neq 0$ for all $x \in S^n$. Then we define $g : S^n \longrightarrow S^{n-1}$ by $g(x) := \dfrac{f(x)}{\|x\|}$ which is antipodal, too, contradicting (b).

   $(c) \Longrightarrow (b)$ The map $\pi(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n)$ is a homeomorphism from the upper hemisphere of S^n to B^n. If we had an antipodal map $f : S^n \longrightarrow S^{n-1}$ then we would have an antipodal map $g : B^n \longrightarrow S^{n-1}$ by $g(x) := f(\pi^{-1}(x))$ which is antipodal on the boundary, contradicting (c).

   $(b) \Longrightarrow (c)$ Assume $g : B^n \longrightarrow S^{n-1}$ is continuous and antipodal on the boundary. Then we define $f : S^n \longrightarrow S^{n-1}$ by $f(x) := g(\pi(x))$ for x in the upper hemisphere and $f(-x) := -g(\pi(x))$. Thus f is antipodal, contradicting (b).

   That all these conditions actually hold, is the theorem of Borsuk-Ulam, which can be proven by topological algebra. E.g. see Theorem 2.2. in https://web.northeastern.edu/suciu/slides/Borsuk-Ulam-tapas05.pdf

   Suppose there exists a continuous function $f : B^n \longrightarrow B^n$ without fixed point. We define $g : B^n \longrightarrow S^{n-1}$ such that $g(x)$ is the point on S^{n-1} which intersects with the ray from $f(x)$ and x. This is a well-defined retraction, as there are no fixed points at which the function would be ill-defined. It is a retraction since a ray from anywhere in B^n to a point on the boundary $x \in \partial B^n = S^{n-1}$ intersects the boundary at x by construction. Now $g(-x) = -g(x)$ for $x \in S^{n-1}$ which is not possible according to (c).

4. Let Y be an affine, complex variety. Prove that Y is irreducible if and only if $I(Y)$ is a prime ideal.

   Reason: Hilbert's Nullstellensatz.

   Solution: Assume Y is irreducible and $f, g \in \mathcal{O}_{\mathbb{C}}(\mathbb{A}^n)$ such that $f \cdot g \in I(Y)$. Then $V(fg) = V(f) \cup V(g)$ and

$$Y = (V(f) \cap Y) \cup (V(g) \cap Y).$$

   Since Y is irreducible, we have w.l.o.g. $Y = V(f) \cap Y$, i.e. $Y \subseteq V(f)$ and $f \in I(Y)$.

   Now let $J := I(Y)$ be a prime ideal. Let $V(J) = Y_1 \cup Y_2$. Thus $J = I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$. Assume $J \neq I(Y_1), I(Y_2)$. Then there are $f_i \in I(Y_i) - J$. Since

$$f_1 f_2 \in I(Y_1) \cdot \mathcal{O}_{\mathbb{C}}(\mathbb{A}^n) \cap \mathcal{O}_{\mathbb{C}}(\mathbb{A}^n) \cdot I(Y_2) = I(Y_1) \cap I(Y_2) = J$$

   is a prime ideal, we have that either $f_1 \in J$ or $f_2 \in J$ contradicting the choice of f_1, f_2. Hence w.o.l.g. we may assume $I(Y) = J = \sqrt{J} = I(Y_1)$. This implies $Y = V(I(Y)) = V(I(Y_1)) = Y_1$ and $V(J)$ is irreducible.

5. Let $p > 5$ be a prime number. Show that

$$\left(\frac{6}{p}\right) = 1 \Longleftrightarrow p \equiv k\,(24) \text{ with } k \in \{1, 5, 19, 23\}.$$

   The parentheses are the Legendre symbol.

   Reason: Quadratic Reciprocity Law.

   Solution:

$$\left(\frac{6}{p}\right) = \left(\frac{2}{p}\right) \cdot \left(\frac{3}{p}\right)$$

   which equals 1 if and only if both factors are $= 1$ or both are $= -1$.

By the quadratic reciprocity law we have

$$\left(\frac{3}{p}\right) = (-1)^{\frac{p-1}{2}}\left(\frac{p}{3}\right) = \begin{cases} 1 & \text{if } p \equiv 1\,(4)\,,\ p \equiv 1\,(3)\,,\text{ i.e. } p \equiv 1\,(12) \\ -1 & \text{if } p \equiv 1\,(4)\,,\ p \equiv -1\,(3)\,,\text{ i.e. } p \equiv 5\,(12) \\ -1 & \text{if } p \equiv -1\,(4)\,,\ p \equiv 1\,(3)\,,\text{ i.e. } p \equiv -5\,(12) \\ 1 & \text{if } p \equiv -1\,(4)\,,\ p \equiv -1\,(3)\,,\text{ i.e. } p \equiv -1\,(12) \end{cases}$$

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}} = \begin{cases} 1 & \text{if } p \equiv \pm 1\,(8) \\ -1 & \text{if } p \equiv \pm 5\,(8) \end{cases}$$

If $p \equiv a\,(n)\,,\ p \equiv b\,(m)$ and $d = (n,m)$, then d can be written as $d = \mu m + \nu n$ with $(d,\mu) = (d,\nu) = 1$ by Bézout's Lemma. This means

$$\begin{aligned} \mu(a-b) &= \mu c_1 m - \mu c_2 n = c_1 \mu m - c_2 \mu n = c_1 d - c_1 \nu n - \mu c_2 n \\ &\Longrightarrow d\,|\,\mu(a-b) \Longrightarrow d\,|\,(a-b) \end{aligned}$$

For $n = 8, m = 12$ we get $d = 4$.

$$\begin{aligned} p \equiv 1\,(8) \wedge p \equiv -1\,(12) &\Longrightarrow 4\,|\,(1-(-1)) = 2 \ \lightning \\ p \equiv -1\,(8) \wedge p \equiv 1\,(12) &\Longrightarrow 4\,|\,(-1-1) = -2 \ \lightning \\ p \equiv 5\,(8) \wedge p \equiv -5\,(12) &\Longrightarrow 4\,|\,(5-(-5)) = 10 \ \lightning \\ p \equiv -5\,(8) \wedge p \equiv 5\,(12) &\Longrightarrow 4\,|\,(-5-5) = -10 \ \lightning \end{aligned}$$

Thus we have $\left(\frac{2}{p}\right) = \left(\frac{3}{p}\right) = \pm 1$ only if p has the same sign modulo 8 and 12

$$\begin{aligned} p = 8m \pm 1 = 12n + pm1 &\Rightarrow 2m = 3n \Rightarrow 2|n, 3|m \Rightarrow p \equiv \pm 1\,(24) \\ p = 8m \pm 5 = 12n + pm5 &\Rightarrow 2m = 3n \Rightarrow 2|n, 3|m \Rightarrow p \equiv \pm 5\,(24) \end{aligned}$$

Thus $p \mod 24 \in \{\pm 1, \pm 5\} = \{1, 5, 19, 23\}$.

6. Let $f \in L^2(\mathbb{R})$ and $g : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ be given as

$$g(t) := t \int_{\mathbb{R}} \chi_{[t,\infty)}(|x|) \exp(-t^2(|x|+1)) f(x)\, dx$$

Show that $g \in L^1(\mathbb{R})$.

Reason: Functional Analysis.

Solution: We observe $\chi_{[t,\infty)}(|x|)=\chi_{(-\infty,|x|]}(t)$ and define $u(x)=\int_{\mathbb{R}}\chi_{[t,\infty)}(|x|)|t|\exp(-t^2(|x|+1))\,dt$. Then

$$\begin{aligned}
u(x)&=\int_{-\infty}^{|x|}|t|\exp(-t^2(|x|+1))\,dt\\
&=-\int_{-\infty}^{0}t\exp(-t^2(|x|+1))\,dt+\int_{0}^{|x|}t\exp(-t^2(|x|+1))\,dt\\
&=\left[\frac{\exp(-t^2(|x|+1))}{-2(|x|+1)}\right]_0^{|x|}-\left[\frac{\exp(-t^2(|x|+1))}{-2(|x|+1)}\right]_{-\infty}^{0}\\
&=\frac{\exp(-|x|^2(|x|+1))-2}{-2(|x|+1)}
\end{aligned}$$

and thus

$$\|u\|_2^2=\int_{\mathbb{R}}\frac{|\exp(-|x|^2(|x|+1))-2|^2}{4(|x|+1)^2}\,dx\leq\int_{\mathbb{R}}\frac{4}{4(|x|+1)^2}\,dx=2$$

Now the integral of $|g|$ yields

$$\|g\|_1=\int_{\mathbb{R}}|g(t)|\,dt\leq\int_{\mathbb{R}}\int_{\mathbb{R}}|t|\chi_{[t,\infty)}(|x|)\exp(-t^2(|x|+1))|f(x)|\,dx\,dt$$

The conditions of the theorem of Tonelli hold, because the function under the integral is positive and as a product of continuous and measurable functions, itself measurable. Thus

$$\begin{aligned}
\|g\|_1&\leq\int_{\mathbb{R}}\int_{\mathbb{R}}|t|\chi_{[t,\infty)}(|x|)\exp(-t^2(|x|+1))|f(x)|\,dt\,dx\\
&=\int_{\mathbb{R}}|u(x)f(x)|\,dx=\|uf\|_1\leq\|u\|_2\|f\|_2<\infty
\end{aligned}$$

by Hölder's inequality and the fact that $u(x),f(x)\in L^2(\mathbb{R})$.

7. (a) Let V be the pyramid with vertices $(0,0,1),(0,1,0),(1,0,0)$ and $(0,0,0)$. Calculate

$$\int_V\exp(x+y+z)\,dV$$

(b) Let $A\in\mathrm{GL}(d,\mathbb{R})$. Calculate

$$\int_{\mathbb{R}^d}\exp(-\|Ax\|_2^2)\,dx$$

Reason: Integration.

Solution:

(a) $V = \{(x,y,z) \in \mathbb{R}^3 | 0 \leq z \leq 1, 0 \leq x \leq 1-z, 0 \leq y \leq 1-z-x\}$ is compact, $g : V \longrightarrow \mathbb{R}$ with $g(x,y,z) = z\exp(x+y+z)$ is continuous, hence integrable, so by Fubini's theorem

$$\begin{aligned}
\int_V \exp(x+y+z)\,dV &= \int_V \exp(x+y+z)\,d(x,y,z) \\
&= \int_0^1 \int_0^{1-z} \int_0^{1-z-x} \exp(x+y+z)\,dy\,dx\,dz \\
&= \int_0^1 \int_0^{1-z} (e - \exp(x+z))\,dx\,dz \\
&= \int_0^1 (e(1-z) - e + \exp(z))\,dz \\
&= e - \frac{1}{2}e - e + (e-1) = \frac{e}{2} - 1
\end{aligned}$$

(b) $\varphi(x) = Ax$ is a C^1-diffeomorphism with $D\varphi = A$ and the transformation theorem can be applied:

$$\begin{aligned}
\int_{\mathbb{R}^d} \exp(-\|Ax\|_2^2)\,dx &= \frac{1}{|\det(D\varphi)|} \int_{\mathbb{R}^d} \exp(-\|x\|_2^2)\,dx \\
&= \frac{1}{|\det(A)|} \int_{\mathbb{R}^d} \prod_{k=1}^{d} \exp(-x_k^2)\,dx_k \\
&\overset{\text{Tonelli}}{=} \frac{1}{|\det(A)|} \prod_{k=1}^{d} \int_{\mathbb{R}} \exp(-x_k^2)\,dx_k \\
&= \frac{\sqrt{\pi}^{\,d}}{|\det(A)|}
\end{aligned}$$

8. Consider the Hilbert space $L^2([0,1])$ and its subspace $K := \operatorname{span}_{\mathbb{C}}\{x, 1\}$. Let $\pi^{\perp} : H \longrightarrow K$ be the orthogonal projection. Give an explicit formula for $\pi^{\perp}$ and calculate $\pi^{\perp}(e^x)$.

Reason: Hilbert Spaces.

Solution: If $\{u, v\}$ is a orthonormal basis of K, then the orthogonal projection is given by $\pi^{\perp}(f) = \langle f, u\rangle u + \langle f, v\rangle v$ and it is sufficient to determine a orthogonal basis. We set $u = 1$ and apply the Gram-Schmidt algorithm to get

$$\bar{v} = x - \langle x, 1\rangle 1 = x - \int_0^1 t\,dt = x - \frac{1}{2}$$

$$\|\bar{v}\|^2 = \int_0^1 \left(t - \frac{1}{2}\right)^2 dt = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12}$$

such that $\{u, v\} = \left\{u, \frac{\bar{v}}{\|\bar{v}\|}\right\} = \left\{1, \sqrt{12}\left(x - \frac{1}{2}\right)\right\}$ and

$$\pi^{\perp}(f) = \langle f, 1\rangle 1 + 12 \left\langle f, x - \frac{1}{2}\right\rangle \left(x - \frac{1}{2}\right)$$

$$\begin{aligned}\pi^{\perp}(e^x) &= \langle e^x, 1\rangle 1 + 12 \left\langle e^x, x - \frac{1}{2}\right\rangle \left(x - \frac{1}{2}\right)\\ &= \int_0^1 e^x \, dx + 3(2x-1) \int_0^1 e^x \,(2x-1)\, dx\\ &= (e-1) + 3(2x-1)(3-e) = 6x(3-e) + 4e - 10\end{aligned}$$

9. Prove $\pi_1(S^n; x) = \{e\}$ for $n \geq 2$.

   Reason: Theorem of Seifert-van Kampen.

   Solution: Assume there is a point $a \in S^n$ such that $\pi_1(S^n; a) \ni g \neq e$. Set
   $$U := S^n \cap \left\{x_n > -\frac{1}{2}\right\}, \; V := S^n \cap \left\{x_n < \frac{1}{2}\right\}$$
   where we may assume without loss of generality that $a \in U \cap V$ by an appropriate choice of coordinates. Then U, V are homeomorphic to an open ball, i.e. $\pi_1(U; x) = \pi_1(V; x) = \{0\}$ because balls are simply connected, $S^n = U \cup V$, and $U \cap V \cong S^{n-1} \times (0, 1)$, which is path connected for $n \geq 2$. Let
   $$\begin{aligned}\iota_U &: \pi_1(U \cap V; a) \longrightarrow \pi_1(U; a)\\ \iota_V &: \pi_1(U \cap V; a) \longrightarrow \pi_1(V; a)\\ \kappa_U &: \pi_1(U; a) \longrightarrow \pi_1(S^n; a)\\ \kappa_V &: \pi_1(V; a) \longrightarrow \pi_1(S^n; a)\end{aligned}$$
   be the embeddings of the according fundamental groups. We can now apply the theorem of Seifert-van Kampen which states, that for every pair $\varphi_U : \pi_1(U; a) \longrightarrow G$, $\varphi_V : \pi_1(V; a) \longrightarrow G$ of group homomorphisms, such that $\varphi_U \circ \iota_U = \varphi_V \circ \iota_V$, there is a unique group homomorphism $\varphi : \pi_1(S^n; a) \longrightarrow G$ with $\varphi_U = \varphi \circ \kappa_U$ and $\varphi_V = \varphi \circ \kappa_V$. We

get, however, with $g \in G = \pi_1(S^n; a)$ two different homomorphisms $\varphi_1 = \mathrm{id}_G, \varphi_2 \equiv e$ in case $g \neq e$. These satisfy the conditions $k = 1, 2$

$$\varphi_k(\kappa_U([\gamma])) = \varphi_k(e) = e = \varphi_U([\gamma])$$
$$\varphi_k(\kappa_V([\gamma])) = \varphi_k(e) = e = \varphi_V([\gamma])$$

since both, U, V are simply connected, in contradiction to the theorem of Seifert-van Kampen. Hence $g = e$ and $\pi_1(S^n; a) = \{e\}$.

The same proof works in the more general case:

If $X = U \cup V$ with open sets U, V and $U \cap V$ is path connected, then $\pi_1(U; x) = \pi_1(V; x) = \{e\}$ implies $\pi_1(X; x) = \{e\}$.

10. Let $U \subseteq \mathbb{R}^{2n}$ be an open set and $f \in C^2(U, \mathbb{R})$ a twice continuously differentiable function at a point $\vec{a} \in U$. Prove that if f has a critical point in $\vec{a}$ and the Hessian matrix $Hf(\vec{a})$ has a negative determinant, then f has neither a local maximum nor a local minimum in $\vec{a}$.

    Reason: Taylor Series with Integral Remainder.

    Solution: The Hessian matrix is symmetric, and thus diagonalizable with real eigenvalues. Hence its determinant is negative, if there is at least one positive and one negative eigenvalue, say $Hf.\vec{v}_+ = \lambda_+\vec{v}_+$, $Hf.\vec{v}_- = \lambda_+\vec{v}_-$. Since a is a critical point, we have $Df(\vec{a}) = 0$, so the Taylor series with integral remainder is

$$f(\vec{a} + \vec{h}) = f(\vec{a}) + \int_0^1 (1 - t)\langle \vec{h}, Hf(\vec{a} + t\vec{h})\vec{h}\rangle \, dt$$

    $\langle \vec{h}, Hf(\vec{a}\vec{h})\rangle = \lambda_+\|\vec{h}\|^2 > 0$ for $\vec{h} \sim \vec{v}_+$ and this product remains positive in a neighborhood of $Hf(\vec{a})$ by continuity. So for sufficiently small $\vec{h}$ we have

$$\langle \vec{h}, Hf(a \vec{+} t\vec{h})\vec{h}\rangle > 0 \text{ for all } t \in [0, 1]$$

    which results in $f(\vec{a} + \vec{h}) > f(\vec{a})$. The same argument leads to the inequality $f(\vec{a} + \vec{h}) < f(\vec{a})$ for sufficiently small $\vec{h} \sim \vec{v}_-$.

11. (HS-1) Show that every non-negative real polynomial $p(x)$ can be written as $p(x) = a(x)^2 + b(x)^2$ with $a(x), b(x) \in \mathbb{R}[x]$.

    Reason: Fundamental Theorem of Algebra.

    Solution: Every real polynomial can be written as a product with its complex zeros $z_1, \ldots, z_n$

$$p(x) = \prod_{i=1}^{n} (x - z_i)^{r_i} = \underbrace{\prod_{i \leq m} \underbrace{(x - z_i)^{r_i}}_{z_i \in \mathbb{R}}}_{=: a_0 \in \mathbb{R}[x]} \cdot \underbrace{\prod_{i > m} \underbrace{(x - z_i)^{r_i}}_{z_i \in \mathbb{C}}}_{=: b_0 \in \mathbb{R}[x]}$$

All zeros of b_0 occur as pairs of conjugate complex numbers such that $z_i = u_i + iv_i$ can be are paired as

$$(x - z_i)^{r_i}(x - \bar{z}_i)^{r_i} = (\underbrace{(x - u_i)^2}_{=: b_i(x)^2 \in \mathbb{R}[x]} + v_i^2)^{r_i} > 0 \; \forall \, x \in \mathbb{R}$$

Assume that some power r_j in the first factor a_0 is odd. Then $p(z_j - \varepsilon)$ and $p(z_j + \varepsilon)$ would be real numbers of different sign, because $b_0(z_j \pm \varepsilon) > 0$ and the real roots z_j are discrete. As we excluded this possibility, $a_0(x) =: a_1(x)^2$ is already a square polynomial. Hence with possible repetitions

$$\begin{aligned} p(x) &= a_1(x)^2 \cdot \prod_{i=1}^{k} (b_i(x)^2 + v_i^2) = a_1(x)^2 \cdot \prod_{i=1}^{k} \det \begin{pmatrix} b_i(x) & -v_i(x) \\ v_i(x) & b_i(x) \end{pmatrix} \\ &= a_1(x)^2 \cdot \det \begin{pmatrix} B(x) & -V(x) \\ V(x) & B(x) \end{pmatrix} = a_1(x)^2 \cdot (B(x)^2 + V(x)^2) \\ &= (\underbrace{a_1(x)B(x)}_{=: a(x)})^2 + (\underbrace{a_1(x)V(x)}_{=: b(x)})^2 \end{aligned}$$

since matrix multiplication preserves this form:

$$\begin{aligned} &\begin{pmatrix} b_i(x) & -v_i(x) \\ v_i(x) & b_i(x) \end{pmatrix} \begin{pmatrix} b_j(x) & -v_j(x) \\ v_j(x) & b_j(x) \end{pmatrix} \\ &= \begin{pmatrix} b_i(x)b_j(x) - v_i(x)v_j(x) & -(b_i(x)v_j(x) + b_j(x)v_i(x)) \\ b_j(x)v_i(x) + b_i(x)v_j(x) & b_i(x)b_j(x) - v_i(x)v_j(x) \end{pmatrix} \end{aligned}$$

12. (HS-2) Show that all Pythagorean triples $x^2 + y^2 = z^2$ can be found by

$$(x, y, z) = d \cdot (u^2 - v^2, 2uv, u^2 + v^2) \text{ with } d, u, v \in \mathbb{N} \,,\, u > v$$

and which are primitive (no common divisor of x, y, z) if and only if u, v are coprime and one is odd and the other one even.

Reason: Pythagorean Triples.

Solution: All such triples form a Pythagorean triple, since

$(u^2-v^2)^2+(2uv)^2 = u^4-2u^2v^2+v^4+4u^2v^2 = u^4+2u^2v^2 = (u^2+v^2)^2$

If we have a Pythagorean triple $x^2+y^2=z^2$ then

$$\left(\frac{x}{z}\right)^2+\left(\frac{y}{z}\right)^2 = 1\ ;\ a:=\frac{x}{z}\ ,\ b:=\frac{y}{z}$$

and the straight from $(-1,0)$ to (a,b) on the unit circle intersects the $y-$axis at $(0,t)$ where $t=\dfrac{b}{a+1}=\dfrac{v}{u}$ with coprime natural numbers u,v is the rational slope of that straight.

$$\begin{aligned}
1 &= a^2+b^2 = a^2+t^2(a+1)^2\\
0 &= (a+1)(a-1)+t^2(a+1)^2\\
0 &= (a-1)+t^2(a+1) = a(t^2+1)+t^2-1\\
a &= \frac{1-t^2}{1+t^2}\ ,\ b = t\cdot(a+1) = \frac{2t}{1+t^2}
\end{aligned}$$

Hence

$$\left(\frac{x}{z},\frac{y}{z}\right) = \left(\frac{1-\dfrac{v^2}{u^2}}{1+\dfrac{v^2}{u^2}}\,,\ \frac{2\dfrac{v}{u}}{1+\dfrac{v^2}{u^2}}\right) = \left(\frac{u^2-v^2}{u^2+v^2}\,,\ \frac{2uv}{u^2+v^2}\right)$$

We can now write $(x,y,z)=d\cdot(u^2-v^2,2uv,u^2+v^2)$ and have the desired form. In case $u<v$ we can simply exchange u,v.

If u,v are both even, or both odd, then 2 divides all three numbers x,y,z and the triple isn't primitive. If u,v are not coprime, say $d|u$ and $d|v$, then $d^2|x,y,z$ and the triple isn't primitive either.

Let us now assume that the conditions hold, i.e. $u>v$ are coprime and one is even and one is odd. Then we have to show that (x,y,z) is primitive. Assume therefore that

$$d|x=u^2-v^2\ ,\ d|y=2uv\ ,\ d|z=u^2+v^2.$$

If d divides two of them, then it automatically divides the third one, too. Let $p|d$ an odd prime divisor. Then

$$\begin{aligned}
p\,|\,(u^2-v^2)\wedge p\,|\,(u^2+v^2) &\Longrightarrow p\,|\,2u^2\wedge p\,|\,2v^2\\
&\Longrightarrow p\,|\,u\wedge p\,|\,v
\end{aligned}$$

which cannot be as we assumed that u, v are coprime. So 2 is the only possible divisor of d and $u^2 - v^2 = (u-v)(u+v)$ is even, which can only be, if u, v are either both odd or both even, which we assumed is not true. Finally only $d = 1$ is possible if those conditions on u, v hold, i.e. (x, y, z) is primitive.

13. (HS-3) Write

$$\sqrt[8]{2207 - \cfrac{1}{2207 - \cfrac{1}{2207 - \cfrac{1}{2207 - \ldots}}}}$$

as $\dfrac{a + b\sqrt{c}}{d}$.

Reason: Continued Fraction.

Solution: Let's first consider the continued fraction. We define $L_0 = 2207\,,\ L_{n+1} = 2207 - 1/L_n$. It is a strictly decreasing sequence within the bounds $2206 < L_n < 2207$ for $n > 0$.

$2206 < L_1 = 2207 - \dfrac{1}{2207} < 2207 = L_0$ and by induction

$$0 < 1 < L_n \Longrightarrow 0 < \frac{1}{L_n} < 1 \Longrightarrow 2207 > 2207 - \frac{1}{L_n} = L_{n+1} > 2206$$

Set $L_0 = 2207 > 987\sqrt{5} =: C$ and assume n is minimal such that $L_0 > L_1 > \ldots > L_{n-1} > L_n > (1/2)(L_0 + C) \geq L_{n+1} = L_0 - 1/L_n$.

$$\begin{aligned} L_n > \frac{1}{2}(L_0 + C) &\Longrightarrow \frac{1}{L_n} < \frac{2}{L_0 + C} \\ &\Longrightarrow -\frac{1}{L_n} > -\frac{2}{L_0 + C} \\ &\Longrightarrow L_0 - \frac{1}{L_n} > L_0 - \frac{2}{L_0 + C} = \frac{L_0^2 + L_0 C - 2}{L_0 + C} \end{aligned}$$

Therfore $\frac{1}{2}(L_0 + C)^2 = \frac{1}{2}L_0^2 + \frac{1}{2}C^2 > L_0^2 - 2$ or $C^2 > L_0^2 - 4$ which cannot be true, since $C^2 = L_0^2 - 4$. Hence $(1/2)(L_0 + C)$ is a strict lower bound for all L_n. But this implies $L_n > L_{n+1}$:

$$L_n > L_{n+1} \Leftrightarrow L_n^2 - L_0 L_n + 1 > 0 \Leftrightarrow L_n > \frac{1}{2}\left(L_0 + \sqrt{L_0^2 - 4}\right) = \frac{1}{2}(L_0 + C)$$

This means we have a strictly decreasing sequence of real numbers which are bounded from below, i.e. its limit exists. Thus

$$
\begin{aligned}
L &:= \lim_{n\to\infty} L_n = \lim_{n\to\infty}\left(L_0 - \frac{1}{L_n}\right) = L_0 - \frac{1}{\lim_{n\to\infty} L_n} = L_0 - \frac{1}{L} \\
0 &= L^2 - L_0L + 1 = L^2 - 2207L + 1 \Longleftrightarrow L = \frac{1}{2}\left(L_0 + C\right)
\end{aligned}
$$

because $2206 < L \leq L_1 < 2207 = L_0$. This means we have to calculate $\sqrt[8]{\frac{1}{2}(L_0 + C)} = \sqrt{\sqrt{\sqrt{\frac{1}{2}(L_0 + \sqrt{L_0^2 - 4})}}}$.

If $0 = x^2 - ax + 1$ then

$$
0 = (x^2 - ax + 1)(x^2 + ax + 1) = (x^2 + 1)^2 - a^2x^2 = x^4 - (a^2 - 2)x^2 + 1
$$

so the positive square root of $y^2 - by + 1$ satisfies $x^2 - \sqrt{b+2}x + 1 = 0$.

$$
\begin{aligned}
y^2 - 2207y + 1 = 0 &\Longrightarrow x^2 - \sqrt{2207 + 2}x + 1 = x^2 - 47x + 1 = 0 \\
y^2 - 47y + 1 = 0 &\Longrightarrow x^2 - \sqrt{47 + 2}x + 1 = x^2 - 7x + 1 \\
y^2 - 7y + 1 = 0 &\Longrightarrow x^2 - \sqrt{7 + 2}x + 1 = x^2 - 3x + 1 \\
&\Longrightarrow L = \frac{1}{2}\left(3 + \sqrt{5}\right)
\end{aligned}
$$

14. (HS-4) To each positive integer with n^2 decimal digits, we associate the determinant of the matrix obtained by writing the digits in order across the rows. For example, for $n = 2$, to the integer 8617 we associate $\det\left(\begin{bmatrix} 8 & 6 \\ 1 & 7 \end{bmatrix}\right) = 50$. Find, as a function of n, the sum of all the determinants associated with n^2-digit integers. Leading digits are assumed to be nonzero; for example, for $n = 2$, there are 9000 determinants: $f(2) = \sum_{1000\leq N\leq 9999} \det(N)$.

    Reason: Determinants.

    Solution: If $n > 2$ then all determinants appear as positive and equal negative value in the sum: Fix all but the last two columns (c_{n-1}, c_n). Then (c_n, c_{n-1}) is in the sum, too, but of opposite sign. $f(1) = \sum_{k=1}^{9} \det(k) = 1 + \ldots + 9 = 45$. So it remains to determine $f(2)$.

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + \det\left(\begin{bmatrix} a & b \\ e & f \end{bmatrix}\right) = ad - bc + af - be = a(d+f) - b(c+e)$$

By the multilinearity of the determinant, the answer is the determinant of the matrix whose first (resp. second) row is the sum of all possible first (resp. second) rows. There are 90 first rows whose sum is the vector $(450, 405)$, and 100 second rows whose sum is $(450, 450)$. Thus the answer is

$$450 \cdot 450 - 450 \cdot 405 = 450 \cdot 45 = 20250$$

15. (HS-5) All squares on a chessboard are labeled from 1 to 64 in reading order (from left to right, row by row top-down). Then someone places 8 rooks on the board such that none threatens any other. Let S be the sum of all squares which carry a rook. List all possible values of S.

    Reason: Chessboard.

    Solution: The squares on a chessboard are usually labeled by numbers $1, \ldots, 8$ for the ranks and letters $A, \ldots, H$ for the files. Any order of rooks require that each number and each letter is assumed exactly once, so

$$\begin{aligned} S &= 1+2+3+4+5+6+7+8+A+B+C+D+E+F+G+H \\ &= 1+2+3+4+5+6+7+8+0+8+16+24+32+40+48+56 \\ &= 1+2+3+4+5+6+7+8+8\cdot(1+2+3+4+5+6+7) \\ &= 9 \cdot \binom{8}{2} + 8 = 9 \cdot 28 + 8 = 260 \end{aligned}$$

4 March 2021

1. Prove that all derivations $D := \operatorname{Der}(L)$ of a semisimple Lie algebra L are inner derivations $M := \operatorname{ad}(L)$.

   Reason: Semisimple Lie Algebras.

   Solution: Since the center $Z(L) = \ker \operatorname{ad}(L) = \{0\}$ of L is trivial, we have an isomorphism $M \cong L$. Furthermore

$$\begin{aligned}[\delta, \operatorname{ad}(X)](Y) &= \delta(\operatorname{ad}(X)(Y)) - \operatorname{ad}(X)(\delta(Y)) \\ &= [\delta(X), Y] + [X, \delta(Y)] - [X, \delta(Y)] \\ &= \operatorname{ad}(\delta(X))(Y)\end{aligned}$$

   i.e. $M \trianglelefteq D$ is an ideal, and the Killing-form K_M of M is the restriction of the Killing-form K_D of D. Let

$$M^{\perp} = \{\delta \in D \,|\, K_D(\delta, \operatorname{ad}(X)) = 0 \,\forall\, X \in L\}$$

   As the Killing-form of L is non-degenerated, so is K_M, hence $M^{\perp} \cap M = \{0\}$. Since both, $M^{\perp}$ and M are ideals in D, we obtain $[M, M^{\perp}] = \{0\}$. This means $D = M \oplus M^{\perp}$ because $\dim M + \dim M^{\perp} = \dim D$. Let $\delta \in M^{\perp}$. Then by the above equation

$$\{0\} = [M^{\perp}, M] \ni [\delta, \operatorname{ad}(X)] = \operatorname{ad}(\delta(\mathrm{X}))$$

   we get $\delta(X) \in Z(L) = \{0\}$, i.e. $M^{\perp} = \{0\}$ and $D = M$.

2. Give four possible non-isomorphic meanings for $\mathbb{Z}_p$.

   Reason: Localizations, p-adics and Factorization.

   Solution: The most common meaning is probably the factor ring

$$\mathbb{Z}_p = \mathbb{Z}/p \cdot \mathbb{Z} = \mathbb{Z}/\sim_p = \{0, 1, \ldots, p-1 \,|\, p \equiv 0\} = \mathbb{F}_p$$

   with the equivalence relation $a \sim_p b \Longleftrightarrow p \,|\, (a - b)$.

   A subset S of a commutative ring R with 1 is called multiplicative closed, if $1 \in S$ and $a, b \in S$ implies $a \cdot b \in S$. Now $(s, a) \sim_S (t, b) \Longleftrightarrow u \cdot (sb - at) = 0$ for some $u \in S$ defines an equivalence relation on $S \times R$. Its factor ring $S \times R/\sim_S =: S^{-1}R$ is called localization of R according to S. There are two natural multiplicative sets:

   The set $\{1, p, p^2, p^3, \ldots\}$ is obviously multiplicative closed for any given $p \in \mathbb{Z} - \{0\}$. In this case we can write

$$\{p^n \,|\, n \in \mathbb{N}_0\}^{-1}\mathbb{Z} = \mathbb{Z}_p = \left(p^{\mathbb{N}_0}\right)^{-1} \mathbb{Z} = p^{\mathbb{N}_0} \times \mathbb{Z}/\sim_{p^{\mathbb{N}_0}}$$

where $(p^n,a) \sim_{p^{\mathbb{N}_0}} (p^m,b) \Longleftrightarrow p^k(bp^n - ap^m) = 0$ for some $k \in \mathbb{N}_0$. Hence

$$\mathbb{Z}_p = \left\{ \frac{a}{p^n} \,\middle|\, a \in \mathbb{Z} \wedge n \in \mathbb{N}_0 \right\} \subseteq \mathbb{Q}$$

is the localization of $\mathbb{Z}$ at the element p.

Another naturally multiplicative closed set is $S = R - P$ where P is a prime ideal of R, since this is exactly the definition of a prime ideal. We can thus localize R at the prime ideal P and write it R_P. This means in our case for a prime ideal $(p) = p \cdot \mathbb{Z}$ in $\mathbb{Z}$ that

$$\mathbb{Z}_p = (p\mathbb{Z})^{-1}\mathbb{Z} = (p) \times \mathbb{Z}/\sim_{(p)} = \left\{ \frac{a}{b} \,\middle|\, a,b \in \mathbb{Z} \wedge p \nmid b \right\} \subseteq \mathbb{Q}$$

Finally we look at the p-adic numbers $\mathbb{Q}_p$ for a given prime p. They are an extension field of the rationals and can be written as $\sum_{i=-\infty}^{\infty} a_i p^i$ with coefficients $a_i \in \{0,1,\ldots,p-1\}$. Then we have the subring of integers of $p-$adic numbers

$$\mathbb{Q}_p \supseteq \mathbb{Z}_p = \left\{ \sum_{i=-\infty}^{\infty} a_i \cdot p^i \,\middle|\, a_i = 0 \,\forall\, i < 0 \right\} = \varprojlim_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z} \supseteq \mathbb{Z}$$

3. Let $T \subseteq (\mathbb{Z}_+^n, \preccurlyeq)$ with the partial natural ordering. Then there is a finite subset $S \subseteq T$ such that for every $t \in T$ exists a $s \in S$ with $s \preccurlyeq t$.

$$\alpha \preccurlyeq \beta \Longleftrightarrow \alpha_i \leq \beta_i \text{ for all } i = 1,\ldots,n$$

Reason: Gordan-Dickson Lemma.

Solution: There are no infinite descending chains under the natural ordering in $(\mathbb{Z}_+^n, \preccurlyeq)$, i.e. the set T_{min} of minimal elements of T is the smallest subset with the required property, and we must show that $S = T_{min}$ is finite.

We proceed by induction along n. The case $n = 1$ is obvious, since $|T_{min}| = 1$ in this case. Now assume $n > 1$ and that the statement is true for all $k < n$. For $k \geq 0$ we define

$$U_k = \{t' \in \mathbb{Z}_+^{n-1} \,|\, (t',k) \in T\} \;\wedge\; U := \bigcup_{k\geq 0} U_k$$

The sets $(U_k)_{min}$ and U_{min} are finite by induction hypothesis. Therefore exists an $m \geq 0$ such that $U_{min} \subseteq U_0 \cup \ldots \cup U_m$. Set

$$S := \bigcup_{k=0}^{m} ((U_k)_{min} \times \{k\}) \subseteq T.$$

which is thus finite. Let $t = (t', k) \in T$ with $t' \in \mathbb{Z}_+^{n-1}$ and $k \geq 0$. If $k \leq m$, then there is a $u \in (U_k)_{min}$ with $u \preccurlyeq t'$. Therefore $(u, k) \in S$ and $(u, k) \preccurlyeq (t', k)$. If $k > m$, then there is by the choice of m a $l \leq m$ and a $u \in (U_l)_{min}$ with $u \preccurlyeq t'$, i.e. $(u, l) \preccurlyeq (t', k)$. Since $(t', k) \in T$ was an arbitrary element of T, we have proven that S has the required property.

Another proof is possible by using Hilbert's basis theorem. The monomial ideal $\langle x^\alpha \,|\, \alpha \in T\rangle$ is generated by a finite set $\{x^\alpha \,|\, \alpha \in T\}$ by Hilbert's basis theorem. This set is necessarily of the form $\{x^\alpha \,|\, \alpha \in S\}$ for a finite subset $S \subseteq T$. As a generating set of the ideal, S has the required property. The Lemma of Gordan-Dickson is therefore a corollary of Hilbert's basis theorem.

4. (a) Solve the following linear differential equation system:

$$\dot{y}_1(t) = 11y_1(t) - 80y_2(t) \;\wedge\; \dot{y}_2(t) = y_1(t) - 5y_2(t)$$
$$y_1(0) = 0 \;\wedge\; y_2(0) = 0$$

   (b) Which solutions do $y_1(0) = \pm\varepsilon \;\wedge\; y_2(0) = \pm\varepsilon$ have?

   (c) How does the trajectory for $y_1(0) = 0.001 \;\wedge\; y_2(0) = 0.001$ behave for $t \to \infty$?

   (d) What will change if we substitute the coefficient -80 by -60?

   (e) Calculate (approximately) the radius of the osculating circle at $t = \pi/12$ for both trajectories with initial condition $\mathbf{y}(0) = (-1, 1)$.

Reason: Unstable Vortex and Repeller.

Solution:

(a) The system can be written as $\dot{\mathbf{y}}(t) = A\mathbf{y}(t)$ with $\mathbf{y}(t) = (y_1(t), y_2(t))$ and $A = \begin{bmatrix} 11 & -80 \\ 1 & -5 \end{bmatrix}$. Obviously solves $\mathbf{y}(t) \equiv 0$ the problem, and is a stable solution.

(b) To determine a general solution, we set $\mathbf{y}(t) = e^{\lambda t}\mathbf{u}$ and find

$$\dot{\mathbf{y}}(t) = \lambda e^{\lambda t}\mathbf{u} = A\mathbf{y}(t) = e^{\lambda t}A\mathbf{u}$$

so we have to solve $(A - \lambda)\mathbf{u} = 0$ which yields the eigenvalues $\lambda_{1,2} = 3 \pm 4i$ and eigenvectors

$$\mathbf{u} \in \{(1 - 2i, -i/4)^\tau, \, (-20i, 1 - 2i)^\tau\}$$

The general solution is thus the linear combination

$$\mathbf{y}(t) = c_1 e^{\lambda_1 t}\mathbf{u_1} + c_2 e^{\lambda_2 t}\mathbf{u_2}$$

$$\dot{\mathbf{y}}(t) = c_1\lambda_1 e^{\lambda_1 t}\mathbf{u_1} + c_2\lambda_2 e^{\lambda_2 t}\mathbf{u_2}$$

$$\begin{aligned}
\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} &= c_1 e^{(3+4i)t}\begin{bmatrix} 1-2i \\ -i/4 \end{bmatrix} + c_2 e^{(3-4i)t}\begin{bmatrix} -20i \\ 1-2i \end{bmatrix} \\
\begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix} &= c_1(3+4i)e^{(3+4i)t}\begin{bmatrix} 1-2i \\ -i/4 \end{bmatrix} + c_2(3-4i)e^{(3-4i)t}\begin{bmatrix} -20i \\ 1-2i \end{bmatrix} \\
&= c_1 e^{(3+4i)t}\begin{bmatrix} 11-2i \\ 1-3i/4 \end{bmatrix} + c_2 e^{(3-4i)t}\begin{bmatrix} -80-60i \\ -5-10i \end{bmatrix} \\
&= c_1 e^{(3+4i)t}\begin{bmatrix} 11(1-2i)-80(-i/4) \\ (1-2i)-5(-i/4) \end{bmatrix} \\
&\quad + c_2 e^{(3-4i)t}\begin{bmatrix} 11(-20i)-80(1-2i) \\ (-20i)-5(1-2i) \end{bmatrix} \\
&= c_1 e^{\lambda_1 t}A\mathbf{u_1} + c_2 e^{\lambda_2 t}A\mathbf{u_2} \\
&= c_1\dot{\mathbf{y}}_\mathbf{1}(t) + c_2\dot{\mathbf{y}}_\mathbf{2}(t)
\end{aligned}$$

$$\begin{aligned}
\mathbf{y_1}(t) &= e^{3t}\cdot e^{4it}\cdot\begin{bmatrix} 1-2i \\ -i/4 \end{bmatrix} = e^{3t}\cdot(\cos(4t)+i\sin(4t))\cdot\begin{bmatrix} 1-2i \\ -i/4 \end{bmatrix} \\
\frac{\mathbf{y_1}(t)}{e^{3t}} &= \begin{bmatrix} \cos(4t)+2\sin(4t) \\ (1/4)\sin(4t) \end{bmatrix} + i\begin{bmatrix} \sin(4t)-2\cos(4t) \\ -(1/4)\cos(4t) \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{y_2}(t) &= e^{3t}\cdot e^{-4it}\cdot\begin{bmatrix} -20i \\ 1-2i \end{bmatrix} = e^{3t}\cdot(\cos(4t)-i\sin(4t))\cdot\begin{bmatrix} -20i \\ 1-2i \end{bmatrix} \\
\frac{\mathbf{y_2}(t)}{e^{3t}} &= \begin{bmatrix} -20\sin(4t) \\ \cos(4t)-2\sin(4t) \end{bmatrix} + i\begin{bmatrix} -20\cos(4t) \\ -2\cos(4t)-\sin(4t) \end{bmatrix}
\end{aligned}$$

Hence the real solutions are given by

$$\mathbf{y}(t) = \alpha e^{3t}\begin{bmatrix} \cos(4t)+2\sin(4t) \\ (1/4)\sin(4t) \end{bmatrix} + \beta e^{3t}\begin{bmatrix} -20\sin(4t) \\ \cos(4t)-2\sin(4t) \end{bmatrix}$$

and

$$\begin{bmatrix} y_1(\pm\varepsilon) \\ y_2(\pm\varepsilon) \end{bmatrix} = \alpha e^{\pm 3\varepsilon}\begin{bmatrix} \cos(4\varepsilon)\pm 2\sin(4\varepsilon) \\ \pm(1/4)\sin(4\varepsilon) \end{bmatrix} + \beta e^{\pm 3\varepsilon}\begin{bmatrix} \mp 20\sin(4\varepsilon) \\ \cos(4\varepsilon)\mp 2\sin(4\varepsilon) \end{bmatrix}$$

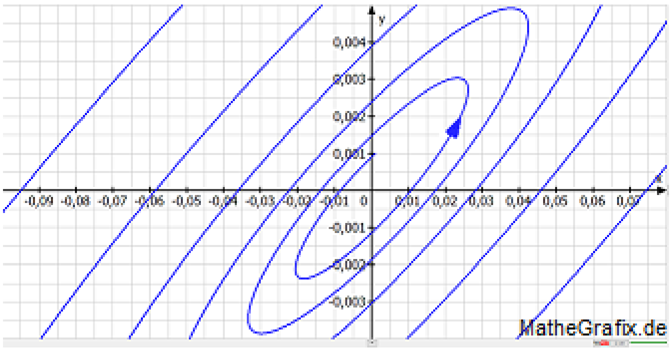
and

$$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} \pm\varepsilon \\ \pm\varepsilon \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \pm\varepsilon\begin{bmatrix} 1 \\ 0 \end{bmatrix} \pm \varepsilon\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This means that the solutions next to the fixed point $(0,0)$ drift away from it, and that any disturbance of the fixed point's initial condition repels from it. The origin is an unstable vortex.

(c) The trajectory from the initial condition $\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 1/1000 \\ 1/1000 \end{bmatrix}$ is

$$\mathbf{y}(t) = \frac{1}{1000} e^{3t} \begin{bmatrix} \cos(4t) - 18\sin(4t) \\ \cos(4t) - (7/4)\sin(4t) \end{bmatrix} \xrightarrow{t\to\infty} \begin{bmatrix} \text{oscillates } \pm\infty \\ \text{oscillates } \pm\infty \end{bmatrix}$$



(Image produced with MatheGrafix.de - not scaled $e^{3t} \to e^{0.3t}$)

(d) The system now writes $\dot{\mathbf{y}}(t) = B\mathbf{y}(t)$ with $\mathbf{y}(t) = (y_1(t), y_2(t))$ and $B = \begin{bmatrix} 11 & -60 \\ 1 & -5 \end{bmatrix}$. Again $\mathbf{y}(t) \equiv 0$ solves the linear problem, and is a stable solution. However we now have the characteristic polynomial

$$t^2 - 6t + 5 = (t-1)(t-5)$$

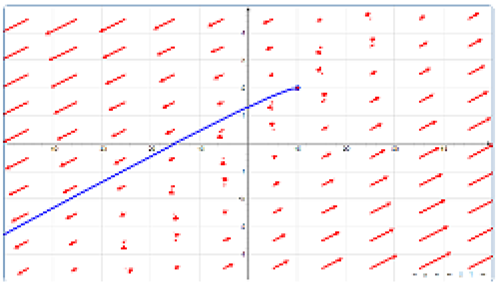
and real eigenvalues $\mu_{1,2}$ with eigenvectors

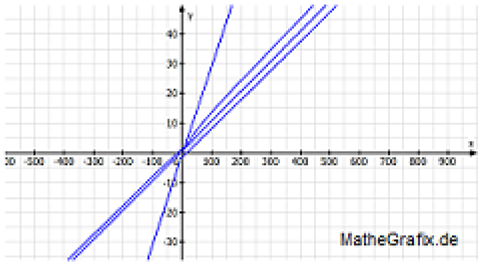
$$\mathbf{u} \in \{(6,1)^\tau, (10,1)^\tau\}$$

and general solution

$$\mathbf{y}(t) = \alpha e^t \begin{bmatrix} 6 \\ 1 \end{bmatrix} + \beta e^{5t} \begin{bmatrix} 10 \\ 1 \end{bmatrix}$$

Thus we not only have two real and positive eigenvalues now, we also don't have a vortex anymore at $(0,0)$. The vector field has now a repeller at the origin.





(e) The radius of the osculating circle is the reciprocal curvature and defined by the formula

$$R(\mathbf{y}(t)) = \left|\frac{(\dot{\mathbf{y}}_1^2(t) + \dot{\mathbf{y}}_2^2(t))^{3/2}}{\dot{\mathbf{y}}_1(t)\cdot\ddot{\mathbf{y}}_2(t) - \ddot{\mathbf{y}}_1(t)\cdot\dot{\mathbf{y}}_2(t)}\right|$$

The initial condition $\mathbf{y}(0) = (-1, 1)$ result in the two trajectories

$$\mathbf{y}_A(t) = e^{3t}\begin{bmatrix} -22\sin(4t) - \cos(4t) \\ -(9/4)\sin(4t) + \cos(4t) \end{bmatrix}$$

$$\dot{\mathbf{y}}_A(t) = e^{3t}\begin{bmatrix} -62\sin(4t) - 91\cos(4t) \\ -(43/4)\sin(4t) - 6\cos(4t) \end{bmatrix}$$

$$\ddot{\mathbf{y}}_A(t) = e^{3t}\begin{bmatrix} 550\sin(4t) - 521\cos(4t) \\ -(33/4)\sin(4t) - 61\cos(4t) \end{bmatrix}$$

$$\mathbf{y}_B(t) = e^{t}\begin{bmatrix} 33/2 \\ 11/4 \end{bmatrix} - e^{5t}\begin{bmatrix} 35/2 \\ 7/4 \end{bmatrix}$$

$$\dot{\mathbf{y}}_B(t) = e^{t}\begin{bmatrix} 33/2 \\ 11/4 \end{bmatrix} - e^{5t}\begin{bmatrix} 175/2 \\ 35/4 \end{bmatrix}$$

$$\ddot{\mathbf{y}}_B(t) = e^{t}\begin{bmatrix} 33/2 \\ 11/4 \end{bmatrix} - e^{5t}\begin{bmatrix} 875/2 \\ 175/4 \end{bmatrix}$$

$$\dot{\mathbf{y}}_A(\pi/12) = e^{\pi/4}\begin{bmatrix} -31\sqrt{3} - 91/2 \\ -(43/8)\sqrt{3} - 3 \end{bmatrix} \approx \begin{bmatrix} -217.56 \\ -27 \end{bmatrix}$$

$$\ddot{\mathbf{y}}_A(\pi/12) = e^{\pi/4}\begin{bmatrix} 275\sqrt{3} - 521/2 \\ -(33/8)\sqrt{3} - (61/2) \end{bmatrix} \approx \begin{bmatrix} 473.34 \\ -82.57 \end{bmatrix}$$

$$R(\mathbf{y}_A(t)) \approx 342.71$$

$$\dot{\mathbf{y}}_B(\pi/12) = e^{\pi/12}\begin{bmatrix} 33/2 \\ 11/4 \end{bmatrix} - e^{5\pi/12}\begin{bmatrix} 175/2 \\ 35/4 \end{bmatrix} \approx \begin{bmatrix} -302.53 \\ -28.82 \end{bmatrix}$$

$$\ddot{\mathbf{y}}_B(\pi/12) = e^{\pi/12}\begin{bmatrix} 33/2 \\ 11/4 \end{bmatrix} - e^{5\pi/12}\begin{bmatrix} 875/2 \\ 175/4 \end{bmatrix} \approx \begin{bmatrix} -1598.39 \\ -158.41 \end{bmatrix}$$

$$R(\mathbf{y}_B(t)) \approx 15,104.4$$

This calculation shows, that both solutions are very different even for small paths ($t \approx 0.2618$). The unstable vortex (A) has a low curvature, but it is still large compared to the curvature of the repeller (B). And all due to a reduction of one single coefficient about 25%. It also shows the importance of stability considerations for the solutions of even simple differential equations.

5. Consider the ideal $I = \langle x^2y + xy, xy^2 + 1\rangle \subseteq \mathbb{R}[x, y]$ and compute a reduced Gröbner basis to determine the number of irreducible components of the algebraic variety $V(I)$.

   Reason: Gröbner Basis.

   Solution: $\mathbb{R}[x, y]$ is partially ordered by $x \prec y$ according to which we define $LT(f)$ as the leading term of the polynomial $f \in \mathbb{R}[x, y]$ and $LC(f)$ as the leading coefficient of f. A **Gröbner basis** of I is a generating system $G = (g_1, \ldots, g_n)$ of polynomials, such that for all $f \in I - \{0\}$ there is a $g \in G$ whose leading term divides the one of f : $LT(g) \,|\, LT(g)$. A Gröbner basis is called **minimal**, if for all $g \in G$

   $$LT(g) \notin \langle LT(G - \{g\})\rangle \;\wedge\; LC(g) = 1.$$

   and **reduced** if no monomial of its elements $g \in G$ is an element of $\langle LT(G - \{g\})\rangle$ and $LC(g) = 1$. Reduced Gröbner basis are automatically minimal. They are also unique whereas the minimal ones do not need to be.

   Gröbner bases can be found by the Buchberger algorithm. We define for two polynomials $p, q \in I - \{0\}$ the division

   $$S(p, q) := \frac{lcm(LT(p), LT(q))}{LT(p)} \cdot p - \frac{lcm(LT(p), LT(q))}{LT(q)} \cdot q$$

Then Buchberger's algorithm can be written as

$$\begin{aligned} \text{INPUT: } & \{I\} = \{f_1, \ldots, f_n\} \\ \text{OUTPUT: } & \text{Gröbner basis } G = (g_1, \ldots, g_m) \\ \text{INIT: } & G := \{I\} \\ & 1.\ \text{DO} \\ & 2.\quad G' := G \\ & 3.\quad \text{FOREACH } p, q \in G'\,,\ p \neq q \\ & 4.\qquad s = remainder(S(p,q), G) \\ & 5.\qquad \text{IF } s \neq 0 \text{ THEN } G := G \cup \{s\} \\ & 6.\quad \text{NEXT} \\ & 7.\ \text{UNTIL } G = G' \end{aligned}$$

We start with $f_1(x,y) = x^2y + xy$, $f_2(x,y) = xy^2 + 1$ and compute

$$\begin{aligned} S(f_1, f_2) &= \frac{lcm(x^2y, xy^2)}{x^2y} f_1 - \frac{lcm(x^2y, xy^2)}{xy^2} f_2 \\ &= yf_1 - xf_2 = xy^2 - x = 1 \cdot f_2 - x - 1 \\ G' &= G \cup \{f_3 := -x - 1\} = \{f_1, f_2, f_3\} \\ S(f_1, f_3) &= \frac{lcm(x^2y, x)}{x^2y} f_1 - \frac{lcm(x^2y, x)}{-x} f_3 \\ &= f_1 + xyf_3 = x^2y + xy + xy(-x - 1) = 0 \\ S(f_2, f_3) &= \frac{lcm(xy^2, x)}{xy^2} f_2 - \frac{lcm(xy^2, x)}{-x} f_3 \\ &= f_2 + y^2 f_3 = xy^2 + 1 + y^2(-x - 1) = -y^2 + 1 \\ G' &= G \cup \{f_4 := -y^2 + 1\} = \{f_1, f_2, f_3, f_4\} \\ S(f_1, f_4) &= \frac{lcm(x^2y, y^2)}{x^2y} f_1 - \frac{lcm(x^2y, y^2)}{-y^2} f_4 = yf_1 + x^2 f_4 \\ &= x^2y^2 + xy^2 - x^2y^2 + x^2 = xy^2 + 1 + x^2 - 1 \\ &= f_2 - (x - 1)(-x - 1) = f_2 - xf_3 + f_3 \equiv 0 \mod G \\ S(f_2, f_4) &= \frac{lcm(xy^2, y^2)}{xy^2} f_2 - \frac{lcm(xy^2, y^2)}{-y^2} f_4 = f_2 + xf_4 \\ &= xy^2 + 1 - xy^2 + x = x + 1 = -f_3 \equiv 0 \mod G \\ S(f_3, f_4) &= \frac{lcm(x, y^2)}{-x} f_3 - \frac{lcm(x, y^2)}{-y^2} f_4 = -y^2 f_3 + xf_4 \\ &= y^2(x + 1) - xy^2 + x = y^2 + x \\ &= -f_2 - y^2 f_3 - f_3 \equiv 0 \mod G \end{aligned}$$

Hence we get a Gröbner basis $\{x^2y+xy, xy^2+1, -x-1, -y^2+1\}$ of I.

$$LT(f_1) = x^2y = (-xy)\cdot(-x) = (-xy)\cdot LT(f_3)$$
$$LT(f_2) = xy^2 = (-x)\cdot(-y^2) = (-x)\cdot LT(f_4)$$

means that $\{x+1, y^2-1\}$ is a minimal Gröbner basis, which is already reduced, because we cannot omit another leading term and the leading coefficients are normed to 1. The vanishing variety are thus the points $\{(-1,-1),(-1,1)\}$ which are two separated points, i.e. two irreducible components.

6. Define the complex function

$$\Gamma(z) := \lim_{n\to\infty} \frac{n!\,n^z}{z(z+1)\cdot\ldots\cdot(z+n)}$$

and prove

(a) $\Gamma(z) = \displaystyle\int_0^\infty e^{-t}\,t^{z-1}\,dt\ ;\quad \mathfrak{R}(z) > 0$

(b) $\Gamma(z)^{-1} = e^{\gamma z} z \displaystyle\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}$

where $\gamma := \displaystyle\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\log(n)\right)$ is the Euler-Mascheroni constant.

Reason: Gamma Function.

Solution:

(a) From $e^{-t} = \displaystyle\lim_{n\to\infty}\left(1-\frac{t}{n}\right)^n$ we get

$$\begin{aligned}
\int_0^\infty e^{-t}\,t^{z-1}\,dt &= \lim_{n\to\infty}\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\,dt \\
&\overset{t=ns}{=} \lim_{n\to\infty}\int_0^1 (1-s)^n n^z s^{z-1}\,ds \\
&= \lim_{n\to\infty} n^z \left(\left[\frac{s^z}{z}(1-s)^n\right]_0^1 + \frac{n}{z}\int_0^1 (1-s)^{n-1}s^z\,ds\right) \\
&\ \vdots \\
&= \lim_{n\to\infty} n^z \left(\frac{n}{z}\cdot\frac{n-1}{z+1}\cdot\ldots\cdot\frac{1}{z+n-1}\int_0^1 s^{z+n-1}\,ds\right) \\
&= \Gamma(z)
\end{aligned}$$

(b) Weierstraß expression.

$$\begin{aligned}\frac{z(z+1)\dots(z+n)}{n!n^z} &= \frac{1}{n^z}\cdot z\cdot\left(1+\frac{z}{1}\right)\cdot\ldots\cdot\left(1+\frac{z}{n}\right)\\ &= \frac{e^{\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)z}}{e^{(\log n)z}}\cdot z\cdot\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}}\\ &= e^{\left(1+\frac{1}{2}+\ldots+\frac{1}{n}-\log n\right)\cdot z}\cdot z\cdot\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}}\\ &\stackrel{n\to\infty}{\longrightarrow} e^{\gamma z}z\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}}\end{aligned}$$

7. Let $u\,:\,[0,1]\times[a,b]\longrightarrow\mathbb{C}$ be a continuous function, such that the partial derivative in the first coordinate exists everywhere and is continuous. Define

$$U(\lambda):=\int_a^b u(\lambda,t)\,dt\ ,\ V(\lambda):=\int_a^b\frac{\partial u}{\partial\lambda}(\lambda,t)\,dt.$$

Show that U is continuously differentiable and $U'(\lambda)=V(\lambda)$ for all $0\leq\lambda\leq 1$.

Reason: Complex Integration.

Solution: Let $\varepsilon>0$. We have to show that there is a $\delta>0$ such that for all $\lambda,h\in\mathbb{R}$ with $0<|h|<\delta$, $0\leq\lambda\leq 1$, and $0\leq\lambda+h\leq 1$

$$|V(\lambda+h)-V(\lambda)|<\varepsilon\,,\ \left|\frac{U(\lambda+h)-U(\lambda)}{h}-V(\lambda)\right|<\varepsilon.$$

Every continuous function on a compact interval is uniformly continuous, hence there is a $\delta>0$ such that for all $(\lambda,t),(\lambda',t')\in[0,1]\times[a,b]$

$$|\lambda'-\lambda|+|t'-t|<\delta\Longrightarrow\left|\frac{\partial u}{\partial\lambda}(\lambda',t')-\frac{\partial u}{\partial\lambda}(\lambda,t)\right|<\frac{\varepsilon}{b-a}.$$

By definition of V we have

$$\begin{aligned}|V(\lambda+h)-V(\lambda)| &= \left|\int_a^b\left(\frac{\partial u}{\partial\lambda}(\lambda+h,t)-\frac{\partial u}{\partial\lambda}(\lambda,t)\right)\,dt\right|\\ &\leq\int_a^b\left|\frac{\partial u}{\partial\lambda}(\lambda+h,t)-\frac{\partial u}{\partial\lambda}(\lambda,t)\right|\,dt\\ &<\varepsilon\end{aligned}$$

since the integrand is continuous and takes its maximum in $[a, b]$.

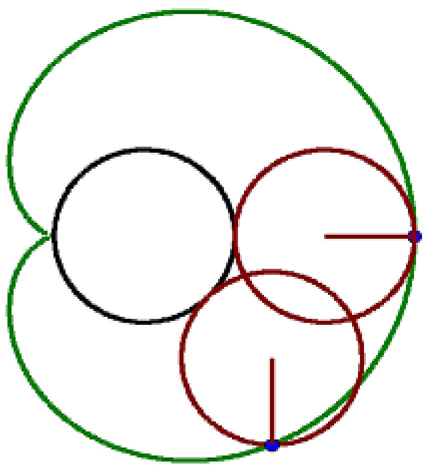
Now assume $0 < h < \delta$ such that $0 \leq \lambda < \lambda + h \leq 1$. (The case $h < 0$ is proven accordingly.) Therefore

$$\begin{aligned}\left|\frac{u(\lambda+h,t)-u(\lambda,t)}{h} - \frac{\partial u}{\partial \lambda}(\lambda,t)\right| &= \left|\frac{1}{h}\int_{\lambda}^{\lambda+h}\left(\frac{\partial u}{\partial \lambda}(\lambda',t) - \frac{\partial u}{\partial \lambda}(\lambda,t)\right)d\lambda'\right| \\ &\leq \frac{1}{h}\int_{\lambda}^{\lambda+h}\left|\frac{\partial u}{\partial \lambda}(\lambda',t) - \frac{\partial u}{\partial \lambda}(\lambda,t)\right| d\lambda' \\ &< \frac{\varepsilon}{b-a}\end{aligned}$$

and so

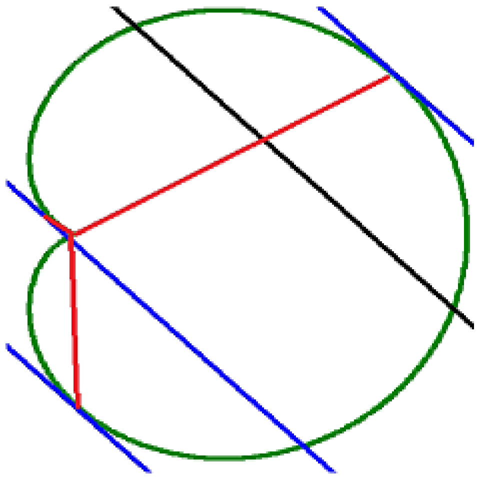
$$\begin{aligned}\left|\frac{U(\lambda+h)-U(\lambda)}{h} - V(\lambda)\right| &= \left|\int_a^b \left(\frac{u(\lambda+h,t)-u(\lambda,t)}{h} - \frac{\partial u}{\partial \lambda}(\lambda,t)\right) dt\right| \\ &\leq \int_a^b \left|\frac{u(\lambda+h,t)-u(\lambda,t)}{h} - \frac{\partial u}{\partial \lambda}(\lambda,t)\right| dt \\ &< \int_a^b \frac{\varepsilon}{b-a}\, dt = \varepsilon\end{aligned}$$

8. A cardiod is defined as the trace of a point on a circle that rolls around a fixed circle of the same size without slipping.



It can be described by $(x^2+y^2)^2 + 4x(x^2+y^2) - 4y^2 = 0$ or in polar coordinates by $r(\varphi) = 2(1-\cos\varphi)$. Show that:

(a) Given any line, there are exactly three tangents parallel to it. If we connect the points of tangency to the cusp, the three segments meet at equal angles of $2\pi/3$.



(b) The length of a chord through the cusp is 4.

(c) The midpoints of chords through the cusp lie on the perimeter of the fixed generator circle (black one in the first picture).

(d) Calculate length, area and curvature.

Reason: Cardioid.

Solution: The cartesian coordinates are given by

$$\begin{bmatrix} x(\varphi) \\ y(\varphi) \end{bmatrix} = r(\varphi) \cdot \begin{bmatrix} \cos(\varphi) \\ \sin(\varphi) \end{bmatrix} = 2 \cdot \begin{bmatrix} \cos(\varphi) - \cos^2(\varphi) \\ \sin(\varphi) - \cos(\varphi)\sin(\varphi) \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}(\varphi) \\ \dot{y}(\varphi) \end{bmatrix} = 2 \cdot \begin{bmatrix} \sin(2\varphi) - \sin(\varphi) \\ \cos(\varphi) - \cos(2\varphi) \end{bmatrix}, \left\| \begin{bmatrix} \dot{x}(\varphi) \\ \dot{y}(\varphi) \end{bmatrix} \right\|^2 = 8 - 8\cos(\varphi)$$

$$(x+1)^2 + y^2 = 1 \text{ generating circle}$$

This yields three vertical tangents $(\dot{x}(\varphi) = 0)$ at $\varphi = \pi/3, \varphi = \pi, \varphi = 5\pi/3$ and three horizontal tangents $(\dot{y}(\varphi) = 0)$ at $\varphi = 0$ (degenerated), $\varphi = 2\pi/3, \varphi = 4\pi/3$ and proves the first statement for tangent slopes $\{0, \infty\}$. In general we have with $\alpha = 2\pi/3$

(a)

$$\begin{aligned}\frac{\cos(\varphi+\alpha)-\cos(2\varphi+2\alpha)}{\sin(2\varphi+2\alpha)-\sin(\varphi+\alpha)} &= \frac{-\sqrt{3}\sin(\varphi)-\cos(\varphi)+\cos(2\varphi)-\sqrt{3}\sin(2\varphi)}{-\sqrt{3}\cos(2\varphi)-\sin(2\varphi)-\sqrt{3}\cos(\varphi)+\sin(\varphi)}\\ &= \frac{\sqrt{3}(\sin(2\varphi)+\sin(\varphi))+(\cos(\varphi)-\cos(2\varphi))}{\sqrt{3}(\cos(\varphi)+\cos(2\varphi))+(\sin(2\varphi)-\sin(\varphi))}\\ &\stackrel{(*)}{=} \frac{\cos(\varphi)-\cos(2\varphi)}{\sin(2\varphi)-\sin(\varphi)}\end{aligned}$$

We want to prove $(*)$ which is equivalent to show

$$\begin{aligned}\frac{\sin(2\varphi)+\sin(\varphi)}{\cos(\varphi)+\cos(2\varphi)} &= \frac{(\sin(2\varphi)+\sin(\varphi))(\cos(\varphi)-\cos(2\varphi))}{\cos^2(\varphi)-\cos^2(2\varphi)}\\ &= \frac{(\sin(2\varphi)+\sin(\varphi))(\cos(\varphi)-\cos(2\varphi))}{\sin^2(2\varphi)-\sin^2(\varphi)}\\ &= \frac{\cos(\varphi)-\cos(2\varphi)}{\sin(2\varphi)-\sin(\varphi)}\end{aligned}$$

(b) A chord through the cusp (origin) intersects the cardiod in $P = 2(1-\cos\varphi)$ and $Q = 2(1-\cos(\varphi+\pi))$. Thus

$$\begin{aligned}|PQ| &= r(\varphi)+r(\varphi+\pi)\\ &= 4-2\left(\cos(\varphi)+\cos(\varphi)\cos(\pi)-\sin(\varphi)\sin(\pi)\right)\\ &= 4-2(\cos(\varphi)-\cos(\varphi)-0) = 4\end{aligned}$$

(c)

$$\begin{aligned}(1/2)\overline{PQ} &= 1-\cos(\varphi)-(1-\cos(\varphi+\pi))\\ &= -\cos(\varphi)+\cos(\varphi)\cos(\pi)-\sin(\varphi)\sin(\pi)\\ &= -2\cos(\varphi)\\ (x+1)^2+y^2 &= (1-2\cos(\varphi)\cos(\varphi))^2+(-2\cos(\varphi)\sin(\varphi))^2\\ &= 1-4\cos^2(\varphi)+4\cos^4(\varphi)+4\cos^2(\varphi)\sin^2(\varphi)\\ &= 1-4\cos^2(\varphi)(1-\cos^2(\varphi)-\sin^2(\varphi)=1\end{aligned}$$

(d)

$$\begin{aligned}A &= 2\cdot\frac{1}{2}\int_0^\pi |r(\varphi)\times\dot{r}(\varphi)|\;d\varphi\\ &= 4\int_0^\pi (1-\cos(\varphi))^2\left|\begin{bmatrix}\cos(\varphi) & -\sin(\varphi)\\ \sin(\varphi) & \cos(\varphi)\end{bmatrix}\right|\;d\varphi\\ &= 4\cdot\left[\varphi-2\sin(\varphi)+(\varphi/2)+(1/2)\sin(\varphi)\cos(\varphi)\right]_0^\pi\\ &= 4(\pi+\pi/2) = 6\pi\end{aligned}$$

$$
\begin{aligned}
L &= 2\int_0^{\pi} \sqrt{r(\varphi)^2+\dot r(\varphi)^2}\,d\varphi \\
&= 4\int_0^{\pi} \sqrt{(1-\cos(\varphi))^2+(\sin(\varphi))^2}\,d\varphi \\
&= 4\int_0^{\pi} \sqrt{2-2\cos(\varphi)}\,d\varphi = 8\int_0^{\pi} \sin(\varphi/2)\,d\varphi \\
&= -16\,[\cos(\varphi/2)]_0^{\pi} = 16
\end{aligned}
$$

$$
\begin{aligned}
\kappa(\varphi) &= \frac{|r(\varphi)^2+2\dot r(\varphi)^2-r(\varphi)\ddot r(\varphi)|}{(r(\varphi)^2+\dot r(\varphi)^2)^{3/2}} \\
&= \frac{1}{2}\cdot\frac{|(1-\cos(\varphi))^2+2\sin^2(\varphi)-(1-\cos(\varphi))\cos(\varphi)|}{((1-\cos(\varphi))^2+(\sin(\varphi))^2)^{3/2}} \\
&= \frac{1}{2}\cdot\frac{3-3\cos(\varphi)}{(2-2\cos(\varphi))^{3/2}} = \frac{3}{4\sqrt{2}}\cdot\frac{1}{\sqrt{(1-\cos(\varphi))}} \\
&= \frac{3}{4\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sin(\varphi/2)} = \frac{3}{8}\cdot\frac{1}{\sin\left(\frac{\varphi}{2}\right)}
\end{aligned}
$$

9. Let A be a complex Banach algebra with 1. Prove that the spectrum

$$\sigma(a) = \{\lambda\in\mathbb{C}\,|\,\lambda\cdot 1 - a \text{ is not invertible }\} \subseteq \{\lambda\in\mathbb{C}\,|\,|\lambda|\leq\|a\|\}$$

for any $a\in A$ is not empty, bounded and closed.

Reason: Theorem of Gelfand.

Solution: Let $G(A) := \{a\in A\,|\,a \text{ is invertible }\}$. We first show that in case $\|a\|<1$ we have

$$1-a\in G(A)\,,\;(1-a)^{-1}=\sum_{k=0}^{\infty} a^k$$

which is referred to as the Neumann series for $(1-a)^{-1}$.

The norm of A is submultiplicative so the series converges absolutely, and by completeness of A it converges in A, say to $s\in A$. Then

$$(1-a)s = (1-a)\lim_{n\to\infty}\sum_{k=0}^{n} a^k = \lim_{n\to\infty}(1-a^{k+1}) = 1$$

because $\|a^{k+1}\|\leq\|a\|^{k+1}\stackrel{k\to\infty}{\longrightarrow} 0$. By the same argument we get the left inverse $s(1-a)=1$, hence $s=(1-a)^{-1}$.

Next we show that $G(A) \subseteq A$ is open and inversion $f : G(A) \longrightarrow A\,,\; f(a) = a^{-1}$ is continuous.

Let $a \in G(A)$ and $b \in A$ such that $\|a-b\| < \|a^{-1}\|^{-1}$. Then

$$\|1 - ba^{-1}\| = \|(a-b)a^{-1}\| \leq \|a-b\| \cdot \|a^{-1}\| < 1$$

so $1-(1-ba^{-1}) = ba^{-1} \in G(A)$ by the previous statement, and thus $b = (ba^{-1})a \in G(A)$ which means that $G(A) \subseteq A$ is open, because $G(A)$ is closed under multiplication.

Let's consider left- and right multiplications $l_b(a) = ba\,,\; r_b(a) = ab$ by $b \in A$. Now we can write inversion as

$$f(a) = (l_{a^{-1}} \circ f \circ r_{a^{-1}})(a)$$

for all $a \in G(A)$. Since $r_{a^{-1}}(a) = 1, f(1) = 1, l_{a^{-1}}(1) = a^{-1}$, and the maps $l_{a^{-1}}, r_{a^{-1}}$ are continuous at 1 and a, respectively for all $a \in G(A)$, it is sufficient to show that f is continuous at 1.
Let $\|1-b\| < \varepsilon < 1/2 < 1$. Then we get for $a := 1-b$ the Neumann series $\sum_{k=0}^{\infty} a^k = (1-a)^{-1} = b^{-1}$ and thus

$$\begin{aligned}\|f(b)-f(1)\| &= \|b^{-1}-1\| \leq \|1-b\| \cdot \|b^{-1}\| \\ &\leq \|a\| \sum_{k=0}^{\infty} \|a\|^k = \|a\|(1-\|a\|)^{-1} \leq 2\varepsilon\end{aligned}$$

which shows that f is continuous at 1.

If $|\lambda| > \|a\|$, then $\|\lambda^{-1}a\| < 1$ and $1 - \lambda^{-1}a \in G(A)$. Hence $\lambda - a = \lambda(1-\lambda^{-1}a) \in G(A)$ and $\lambda \notin \sigma(a)$ which proves the inclusion in the statement and that $\sigma(a)$ is bounded. Now define $g : \mathbb{C} \longrightarrow A$ by setting $g(\lambda) := \lambda \cdot 1 - a$. Then g is continuous, and thus $g^{-1}(G(A)) = \mathbb{C} - \sigma(a)$ is open as the preimage of the open set $G(A)$ which means that $\sigma(a)$ is closed.

It remains to show that $\sigma(a) \neq \emptyset$. Let $\lambda \in \mathbb{C} - \sigma(a)$, i.e. $\lambda - a \in G(A.)$ Since $G(A)$ is open, there is an $r > 0$ such that $\mu - a \in G(A)$ whenever $|\mu - \lambda| < r$.

$$\begin{aligned}(\mu-a)^{-1} - (\lambda-a)^{-1} &= (\mu-a)^{-1}((\lambda-a)(\lambda-a)^{-1}) \\ &\quad - ((\mu-a)^{-1}(\mu-a))(\lambda-a)^{-1} \\ &= (\mu-a)^{-1}((\lambda-a)-(\mu-a))(\lambda-a)^{-1} \\ &= (\lambda-\mu)(\mu-a)^{-1}(\lambda-a)^{-1}\end{aligned}$$

Hence for $\phi \in A^*$ we get by continuity of inversion and continuity of ϕ

$$\begin{aligned}\frac{\phi((\mu-a)^{-1})-\phi((\lambda-a)^{-1})}{\mu-\lambda} &= \frac{\phi((\mu-a)^{-1}-(\lambda-a)^{-1})}{\mu-\lambda}\\ &= \phi(-(\mu-a)^{-1}(\lambda-a)^{-1})\\ &\stackrel{\mu\to\lambda}{\longrightarrow} \phi(-(\lambda-a)^{-2})\end{aligned}$$

which means that $\lambda \longmapsto \phi((\lambda-a)^{-1})$ is analytic on $\mathbb{C}-\sigma(a)$. Using continuity of inversion again yields

$$\phi((\lambda-a)^{-1}) = \lambda^{-1}\phi((1-\lambda^{-1}a)^{-1}) \stackrel{|\lambda|\to\infty}{\longrightarrow} 0$$

Assume $\sigma(A)=\emptyset$. Then $\lambda \longmapsto \phi((\lambda-a)^{-1})$ is analytic on the whole plane $\mathbb{C}$, and bounded as we have seen above. By the theorem of Liouville the function is identically zero. Consequently $\phi(a^{-1})=0$ for all $\phi\in A^*$. By the theorem of Hahn-Banach this implies $a^{-1}=0$, which is a contradiction. Thus $\sigma(a)\neq\emptyset$.

10. (a) Determine all primes which occur as orders of an element from $G := \operatorname{SL}_3(\mathbb{Z})$.
    (b) Let $I \trianglelefteq R$ be a two-sided ideal in a unitary ring with group of unities U. Show by two different methods that
    $$M := \{u\in U \,|\, u-1\in I\} \trianglelefteq U$$
    is a normal subgroup.

Reason: Group Theory.

Solution:

(a) There are eight vectors in $\mathbb{Z}_2^3$. For a basis we can choose among $7=8-1$ of them as first basis vector, $6=8-2$ for the second, and $4=8-4$ for the last one. Hence there are $7\cdot 6\cdot 4 = 168 = 2^3\cdot 3\cdot 7$ possible ordered basis, which equals $|\operatorname{GL}(Z_2^3)| = |\operatorname{GL}_3(\mathbb{Z}_2)|$. Next we consider the induced homomorphism

$$\varphi \,:\, G=\operatorname{SL}_3(\mathbb{Z}) \rightarrowtail \operatorname{GL}_3(\mathbb{Z}) \twoheadrightarrow \operatorname{GL}_3(\mathbb{Z}_2)$$

For all elements $g\in G$ of order n holds $1=\varphi(1)=\varphi(g^n)=\varphi(g)^n\in\operatorname{GL}_3(\mathbb{Z}_2)$ and thus $n|168$ and possible prime orders are $\{2,3,7\}$. With

$$\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}\,,\,\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$$

we have elements of order 2 and 3 in G.

Assume there is an element $a \in G$ of order 7. Then a is a zero of

$$\mathbb{Z}[x] \ni p(x) := X^7 - 1 = \prod_{k=1}^{7}(x-\zeta_k) = \prod_{k=1}^{7}\left(x - e^{\frac{2k\pi i}{7}}\right)$$

The Jordan normal form of a states that a is a conjugate of a complex upper triangular matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ on its diagonal. Those eigenvalues are all among the $7-$th roots of unity, the zeros of $p(x)$, because the characteristic polynomial of a divides $p(x)$. We may assume that $\lambda_1 \neq 1$, since if $\lambda_1 = \lambda_2 = \lambda_3 = 1$ then a is a conjugate to a matrix of the Heisenberg group, and the Heisenberg group is torsion free. Thus λ_1 is a primitive $7-$th root of unity, and by choosing a suitable power of a we may assume that

$$\lambda_1 = \zeta_1 = e^{\frac{2\pi i}{7}} = \cos\left(\frac{2\pi}{7}\right) + i \cdot \sin\left(\frac{2\pi}{7}\right)$$

Moreover we have $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = \det(a) = 1$ and $\lambda_1 + \lambda_2 + \lambda_3 = \operatorname{tr}(a) \in \mathbb{Z}$. Thus $\lambda_3 = \zeta_1^{-1}\lambda_2^{-1}$ and so $\zeta_1 + \lambda_2 + \zeta_1^{-1}\lambda_2^{-1} \in \mathbb{Z}$. As λ_2 is also a $7-$th root of unity, we have only seven possible values for $\zeta_1 + \lambda_2 + \zeta_1^{-1}\lambda_2^{-1}$. Those values are only real if $\lambda_2 = 1$ or $\lambda_2 = \zeta_1^{-1}$. In either case we have

$$\zeta_1 + 1 + \zeta^{-1} \approx 2.247 \notin \mathbb{Z}$$

which disproves our assumption that there is an element of order 7. The only possible (and occurring) prime orders are 2 and 3.

(b) Let $n, m \in M$. Obviously we have $0 \in I$ which means $1 \in M$. Moreover $nm - 1 = (n-1)m + (m-1) \in I$ so $nm \in M$. Thirdly $n^{-1} - 1 = n^{-1}(1-n) \in I$ so $n^{-1} \in M$ which shows that $M \leq U$ is a subgroup. With $u \in U$ we have $unu^{-1} - 1 = u(n-1)u^{-1} \in I$ hence $unu^{-1} \in M$.

Alternatively we can consider the projection $\pi : R \twoheadrightarrow R/I$. Since π is a ring homomorphism, we have $1 = \pi(1) = \pi(n \cdot n^{-1}) = \pi(n) \cdot \pi^{-1}(n)$. Thus π induces a group homomorphism on the according groups of units:

$$\overline{\pi} : R^* = U \longrightarrow (R/I)^*$$

i.e. $\ker \overline{\pi} \trianglelefteq U$ is a normal subgroup. But $\ker \overline{\pi} = \{u \in U \mid \overline{\pi}(u) = u + I = 1 + I\} = \{u \in U \mid u - 1 \in I\} = M$.

11. (HS-1) If a, b, c are real numbers such that $a+b+c = 2$ and $ab+ac+bc = 1$, show that $0 \leq a, b, c \leq \frac{4}{3}$.

    Reason: Discriminant.

    Solution: We get the quadratic equation

    $$ab+(a+b)(2-(a+b)) = -(b^2+(a-2)b+(a^2-2a)) = 1 \text{ or } 0 = b^2+(a-2)b+(a-1)^2$$

    from the given conditions, which must have a non-negative discriminant

    $$\begin{aligned}(2-a)^2 - 4(a-1)^2 = -3a^2 + 4a \geq 0 &\Longrightarrow 4a \geq 3a^2 \geq 0 \\ &\Longrightarrow \frac{4}{3} \geq a \geq 0\end{aligned}$$

    which holds for symmetry reasons for b, c, too.

12. (HS-2) Determine all pairs (m, n) of (positive) natural numbers such that $2022^m - 2021^n$ is a square.

    Reason: Modular Arithmetics.

    Solution: $(1, 1)$ is obviously a solution. If $m = 1$ then $n = 1$ for otherwise the sum would be negative and cannot be a square. Hence we may assume $m \geq 2$. As a consequence $4 \,|\, 2022^m$ and

    $$x^2 := 2022^m - 2021^n \equiv -1 \equiv 3 \mod 4$$

    If x is even, then $x^2 \equiv 0 \mod 4$, and if $x = 2k+1$ is odd, then $x^2 = 4k^2 + 4k + 1 \equiv 1 \mod 4$. The only possible solution is thus $(n, m) = (1, 1)$.

13. (HS-3)

    (a) Prove for any $n \in \mathbb{N}$, $n \geq 4$

    $$Q(n) := \frac{4^2-9}{4^2-4} \cdot \frac{5^2-9}{5^2-4} \cdot \ldots \cdot \frac{n^2-9}{n^2-4} > \frac{1}{6}.$$

    (b) Is the above statement still true, if we replace $1/6$ on the right hand side by 0.1667?

    Reason: Inequality.

    Solution:

(a)

$$\begin{aligned}
&\frac{4^2-9}{4^2-4}\cdot\frac{5^2-9}{5^2-4}\cdot\ldots\cdot\frac{n^2-9}{n^2-4}\\
&\quad=\frac{(4-3)(4+3)}{(4-2)(4+2)}\cdot\frac{(5-3)(5+3)}{(5-2)(5+2)}\cdot\ldots\cdot\frac{(n-3)(n+3)}{(n-2)(n+2)}\\
&\quad=\frac{1\cdot 2\cdots(n-3)}{2\cdot 3\cdots(n-2)}\cdot\frac{7\cdot 8\cdots(n+3)}{6\cdot 7\cdots(n+2)}\\
&\quad=\frac{1}{n-2}\cdot\frac{n+3}{6}>\frac{1}{n}\cdot\frac{n}{6}=\frac{1}{6}
\end{aligned}$$

(b) For any $n > 25,002$ we get

$$\begin{aligned}
0.0002n > 5.00040 \Longrightarrow 1.0002(n-2) > n+3 &\Longrightarrow \frac{n+3}{n-2} < 1.0002\\
&\Longrightarrow \frac{1}{n-2}\cdot\frac{n+3}{6} < 0.1667\\
&\Longrightarrow 0.1667 > Q(25,002) > \frac{1}{6}
\end{aligned}$$

Alternatively consider the sequence $Q(n) := \dfrac{n+3}{6(n-2)}$. This sequence converges $\lim_{n\to\infty} Q(n) = \dfrac{1}{6}$. Hence for any $\varepsilon > 0$, especially for $\varepsilon = 0.1667 - (1/6)$, there are only finitely many exceptions n to

$$\left|Q(n)-\frac{1}{6}\right| < \varepsilon \Longleftrightarrow \frac{1}{6}-\varepsilon < Q(n) < \frac{1}{6}+\varepsilon = 0.1667$$

The answer is therefore 'no'. The lower limit cannot be improved.

14. (HS-4) Determine all pairs $(x, y) \in \mathbb{R}^2$ such that

$$\begin{aligned}
5 &= \sqrt{1+x+y}+\sqrt{2+x-y}\\
2-x+y &= \sqrt{18+x-y}
\end{aligned}$$

Reason: Quadratic Equations.

Solution: Set $u := x + y, v := x - y$. Then the equations become

$$\begin{aligned}
5 &= \sqrt{1+u}+\sqrt{2+v}\\
2-v &= \sqrt{18+v}
\end{aligned}$$

From the second equation we get $2-v\geq 0$ and $v^2-5v-14=0$, i.e.

$$v_{1,2}=\frac{5}{2}\pm\frac{1}{2}\sqrt{25+56}\in\{-2,7\}$$

hence $v=-2$ since $2-7<0$. Thus $\sqrt{1+u}=5-\sqrt{2+(-2)}=5$ and $u=24$. This means $x=11, y=13$ which also satisfy the initial equation system.

15. (HS-5) Given a real, continuous function $f:\mathbb{R}\longrightarrow\mathbb{R}$ such that $f(f(f(x)))=x$. Prove that $f(x)=x$ for all $x\in\mathbb{R}$.

Reason: Functions.

Solution: $f(x)$ is injective (into, one-to-one), i.e.

$$f(r)=f(s)\Longrightarrow r=f(f(f(r)))=f(f(f(s)))=s$$

and monotone. Assume that given $r<sr<t$ we have either

$$\begin{aligned}&f(r)<f(s)\,,\,f(s)>f(t)\quad\text{or}\\&f(r)>f(s)\,,\,f(s)<f(t)\end{aligned}$$

then there would be a real number $y\in\mathbb{R}$ with

$$\begin{aligned}&f(r)<y<f(s)\,\wedge\,f(s)>y>f(t)\quad\text{or}\\&f(r)>y>f(s)\,\wedge\,f(s)<y<f(t)\end{aligned}$$

and therefore also real numbers $x,x'\in\mathbb{R}$ such that

$$r<x<s<x'<t\,\wedge\,f(x)=f(x')=y$$

by the meanvalue theorem for continuous functions. By the previous part, it follows $x=x'$, a contradiction. $f(x)$ is therefore strictly monotone. Assume $f(x)$ is strictly monotone decreasing, i.e. $f(x)\neq x$ for an $x\in\mathbb{R}$, as $f(x)=x$ would be strictly monotone increasing. W.l.o.g. let

$$\begin{aligned}x<f(x)&\Longrightarrow f(x)>f(f(x))\\&\Longrightarrow f(f(x))<f(f(f(x)))=x\\&\Longrightarrow f(f(f(x)))=x>f(x)\end{aligned}$$

which is impossible. Hence $f(x)$ is strictly monotone increasing and one-to-one. Finally assume that $f(x)\neq x$ for an $x\in\mathbb{R}$ and $x<f(x)$ or $x>f(x)$. Then we apply f again twice and get $x<f(x)<f(f(x))<f(f(f(x)))=x$ or $x>f(x)>f(f(x))>f(f(f(x)))=x$ which is impossible. Therefore $f(x)=x$ for all $x\in\mathbb{R}$.

5 February 2021

1. Let f be a real, differentiable function such that there is no $x \in \mathbb{R}$ with $f(x) = 0 = f'(x)$. Show that f has at most finitely many zeros in the interval $[0, 1]$.

   Reason: Nice Proof.

   Solution: Set $S := \{x \in \mathbb{R} \,|\, f(x) = 0\} = [0, 1] \cap f^{-1}(\{0\})$. Then S is a compact set. If S is infinite, then it has a limit point

   $$S \ni x = \lim_{n\to\infty} x_n$$

   with a sequence $(x_n) \subseteq S$ of distinct points. Therefore $f(x_n) = f(x) = 0$ for all $n \in \mathbb{N}$. Now

   $$f'(x) = \lim_{x_n \to x} \frac{f(x + (x_n - x)) - f(x)}{x_n - x} = \lim_{n\to\infty} \frac{f(x_n) - f(x)}{x_n - x} = 0$$

   which contradicts our assumption.

2. Let (X, Ω, ω) be a measure space and f be a $\omega-$integrable function. Show that for every $\varepsilon > 0$ there is a set $W \in \Omega$ such that $\omega(W) < \infty$ and $\int_{X-W} |f| \, d\omega < \varepsilon$.

   Reason: Measure Theory.

   Solution: We define $A_n := \{x \in X \,|\, 1/n \leq |f(x)| < n\}$ for $n \in \mathbb{N}$ and $A_1 \subseteq A_2 \subseteq \ldots =: A = \cup_{n=1}^{\infty} A_n$. All $A_n = v^{-1}([1/n, n])$ with the continuous function $v(x) = |f(x)|$ are measurable. If we add $A_0 := \{x \in X \,|\, f(x) = 0\}$ and $A_\infty := \{x \in X \,|\, |f(x)| = \infty\}$ then $X = A_0 \cup A \cup A_\infty$ is a disjoint union and

   $$\int_X |f| \, d\omega = \int_{A_0} |f| \, d\omega + \int_A |f| \, d\omega + \int_{A_\infty} |f| \, d\omega = \int_A |f| \, d\omega$$

   so it is sufficient to find $W \subseteq A$.

   With $f_n := |f| \circ \chi_{A_n}$ we get a sequence of non-negative measurable functions, which converge pointwise to $|f| \circ \chi_A$. As $A_n \subseteq A_{n+1}$, we have $0 \leq f_1(x) \leq f_2(x) \leq \ldots$ and by the monotone convergence theorem

   $$\lim_{n\to\infty} \int_X f_n \, d\omega = \lim_{n\to\infty} \int_{A_n} |f| \, d\omega = \int_A |f| \, d\omega = \int_X |f| \, d\omega \,.$$

   Hence there is some $N > 0$ for which

   $$\int_{X-A_N} |f| \, d\omega < \varepsilon$$

and since $1/N \leq |f| < N$ on $W := A_N$

$$\omega(W) \leq N \int_W |f|\, d\omega \leq N \int_X |f|\, d\omega < \infty\,.$$

3. Prove or find a counterexample to:

   (a) L^2 convergence implies pointwise convergence.

   (b) $\displaystyle\lim_{n\to\infty} \int_0^\infty \frac{\sin x^n}{x^n}\, dx = 1$

   (c) Let (f_n) be a sequence of measurable functions which converge uniformly to zero on $[0,\infty)$. Then

$$\lim_{n\to\infty} \int_{[0,\infty)} f_n(x)\, dx = \int_{[0,\infty)} \lim_{n\to\infty} f_n(x)\, dx\,.$$

Reason: Convergence.

Solution: False - True - False.

(a) For each $k \in \mathbb{N}$ and $1 \leq i \leq k$ we define with the characteristic function $\chi(.)$ the functions $f_{k,i} := \chi\left(\left[\frac{i-1}{k}, \frac{i}{k}\right)\right)$ and the sequence (g_n) defined as

$$\begin{aligned}
&g_1 = f_{1,1}\,,\\
&g_2 = f_{2,1}\,,\; g_3 = f_{2,2}\,,\\
&g_4 = f_{3,1}\,,\; g_5 = f_{3,2}\,,\; g_6 = f_{3,3}\\
&g_7 = f_{4,1}\,,\; \ldots
\end{aligned}$$

Then $\int |f_{k,i}|^2\, d\mu = 1/k$ for each $1 \leq i \leq k$, i.e.

$$\lim_{k\to\infty} \|f_{k,i}\|_2 = \lim_{k\to\infty} \frac{1}{k} = 0 \Longrightarrow \lim_{n\to\infty} \|g_n\|_2 = 0$$

But (g_n) does not converge pointwise:

For every $N \in \mathbb{N}$ and every $x \in [0,1]$ there is a pair (k,i) such that $g_n(x) = f_{k,i}(x) = 1$ for all $n \geq N$, and we can find a pair (k',i') such that $g_{n'}(x) = f_{k',i'}(x) = 0$ for all $n' \geq N$.

(b) $\sin\varphi = \varphi - \dfrac{\varphi^3}{3!} + \dfrac{\varphi^5}{5!} \mp \ldots$ so for $0 < x < 1$ we get $\displaystyle\lim_{n\to\infty} \frac{\sin x^n}{x^n} = 1$, and for $\varphi \geq 0$

$$|\sin\varphi| \leq \int_0^\varphi |\cos x|\, dx \leq \int_0^\varphi 1\, dx = \varphi\,.$$

For $\varphi < 0$ we have $|\sin\varphi| = |\sin(-\varphi)| \leq -\varphi = |\varphi|$.

In particular we have $\left|\dfrac{\sin x^n}{x^n}\right| \leq 1$ on $(0,1)$ and by the dominant convergence theorem

$$\lim_{n\to\infty}\int_0^1 \frac{\sin x^n}{x^n}\,dx = \int_0^1 \lim_{n\to\infty}\frac{\sin x^n}{x^n}\,dx = \int_0^1 1\,dx = 1$$

Substitute $u = x^n$ and $du = nx^{n-1}dx = nu^{1-\frac{1}{n}}dx$ for

$$\int_1^N \frac{\sin x^n}{x^n}\,dx = \int_1^{N^n}\frac{\sin u}{u}\,\frac{du}{nu^{1-\frac{1}{n}}} = \frac{1}{n}\int_1^{N^n}\frac{\sin u}{u^{2-\frac{1}{n}}}\,du$$

Now for $n \geq 2$

$$\begin{aligned}\left|\int_1^\infty \frac{\sin x^n}{x^n}\,dx\right| &= \lim_{N\to\infty}\frac{1}{n}\left|\int_1^{N^n}\frac{\sin u}{u^{2-\frac{1}{n}}}\,du\right| \leq \lim_{N\to\infty}\frac{1}{n}\,u^{\frac{1}{n}-2}\,du\\ &= \lim_{N\to\infty}\frac{1}{n}\,\frac{u^{\frac{1}{n}-1}}{\frac{1}{n}-1}\Bigg|_1^{N^n} = \frac{1}{1-n}\,\left(N^{1-n}-1\right) = \frac{1}{n-1}\end{aligned}$$

which implies

$$\lim_{n\to\infty}\int_1^\infty \frac{\sin x^n}{x^n}\,dx = 0$$

and thus

$$\lim_{n\to\infty}\int_0^\infty \frac{\sin x^n}{x^n}\,dx = \lim_{n\to\infty}\int_0^1 \frac{\sin x^n}{x^n}\,dx + \lim_{n\to\infty}\int_1^\infty \frac{\sin x^n}{x^n}\,dx = 1+0 = 1$$

(c) The sequence $f_n = \frac{1}{n}\chi([0,n)$ converges uniformly to zero, i.e. $\int \lim f_n = 0$. But $\int f_n = 1$ for all $n \in \mathbb{N}$, i.e. $\lim \int f_n = 1 \neq 0$.

4. Let (a_n) be a sequence of positive real numbers such that the series $\displaystyle\sum_{n=1}^\infty a_n =: C < \infty$ converges. Show that $\displaystyle\sum_{n=1}^\infty \left(\prod_{k=1}^n a_k\right)^{1/n} \leq e\cdot C.$

Reason: Carleman's inequality.

Solution: We denote the geometric and arithmetic means by

$$\operatorname{GM}(a_1,\ldots,a_n) = \left(\prod_{k=1}^n a_k\right)^{1/n} < \operatorname{AM}(a_1,\ldots,a_n) = \frac{a_1+\ldots+a_n}{n}\,.$$

We first show $e \geq (n+1)(n!)^{-1/n}$. From Stirling's formula we get

$$\begin{aligned}\sqrt{2\pi}n^{n+1/2}e^{-n} &\leq n!\\ \sqrt[n]{2\pi}\cdot\frac{n}{e}\cdot\sqrt[2n]{n} &\leq \sqrt[n]{n!}\\ \frac{1}{\sqrt[n]{n!}} &\leq \frac{1}{\sqrt[n]{2\pi}\cdot\sqrt[2n]{n}}\cdot\frac{e}{n}\end{aligned}$$

and

$$\begin{aligned}\left(1+\frac{1}{n}\right)^{2n} &\leq e^2 \leq n\cdot 4\pi^2\\ 1+\frac{1}{n} &\leq \sqrt[n]{2\pi}\cdot\sqrt[2n]{n}\\ \frac{1}{n\cdot\sqrt[n]{2\pi}\cdot\sqrt[2n]{n}} &\leq \frac{1}{n+1}\end{aligned}$$

Hence combining both we get

$$\frac{1}{\sqrt[n]{n!}} \leq \frac{e}{n+1}\,.$$

With the notation above we have

$$\begin{aligned}\mathrm{GM}(a_1,\ldots,a_n) &= \mathrm{GM}(a_1,2a_2,\ldots,na_n)(n!)^{-1/n}\\ &\leq \mathrm{AM}(a_1,2a_2,\ldots,na_n)(n!)^{-1/n}\\ &\leq \frac{e}{n(n+1)}\sum_{k=1}^{n} ka_k\end{aligned}$$

Thus

$$\sum_{n=1}^{\infty}\mathrm{GM}(a_1,\ldots,a_n) \leq e\sum_{k=1}^{\infty}\underbrace{\left(\sum_{n=k}^{\infty}\frac{k}{n(n+1)}\right)}_{\overset{(*)}{=}1} a_k = e\sum_{k=1}^{\infty}a_k = e\cdot C$$

If the inequality wasn't a strict one, then

$$\mathrm{GM}(a_1,2a_2,\ldots,na_n) = \mathrm{AM}(a_1,2a_2,\ldots,na_n) \Longrightarrow a_k = \frac{a_1}{k}$$

but the harmonic series is divergent.

$$\begin{aligned}(*)\quad \sum_{n=k}^{m}\frac{k}{n(n+1)} &= \sum_{n=1}^{m}\frac{k}{n(n+1)} - \sum_{n=1}^{k-1}\frac{k}{n(n+1)}\\ &= k\cdot\left(\frac{m}{m+1}-\frac{k-1}{k}\right) = 1-\frac{k}{m+1} \overset{m\to\infty}{\longrightarrow} 1\end{aligned}$$

5. Let $(E,\mathcal{T})$ be a normal Hausdorff space, and $U_1,\ldots,U_n$ a finite open covering of E. Then there are continuous functions $g_1,\ldots,g_n\,:\,(E,\mathcal{T})\longrightarrow [0,1]$ such that $g_1+\ldots+g_n=1$ on E and $g_j(E-U_j)=\{0\}$ for all $1\leq j\leq n$.

   Reason: Important Topological Result: Partition of Unity.

   Solution: We first show that there are n closed subsets $F_1,\ldots,F_n\subseteq E$, such that $F_j\subseteq U_j$ for all $1\leq j\leq n$ and $F_1\cup\ldots\cup F_n=E$.

   The set $G_1:=E-(U_2\cup\ldots\cup U_n)\subseteq U_1$ is closed, and we can find an open set V_1 such that $G_1\subseteq V_1\subseteq\overline{V}_1\subseteq U_1$ where $V_1\cup U_2\ldots\cup U_n=E$ is still a finite open covering. Now we proceed by setting $G_2:=E-(V_1\cup U_3\ldots\cup U_n)\subseteq U_2$ which is closed, i.e. again we find an open set V_2 such that $G_2\subseteq V_2\subseteq\overline{V}_2\subseteq U_2$ where $V_1\cup V_2\cup U_3\ldots\cup U_n=E$ is still a finite open covering. Iteration up to n yields the closed sets $F_1:=\overline{V}_1,\ldots,F_n:=\overline{V}_n$, such that $E=V_1\cup\ldots\cup V_n$ is still a finite open covering with $F_j\subseteq U_j$ for all $1\leq j\leq n$.

   Since F_j and $E-U_j$ are disjoint closed sets, we may apply Urysohn's lemma and find continuous functions $f_j\,:\,E\longrightarrow[0,1]$ such that $f_j=1$ on F_j and $f_j=0$ on $E-U_j$ for all $1\leq j\leq n$. We finally define $g_1:=f_1\,,\,g_2:=f_2\cdot(1-f_1)\,,\,\ldots\,,\,g_n:=f_n\cdot(1-f_{n-1})\cdot\ldots\cdot(1-f_1)$. With these functions we get $g_j(E-U_j)=\{0\}$ for all $1\leq j\leq n$ and $1-(g_1+\ldots+g_j)=(1-f_1)\cdot\ldots\cdot(1-f_j)$. The case $j=n$ finishes the proof, since the A_j are a (closed) covering of E.

6. Let (X,Ω,ω) be a measure space and $1\leq p<\infty$. Show that

   (a) $\tilde{L}^p:=L^p(X,\Omega,\omega)$ is a Banach space with respect to $\|\cdot\|_p$.

   (b) The sequence $(\|f_n\|_p)\subseteq\mathbb{R}$ is bounded for every Cauchy sequence $(f_n)\subseteq L^p(X,\Omega,\omega)$.

   Reason: Functional Analysis.

   Solution:

   (a) Let $(f_n)_{n\in\mathbb{N}}\subseteq\tilde{L}^p$ be a Cauchy sequence, i.e. for every $\varepsilon>0$ there is a N_ε such that $\|f_n-f_m\|_p<\varepsilon$ for all $n,m\geq N_\varepsilon$. Thus we have a subsequence $(f_{n_k})_{k\in\mathbb{N}}\subseteq(f_n)_{n\in\mathbb{N}}$ such that $\|f_{n_k}-f_m\|_p<2^{-k}$ for all $m\geq n_k$. If we define $g_k:=f_{n_k}-f_{n_{k+1}}$, then

$$\left\|\sum_{k=1}^n|g_k|\right\|_p\leq\sum_{k=1}^n\|g_k\|_p\leq\sum_{k=1}^n\frac{1}{2^k}<1$$

for all $1 \leq n < \infty$, so the sequence of partial sums is convergent if it is bounded by the theorem of monotone convergence:

$$\left\| \sum_{k=1}^{\infty} |g_k| \right\|_p \leq \lim_{n\to\infty} \sum_{k=1}^{n} \|g_k\|_p \leq \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$

Hence the sequence $(\sum_{k=1}^{n} g_k)_{n\in\mathbb{N}}$ converges $\omega-$almost everywhere (a.e.) absolutely, and $f_{n_1} - f_{n_k} = \sum_{j=1}^{n_{k-1}} g_j$ converges $\omega-$a.e. for $k \to \infty$. So $f_{n_k} = f_{n_1} + \sum_{j=1}^{n_{k-1}} g_j$ converges $\omega-$a.e. for $k \to \infty$ to a function $f = f_{n_1} + \sum_{j=1}^{\infty} g_j$. We have thus found a convergent subsequence and it remains to show that $f \in \tilde{L}^p$, i.e. $\|f\|_p < \infty$, and $\lim_{n\to\infty} \|f_n - f\|_p = 0$.

For $\varepsilon > 0$ we choose $N_\varepsilon \in \mathbb{N}$ such that $\|f_{n_k} - f_m\|_p < \varepsilon$ for all $n_k, m \geq N_\varepsilon$ and apply Fatou's Lemma on the sequence $(|f_{n_k} - f_m|^p)_{k\in\mathbb{N}}$ and get for all $m \geq N_\varepsilon$

$$\int |f - f_m|^p d\omega = \int \liminf_{k\to\infty} |f_{n_k} - f_m|^p d\omega = \liminf_{k\to\infty} \int |f_{n_k} - f_m|^p d\omega \leq \varepsilon^p$$

hence $\lim_{m\to\infty} \|f_m - f\|_p = 0$ and $\|f\|_p \leq \|f - f_{n_1}\|_p + \|f_{n_1}\|_p < \infty$, i.e. $f \in \tilde{L}_p$.

(b) This follows immediately from the previous part, since there is a $N_0 \in \mathbb{N}$ such that $\|f_n - f\|_p \leq C < \infty$ for all $n \geq N_0$ and $\|f_n\|_p \leq \|f_n - f\|_p + \|f\|_p \leq C + \|f\|_p < \infty$.

7. We know that there are only two sets in $(\mathbb{R}, |\cdot|)$, which are open and closed, the empty set and the entire topological space. Prove it.

   Reason: Open And Closed Sets.

   Solution: Let $\emptyset \neq M \subsetneq \mathbb{R}$ be open and closed, and $x \in \mathbb{R} - M, y \in M$. W.l.o.g. we may assume that $x > y$. Then x is an upper bound of

   $$N := \{z \in \mathbb{R} \,|\, [y, z] \subseteq M\}$$

   and the real number $s := \sup N$ exists. Then $s - 1/n$ is no upper bound for all $n \in \mathbb{N}$ and we can find numbers $z_n \in N$ such that $s - 1/n < z_n \leq s$, i.e. $\lim_{n\to\infty} z_n = s$. Since $N \subseteq M$ we have $z_n \in M$, and since M is closed, $s \in M$. Now M is open as well, so there is a $r > 0$ such that

   $$[s - r, s + r] \subseteq M.$$

   From $z_n \in N$, i.e. $[y, z_n] \subseteq M$, and $\lim_{n\to\infty} z_n = s$ we get $[y, s+r] \subseteq M$, i.e. $s + r \in N$. However, s is an upper bound of N and there cannot be an element greater than s in N. This means that one of our assumptions was wrong and either $M = \emptyset$ or $\mathbb{R} - M = \emptyset$.

8. Let (V, α) and (W, β) be irreducible representations of an associative, complex algebra $\mathcal{A}$. Assume that V and W are complex and of countable dimension. Then

$$\dim \operatorname{Hom}_{\mathcal{A}}(V, W) = \begin{cases} 1 & \text{if } (V, \alpha) \cong (W, \beta) \\ 0 & \text{otherwise} \end{cases}$$

Reason: Schur's Lemma.

Solution: Given a $\mathcal{A}-$homomorphism $\varphi : V \longrightarrow W$ kernel and range are invariant subspaces of V, W, resp. If $\varphi \neq 0$ then $\ker \varphi \neq V$ and $\operatorname{range} \varphi \neq \{0\}$. By irreducibility we get $\ker \varphi = \{0\}$ and $\operatorname{range} \varphi = V$, which means that φ is a linear isomorphism. Hence $\operatorname{Hom}_{\mathcal{A}}(V, W) \neq \{0\}$ if and only if $(V, \alpha) \cong (W, \beta)$.

Next we have to show that $\operatorname{Hom}_{\mathcal{A}}(V, W)$ is one dimensional in case the representations are equivalent, which means that the only possible homomorphism is a multiple of the identity operator. Let $\varphi, \psi \in \operatorname{Hom}_{\mathcal{A}}(V, W) - \{0\}$ and $\rho := \psi^{-1}\varphi \in \operatorname{End}_{\mathcal{A}}(V)$. Assume further that $\rho \notin \mathbb{C} \cdot \operatorname{Id}_V$. This means that $\rho - \lambda I$ is non zero for any $\lambda \in \mathbb{C}$ and thus invertible. We will show that the set

$$\{(\rho - \lambda_k I)^{-1}(v) \,|\, 1 \leq k \leq m\}$$

is linear independent for any $v \in V - \{0\}$ and pairwise distinct complex numbers $\lambda_1, \ldots, \lambda_m$, which contradicts the countable dimensionality of V.

Let

$$\sum_{k=1}^{m} c_k (\rho - \lambda_k I)^{-1}(v) = 0 \text{ and } f(x) := \sum_{k=1}^{m} c_k \prod_{l \neq k} (x - \lambda_l I)$$

Then

$$f(\rho)(v) = \sum_{k=1}^{m} c_k \prod_{l \neq k} (\rho - \lambda_l I)(v) = \prod_{l=1}^{m} (\rho - \lambda_l I) \cdot \sum_{k=1}^{m} c_k (\rho - \lambda_k)^{-1}(v) = 0$$

Now $f(\lambda_j) = c_j \prod_{l \neq j} (\lambda_j - \lambda_l)$. If $c_j \neq 0$, then $f(x)$ is a nonzero polynomial and has a factorization $f(x) = c(x - z_1) \ldots (x - z_{m-1})$ with $c \neq 0$ and $z_i \in \mathbb{C}$. We know that all $\rho - z_i I$ are invertible and so is $f(\rho)$, in which case $f(\rho)(v)$ cannot be zero for $v \neq 0$. Hence $c_j = 0$ for all $1 \leq j \leq m$.

9. Let $U \subseteq \mathbb{C}$ be an open connected neighborhood and $f : U \longrightarrow \mathbb{C}$ a holomorphic function. If $|f|$ has a local maximum in $z_0 \in U$, i.e. there is an open neighborhood $z_0 \in U_0 \subseteq U$ with $|f(z_0)| \geq |f(z)|$ for all $z \in U_0$, then f is constant.

   Reason: Maximum Principle.

   Solution: Since f is holomorphic, we can write

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

f as a power series in a neighborhood of z_0 (**Cauchy-Taylor**). Assume s is its radius of convergence and $0 < r < s$. Note that for $m, n \in \mathbb{N}_0$

$$\int_0^{2\pi} e^{it(n-m)}\, dt = \begin{cases} 0 & \text{if } m \neq n \\ 2\pi & \text{if } m = n \end{cases}$$

so uniform convergence and $f(z_0 + re^{it}) = \sum_{n \in \mathbb{N}_0} a_n r^n e^{itn}$ yields

$$\int_0^{2\pi} f(z_0 + re^{it}) e^{-itm}\, dt = \sum_{n \in \mathbb{N}_0} a_n r^n \int_0^{2\pi} e^{it(n-m)}\, dt = 2\pi a_m r^m$$

From

$$|f(z_0 + re^{it})|^2 = f(z_0 + re^{it})\overline{f(z_0 + re^{it})} = \sum_{n \in \mathbb{N}_0} \overline{a}_n r^n f(z_0 + re^{it}) e^{-itn}$$

we get (again with uniform convergence) the **Gutzmersche formula**

$$\begin{aligned} \int_0^{2\pi} |f(z_0 + re^{it})|^2\, dt &= \sum_{n=0}^{\infty} \overline{a}_n r^n \int_0^{2\pi} f(z_0 + re^{it}) e^{-itn}\, dt \\ &= 2\pi \sum_{n=0}^{\infty} |a_n|^2 r^{2n} \\ &\leq 2\pi (\max\{|f(z)| \,:\, |z - z_0| = r\})^2 \end{aligned}$$

From $|f(z_0)| \geq |f(z)|$ we get for small $r > 0$

$$\begin{aligned} (\max\{|f(z)| \,:\, |z - z_0| = r\})^2 + \sum_{n=1}^{\infty} |a_n|^2 r^{2n} &\leq |a_0|^2 + \sum_{n=1}^{\infty} |a_n|^2 r^{2n} \\ &\leq (\max\{|f(z)| \,:\, |z - z_0| = r\})^2 \\ &\Longrightarrow \end{aligned}$$

$$a_n = 0 \quad (n \geq 1) \quad \wedge \quad f(z) = a_0 \quad (|z - z_0| < r)$$

This means that the set $\{z \in U \, : \, f(z) = g(z)\}$ with $g(z) := a_0$ has a limit point in U. Since U is connected we can apply the **identity theorem** $(*)$ and conclude

$$f(z) \equiv g(z) \equiv a_0 \quad (z \in U)$$

Proof of the identity theorem $(*)$:

Let z_1 be a limit point of the coincidence set $V := \{z \in U \, : \, f(z) = g(z)\}$. For the sake of simplicity in notation we choose $z_1 = 0$. Now assume that there is a natural number $n \in \mathbb{N}_0$ such that $f^{(n)}(0) \neq g^{(n)}(0)$ and that n is the smallest among them. Then we have in a neighborhood of $z = 0$

$$f(z) - g(z) = z^n \underbrace{\sum_{k=0}^{\infty} \frac{f^{(n+k)}(0) - g^{(n+k)}(0)}{(n+k)!} z^k}_{=:h(z)}$$

and $V = \{z \, : \, h(z) = 0\}$ since h is continuous. In particular we have $0 = h(0) = \dfrac{f^{(n)} - g^{(n)}}{n!}$ contradicting minimality and choice of n. This means that $f^{(n)}(z_1) = g^{(n)}(z_1)$ for all $n \in \mathbb{N}_0$.

Because U is connected, it is sufficient to show that

$$A := \{z \in U \, : \, f^{(n)}(z) = g^{(n)}(z) \, \forall \, n \in \mathbb{N}_0 \}$$

is nonempty, open and closed in U, in order to conclude $f = g$ on U. With $z_1 \in A$ we have $A \neq \emptyset$ and by

$$A_n := \{z \in U \, : \, f^{(n)}(z) = g^{(n)}(z) \} = (f^{(n)} - g^{(n)})^{-1}(\{0\})$$

we see that $A = \cap_{n \in \mathbb{N}_0} A_n$ is closed. If $z \in A$ then the holomorphic function $f - g$ is identical to its Taylor series in a neighborhood of z (**Cauchy-Taylor**), i.e. identical zero. However, this neighborhood is part of A, hence A is also open.

10. Show that all groups G of order pqr with pairwise distinct primes $p < q < r$ are solvable.

    Reason: Groups of Order pqr.

    Solution: The number of $r-$Sylow-subgroups of G is congruent 1 modulo r and a divisor of pq according to the Sylow theorems, i.e. it equals either 1 or pq. If there is only one $r-$Sylow-subgroup H of G, then H is a normal subgroup of prime order, thus cyclic and solvable. But G/H is a group of order pq, which are solvable as well (see problem no. 10 from Dec. 2018). Thus G is solvable.

Now we assume that there are pq many $r-$Sylow-subgroups of G. The number of elements in an intersection of two different $r-$Sylow-subgroups is a proper divisor of the prime r, hence 1. Every nontrivial element of a $r-$Sylow-subgroup is of order r, and all elements of order r in G are contained in one $r-$Sylow-subgroup. Thus G contains exactly $pq(r-1)$ elements of order r. The number m of $q-$Sylow-subgroups of G is congruent 1 modulo q and a divisor of pr. The only possibilities are $1, r, pr$, since $1 < p < q < r$. ($1 < p = m \cdot q + 1 < q \Longrightarrow m = 0$ ↯)

If $m = 1$ then this $q-$Sylow-subgroup H is normal, of prime order q, hence cyclic and thus solvable, and as $|G/H| = pr$, G/H is solvable as well (see problem no. 10 from Dec. 2018), hence G is solvable. If $m \in \{r, pr\}$ then there are exactly $r(q-1)$, or $pr(q-1)$ resp. many elements of order q according to the same argument as above. Now we have $pq(r-1)$ elements of order r and at least $r(q-1)$ many elements of order q, i.e.

$$|G| = pqr \geq pq(r-1) + r(q-1) \geq pq(r-1) + rp = pqr + p(r-q) > pqr$$

which is not possible. So the cases $m \in \{r, pr\}$ do not exist and G is solvable.

11. (HS-1) Let z be a natural number with 1995 decimal digits and $1 \leq n \leq 1994$. Then we note the number, which we get by cutting off the first n digits and append them in the same order at the end of z by $z^{[n]}$. Show that if z is divisible by 27, then all $z^{[n]}$ are divisible by 27, too.

    Reason: Puzzle with 1995.

    Solution: It is sufficient to show the statement for $n = 1$, because if it is true for $n = 1$ we can repeat the process as long as we need for any $1 \leq n \leq 1994$. Let a be the first digit of z, i.e. $z = 10^{1994} + b$ and $z^{[1]} = 10b + a$ for some $b \in \mathbb{N}$.

$$10z - z^{[1]} = a \cdot (10^{1995} - 1) = a \cdot (1000^{665} - 1)$$

From $1000 = 37 \cdot 27 + 1$ we get $1000^{665} \equiv 1^{665} \equiv 1 \mod 27$. Hence $27 \,|\, (10z - z^{[1]})$ and if $27 \,|\, z$ then $27 \,|\, z^{[1]} = 10z - (10z - z^{[1]})$.

12. (HS-2) Let a, b, c, d be positive real numbers. Prove (in the logically correct order)
$$\frac{1}{\dfrac{1}{a}+\dfrac{1}{b}}+\frac{1}{\dfrac{1}{c}+\dfrac{1}{d}} \leq \frac{1}{\dfrac{1}{a+c}+\dfrac{1}{b+d}}$$

Reason: Inequality.

Solution:

$$\begin{aligned}
0 &\leq (ad-bc)^2 \\
2abcd &\leq a^2d^2+b^2c^2 \|_{+ab(c^2+d^2)+cd(a^2+2ab+b^2)} \\
ab(c+d)^2+cd(a+b)^2 &\leq (ad+bc)(ac+ad+bc+bd) \|_{+(ab+cd)(a+b)(c+d)} \\
ab(c+d)(a+b+c+d)+cd(a+b)(a+b+c+d) & \\
&\leq (a+c)(b+d)(a+b)(c+d) \|_{:[(a+b)(c+d)(a+b+c+d)]} \\
\frac{ab}{a+b}+\frac{cd}{c+d} &\leq \frac{(a+c)(b+d)}{a+b+c+d} \\
\frac{1}{\dfrac{1}{a}+\dfrac{1}{b}}+\frac{1}{\dfrac{1}{c}+\dfrac{1}{d}} &\leq \frac{1}{\dfrac{1}{a+c}+\dfrac{1}{b+d}}
\end{aligned}$$

13. (HS-3) Let $m \geq 2$ be a given natural number. We define a sequence $(x_0, x_1, x_2, \ldots)$ of numbers by $x_0 = 0, x_1 = 1$, and for $n \geq 0$ we set x_{n+2} to be the remainder of $x_{n+1} + x_n$ by division by m, chosen such that $0 \leq x_{n+2} < m$. Decide whether for every $m \geq 2$ there exists a natural number $k \geq 1$, such that $x_{k+2} = 1\,,\ x_{k+1} = 1\,,\ x_k = 0$.

Reason: Sequence.

Solution: There are at most m^3 possible triplets $(x_j, x_{j+1}, x_{j+2})_{j\in\mathbb{N}} \in \{0, 1, \ldots, m-1\}^3$ so there have to be repetitions. Hence there are $j \geq 0, k \geq 1$ with

$$x_j = x_{k+j}\,,\ x_{j+1} = x_{k+j+1}\,,\ x_{j+2} = x_{k+j+2}$$

Assume $j > 0$. Per construction we know that $x_{j+1} \equiv x_j + x_{j-1} \mod m$ and $x_{k+j+1} \equiv x_{k+j} + x_{k+j-1} \mod m$, hence subtraction yields

$$x_{j+1} - x_{k+j+1} = 0 = x_j + x_{j-1} - x_{k+j} - x_{k+j-1} \equiv x_{j-1} - x_{k+j-1} \mod m \Longrightarrow m \mid (x_{j-1} - x_{k+j-1})$$

which is only possible if $x_{j-1} = x_{k+j-1}$ for number between 0 and $m-1$. We can repeat this argument until

$$0 = x_0 = x_k\ ,\ 1 = x_1 = x_{k+1}\ ,\ 1 = x_2 = x_{k+2}$$

14. (HS-4) We define real functions

$$f_n(x) := x^3 + (n+3)\cdot x^2 + 2n\cdot x - \frac{n}{n+1}$$

for every non-negative integer $n \geq 0$. Determine all values of n, such that all zeros of $f_n(x)$ are contained in an interval of length 3.

Reason: Polynomial.

Solution: For $n=0$ we have $f_0(x) = x^3 + 3x^2 = x^2(x-(-3))$ with the zeros $0, -3 \in [-3, 0]$. Now assume $n > 0$. Here we have

$$\begin{aligned} f_n(-n-3) &= 2n(-n-3) - \frac{n}{n+1} = -2n^2 - 6n - \frac{n}{n+1} < 0 \\ f_n(-2) &= -8 + 4(n+3) - 4n - \frac{n}{+n+1} = 4 - \frac{n}{n+1} > 0 \\ f_n(0) &= -\frac{n}{n+1} < 0 \\ f_n(1) &= 1 + (n+3) + 2n - \frac{n}{n+1} = 3n + 4 - \frac{n}{n+1} > 0 \end{aligned}$$

hence all f_n have three pairwise distinct real zeros, say $a < b < c$. Vieta's formulas are thus

$$a+b+c = -(n+3)\,,\; ab+ac+bc = 2n\,,\; abc = \frac{n}{n+1}$$

Now

$$\begin{aligned} (c-a)^2 &= (a+b+c)^2 - 3(ab+ac+bc) + \underbrace{(c-b)(b-a)}_{>0} \\ &> (n+3)^2 - 6n = n^2 + 9 > 9 \implies c - a > 3 \end{aligned}$$

Thus we have only for $n=0$ that all zeros of f_0 are within a distance of three, whereas they are further apart for all other f_n $(n \geq 1)$.

15. (HS-5)

    (a) Determine the number of all pairs of integers $(x,y) \in \mathbb{N}_0^2$ with $\sqrt{x} + \sqrt{y} = 1993$.

    (b) Determine for every $n \in \mathbb{N}$ the greatest power of 2 which divides $[(4+\sqrt{18})^n]$.

Reason: Calculus.

Solution:

(a) From $\sqrt{x}+\sqrt{y}=1993$ we get

$$\begin{aligned} y &= (1993-\sqrt{x})^2 = 1993^2 - 3986\sqrt{x}+x \\ \sqrt{x} &= \frac{1993^2+x-y}{3986} \in \mathbb{Q} \\ x &= \left(\frac{u}{v}\right)^2 \text{ for some } u,v \in \mathbb{N}_0 \\ u^2 &= v^2 \cdot x \end{aligned}$$

Because u^2, v^2 have an even number of primes, so does x, i.e. $x=a^2$ for some $a\in\mathbb{N}_0$. For the same reason is $y=b^2$ for some $b\in\mathbb{N}_0$, hence combined: $a+b=1993$.

On the other hand are any two integers $a,b\geq 0$ for which $a+b=1993$ holds, a solution to $\sqrt{x}+\sqrt{y}=1993$ with $x=a^2, y=b^2$.

Thus we have shown that there are as many integer solutions as there are pairs (a,b), which are the following

$$(0,1993)\,,\,(1,1992)\,,\,\ldots\,,\,(1993,0)$$

1994 possible pairs.

(b) For the sequences $(a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0}, (c_n)_{n\in\mathbb{N}_0}$ defined by

$$a_n := (4+\sqrt{18})^n\,,\, b_n := (4-\sqrt{18})^n\,,\, c_n := a_n+b_n \in \mathbb{Z}$$

we have the following recursions

$$\begin{aligned} a_{n+2} &= a_n(4+\sqrt{18})^2 = a_n(34+8\sqrt{18}) \\ &= 2a_n(4(4+\sqrt{18})+1) = 2\cdot(4a_{n+1}+a_n) \\ b_{n+2} &= b_n(4-\sqrt{18})^2 = b_n(34-8\sqrt{18}) \\ &= 2b_n(4(4-\sqrt{18})+1) = 2\cdot(4b_{n+1}+b_n) \\ c_{n+2} &= 2\cdot(4c_{n+1}+c_n) \in \mathbb{Z} \end{aligned}$$

with $c_0=2\,,\, c_1=8$. Next we prove that for all integers c_n holds:

$$S(k) := \begin{cases} 2^{k+1} \mid c_{2k} & \wedge \quad 2^{k+2} \nmid c_{2k} \\ 2^{k+3} \mid c_{2k+1} & \wedge \quad 2^{k+4} \nmid c_{2k+1} \end{cases}$$

We know already that $S(0)$ is true. Now assume $S(k)$ is also true. Then there are odd integers s,t such that $c_{2k}=s\cdot 2^{k+1}\,,\, c_{2k+1}=$

$t \cdot 2^{k+3}$. Thus

$$\begin{aligned}
c_{2k+2} &= 2 \cdot (4c_{2k+1} + c_{2k}) = 2 \cdot (4(t \cdot 2^{k+3}) + (s \cdot 2^{k+1})) \\
&= t \cdot 2^{k+6} + s \cdot 2^{k+2} \equiv \begin{cases} 0 & \bmod 2^{(k+1)+1} \\ 1 & \bmod 2^{(k+1)+2} \end{cases} \\
c_{2k+3} &= 2 \cdot (4c_{2k+2} + c_{2k+1}) \\
&= 2 \cdot (4(t \cdot 2^{k+6} + s \cdot 2^{k+2}) + t \cdot 2^{k+3}) \\
&= t \cdot (2^{k+9} + 2^{k+5}) + s \cdot 2^{k+4} \\
&\equiv \begin{cases} 0 & \bmod 2^{(k+1)+3} \\ 1 & \bmod 2^{(k+1)+4} \end{cases}
\end{aligned}$$

which proves $S(k+1)$ and the truth of the statement by induction.

$$\begin{aligned}
&4 < \sqrt{18} < 5 \\
&\implies -1 < 4 - \sqrt{18} < 0 \\
&\implies \begin{cases} 0 < b_n < 1 & \text{for } n = 2k \\ -1 < b_n < 0 & \text{for } n = 2k+1 \end{cases} \\
&\implies \begin{cases} a_n < c_n < a_n + 1 & \text{for } n = 2k \\ a_n - 1 < c_n < a_n & \text{for } n = 2k+1 \end{cases} \\
&\implies \begin{cases} c_n - 1 < a_n < c_n & \text{for } n = 2k \\ c_n < a_n < c_n + 1 & \text{for } n = 2k+1 \end{cases} \\
&\implies \begin{cases} [a_n] = c_n - 1 & \text{for } n = 2k \\ [a_n] = c_n & \text{for } n = 2k+1 \end{cases}
\end{aligned}$$

The greatest power of 2 which divides $[a_n] = [(4+\sqrt{18})^n]$ is thus for even n according to $S(2k)$ the number 0, because a_n is odd, and for odd n according to $S(2k+1)$ the number $2^{k+3} = 2^{(n+5)/2}$.

6 January 2021

1. Let $A \in \mathbb{M}_{m,n}(\mathbb{R})$ and $b \in \mathbb{R}^m$. Then exactly one of the following two statements is true:

   - $Ax = b\,,\; x \geq 0\,,$ is solvable for a $x \in \mathbb{R}^n$.
   - $A^\tau y \leq 0\,,\; b^\tau y > 0\,,$ is solvable for some $y \in \mathbb{R}^m$.

   The ordering is meant componentwise.

   Reason: Farkas Lemma.

   Solution:

   Both statements cannot be simultaneously true, as

   $$0 < y^\tau b = y^\tau (Ax) = (y^\tau A)x = (A^\tau y)^\tau x \leq 0$$

   The statement can be proven with the **strict separation theorem**:

   Let $K \subseteq \mathbb{R}^n$ be convex, nonempty, closed, and $x \notin K$. Then there is a hyperplane $H = \{y \in \mathbb{R}^n \,|\, a^\tau y = \gamma\}$ with $a \in \mathbb{R}^n - \{0\}\,,\, \gamma \in \mathbb{R}$, which separates x and K, i.e. $a^\tau z \leq \gamma < a^\tau x$ for all $z \in K$. Moreover if K is additionally a cone, then we may choose $\gamma = 0$.

   or the **strong duality theorem**:

   Let $z = c^\tau x \longrightarrow \min!$ with $Ax = b\,,\; x \geq 0$ for $c, x \in \mathbb{R}^n$. the primal optimization problem, and $\tilde{z} := b^\tau y \longrightarrow \max!$ with $A^\tau y \leq c$ for $c \in \mathbb{R}^n\,,\; y \in \mathbb{R}^m$ its dual problem.

   The primal problem has a finite optimal solution if and only if its dual problem has a finite optimal solution, in which case $z_{\min} = \tilde{z}_{\max}\,.$

   (a) (strict separation theorem)

   Assume that the first statement is false. Then $b \notin K := \{Ax \,|\, x \in \mathbb{R}^n, x \geq 0\}$, which is a convex, polyhedral, closed cone. Thus we can separate b and K, i.e. there is a vector $y \in \mathbb{R}^m - \{0\}$ such that

   $$y^\tau Ax \leq 0 < y^\tau b$$

   If we choose subsequently all unit vectors $x = (0, \ldots, 0, 1, 0, \ldots, 0)$ then $A^\tau y \leq 0$ which had to be shown.

   (b) (strong duality theorem)

   Set $M := \{x \in \mathbb{R}^n \,|\, Ax = b, x \geq 0\}$. The statement can thus be rephrased by

$$M \neq \emptyset \Longleftrightarrow \forall\, y \in \mathbb{R}^m \,(A^\tau y \leq 0 \Longrightarrow b^\tau y \leq 0)$$

"$\Longrightarrow$":

If $M \neq \emptyset$ then there is a solution $x^* \in M$ to the primal problem

$$c^\tau x \;(x \in M) \longrightarrow \min! \quad (P)$$

with $c = 0$. This means by the strict duality theorem, that the dual problem

$$b^\tau y \;(y \in N := \{y \in \mathbb{R}^m \,|\, A^\tau y \leq c = 0\}) \longrightarrow \max! \quad (D)$$

has a solution $y^* \in N$, too, since $0 \in N$, i.e. $N \neq \emptyset$. Moreover $\min(P) = \max(D)$ and so for all $y \in \mathbb{R}^m$ with $A^\tau y \leq 0$

$$b^\tau y \leq b^\tau y^* = \max(D) = \min(P) = 0^\tau x^* = 0$$

"$\Longleftarrow$":

We consider again the dual problem (D), which is feasible since $0 \in N$ and $b^\tau y \leq 0$ for all $y \in N$. Hence $\sup(D) < \infty$. Assume that $M = \emptyset$. By the strong duality theorem we then had $\sup(D) = \infty$, a contradiction, so M cannot be empty.

2. Prove $\pi = \lim\limits_{n\to\infty} 2^n \underbrace{\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}}}_{n \text{ square roots}}.$

Reason: Viète's formula.

Solution: We first prove that

$$\sin x = 2^n \sin \frac{x}{2^n} \left(\prod_{k=1}^{n} \cos \frac{x}{2^k} \right)$$

This is a simple induction on n. For $n = 1$ we have the known formula for half angles

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$$

and the induction step is

$$\begin{aligned}
2^{n+1}\sin\frac{x}{2^{n+1}}&\left(\prod_{k=1}^{n+1}\cos\frac{x}{2^k}\right)\\
&=2\cdot 2^n\sin\left(\frac{1}{2}\cdot\frac{x}{2^n}\right)\cos\left(\frac{1}{2}\cdot\frac{x}{2^n}\right)\left(\prod_{k=1}^{n}\cos\frac{x}{2^k}\right)\\
&=2^n\sin\frac{x}{2^n}\left(\prod_{k=1}^{n}\cos\frac{x}{2^k}\right)\\
&=\sin x
\end{aligned}$$

From the Taylor expansion of the exponential formula and Euler's formula $e^{\mathbf{i}x}=\cos x+\mathbf{i}\sin x$ we get the series expansion of the sine function $\sin x = x-\dfrac{x^3}{3!}\pm\ldots$ which shows that

$$\sin x=\lim_{n\to\infty}\sin x=\lim_{n\to\infty}\left(2^n\sin\frac{x}{2^n}\right)\cdot\lim_{n\to\infty}\prod_{k=1}^{n}\cos\frac{x}{2^k}=x\cdot\prod_{k=1}^{\infty}\cos\frac{x}{2^k}$$

For $x=\pi/2$ we get the formula

$$\frac{2}{\pi}=\prod_{k=1}^{\infty}\cos\frac{\pi}{2^{k+1}}$$

or

$$\pi=2\cdot\prod_{k=1}^{\infty}\left(\cos\frac{\pi}{2^{k+1}}\right)^{-1}$$

The half angle formula for the cosine function is

$$\begin{aligned}
\cos(2x)&=\cos^2 x-\sin^2 x=1-2\sin^2 x=2\cos^2 x-1\\
&\Longrightarrow \cos\frac{x}{2}=\sqrt{\frac{1}{2}+\frac{1}{2}\cdot\cos x}\Longrightarrow\left(\cos\frac{x}{2}\right)^{-1}=\sqrt{\frac{2}{1+\cos x}}
\end{aligned}$$

Set $a_0=0\,,\ a_n=\sqrt{\dfrac{1}{2}+\dfrac{1}{2}\cdot a_{n-1}}\,,\ 2a_n^2=1+a_{n-1}\quad(n\geq 1)$. We show

$$\prod_{k=1}^{n}\frac{1}{a_k}=2^n\sqrt{1-a_n^2}$$

which is true for $n = 0, 1$.

$$\prod_{k=1}^{n+1} \frac{1}{a_k} = 2^n\sqrt{1-a_n^2}\cdot\frac{1}{a_{n+1}} = 2^n\sqrt{1-a_n^2}\cdot\sqrt{\frac{2}{1+a_n}}$$
$$= 2^n\sqrt{2(1-a_n)} = 2^{n+1}\sqrt{1-\left(\frac{1}{2}+\frac{1}{2}a_n\right)}$$
$$= 2^{n+1}\sqrt{1-a_{n+1}^2}$$

Note that $\cos\frac{\pi}{2^{n+1}} = a_n$ since

$$a_0 = 0 = \cos\frac{\pi}{2}$$
$$a_1 = \sqrt{\frac{1}{2}} = \cos\frac{\pi}{4}$$
$$a_{n+1} = \sqrt{\frac{1}{2}+\frac{1}{2}\cdot a_n} = \sqrt{\frac{1}{2}+\frac{1}{2}\cdot\cos\frac{\pi}{2^{n+1}}} = \cos\frac{\pi}{2^{n+2}}$$

which combines to

$$\pi = 2\cdot\prod_{k=1}^{\infty}\left(\cos\frac{\pi}{2^{k+1}}\right)^{-1} = 2\lim_{n\to\infty}\prod_{k=1}^{n}\left(\cos\frac{\pi}{2^{k+1}}\right)^{-1} = \lim_{n\to\infty}2^{n+1}\sqrt{1-a_n^2}$$
$$= \lim_{n\to\infty}2^n\sqrt{4-(2+2a_{n-1})} = \lim_{n\to\infty}2^n\sqrt{2-2a_{n-1}}$$
$$= \lim_{n\to\infty}2^n\sqrt{2-\sqrt{2+2a_{n-2}}} = \ldots$$
$$= \lim_{n\to\infty}2^n\underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{n \text{ square roots}}$$

3. Let $z(t)$ be a non-negative continuous real function on the interval $[a, b]$ and $t_0 \in [a, b]$. Prove that if

$$z(t) \leq C + L\left|\int_{t_0}^{t} z(s)\,ds\right| \quad (*)$$

for all $t \in [a, b]$ with any constants $C, L \geq 0$, then

$$z(t) \leq Ce^{L|t-t_0|} \quad (**)$$

for all $t \in [a,b]$.

Reason: Grönwall-Lemma.

Solution: The Grönwall-Lemma is often stated for $t_0 = a$ in which case all absolute values can be omitted as $t \geq t_0 = a$.

W.l.o.g. we assume $C > 0$. Indeed, if $C = 0$, then the condition $(*)$ holds for any positive $C > 0$ as well. Now if this implies $(**)$, then

$$z(t) \leq \lim_{C \searrow 0} Ce^{L|t-t_0|} = 0$$

We define the function F on $[t_0, b]$ by

$$F(t) := C + L\int_{t_0}^{t} z(s)\,ds$$

which is strictly positive and differentiable with $F' = L \cdot z$. Condition $(*)$ means $z \leq F$ for $t \in [t_0, b]$ and so

$$\begin{aligned} F' = Lz \leq LF &\Longrightarrow \frac{F'}{F} \leq L \\ &\Longrightarrow \log \frac{F(t)}{F(t_0)} = \int_{t_0}^{t} \frac{F'(s)}{F(s)}\,ds \leq \int_{t_0}^{t} L\,ds = L(t-t_0) \\ &\Longrightarrow z(t) \leq F(t) \leq F(t_0)e^{L(t-t_0)} = Ce^{L(t-t_0)} \end{aligned}$$

Since $z \leq F$ we get the inequality $(**)$ for all $t \in [t_0, b]$.

We consider the function

$$G(t) = C + L\int_{t}^{t_0} z(s)\,ds$$

for the interval $[a, t_0]$. which is also positive and differentiable with $G' = -Lz$ and $z \leq G$ by $(*)$, so $G' \geq -LG$. Hence

$$\begin{aligned} \log \frac{G(t_0)}{G(t)} = \int_{t}^{t_0} \frac{G'(s)}{G(s)}\,ds \geq -\int_{t}^{t_0} L\,ds = -L(t_0 - t) = -L|t-t_0| \\ \Longrightarrow z(t) \leq G(t) \leq G(t_0)e^{L|t-t_0|} = Ce^{L|t-t_0|} \end{aligned}$$

4. Solve the partial differential equation

$$\begin{gathered} u : D \longrightarrow \mathbb{R}\,,\; D \subseteq \mathbb{R}^3 \\ xu_x + yu_y + (x^2+y^2)u_z = 0 \\ u(1,0,0) = 0\,,\; u_x(1,0,1) = 0 \\ u_y(-1,1,(\pi+2)/2) = 1\,,\; u_z(-1,1,(\pi+2)/2) = -1 \end{gathered}$$

Reason: Differential Equation. PDE.

Solution: The characteristic system of this PDE is

$$\dot{x} = x \, , \, \dot{y} = y \, , \, \dot{z} = x^2 + y^2$$

with the general (characteristic) solutions

$$x(t) = \alpha e^t \, , \, y(t) = \beta e^t \, , \, z(t) = \frac{1}{2} \left(\alpha^2 + \beta^2 \right) e^{2t} + \gamma$$

The solution of the equation is thus

$$u(x(t), y(t), z(t)) = u \left(\alpha e^t, \beta e^t, \frac{1}{2} \left(\alpha^2 + \beta^2 \right) e^{2t} + \gamma \right) = \text{ constant.}$$

For the characteristic flows we have the relations:

$$e^t = \frac{x(t)}{\alpha} = \frac{y(t)}{\beta} \Longrightarrow \frac{y(t)}{x(t)} = \frac{\alpha}{\beta} =: a \in \mathbb{R}$$

$$z(t) = \frac{1}{2} \left(x^2 + y^2 \right) + \gamma \Longrightarrow z(t) - \frac{1}{2} / (x(t)^2 + y(t)^2) =: b \in \mathbb{R}$$

i.e. the two constants a, b alone define the value of u along the characteristic flows. The representation of the solution is thus

$$u(x, y, z) = \Phi \left(\frac{y}{x}, z - \frac{1}{2}(x^2 + y^2) \right)$$

for any differentiable function $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

This means we have infinitely many possible solutions and the initial values are useless. Those initial values given here correspond to $\Phi(a, b) = a^2 \sin(b)$, but this is not a unique solution.

5. Let $A \in \mathbb{M}(n, \mathbb{R})$ be a real square matrix and $x : \mathbb{R} \longrightarrow \mathbb{R}^n$ a parameterized path. Prove that there exists a unique solution of the differential equation $\dot{x}(t) = Ax(t)$ for any initial condition $x(t_0) = x_0$.

   Reason: Differential Equation. Matrix.

   Solution: We first show that $x(t) := e^{A(t-t_0)} x_0$ is a solution which proves the existence part.

$$x(t_0) = e^0 x_0 = \sum_{k=0}^{\infty} 0^k x_0 \frac{t^k}{k!} = 1 \cdot x_0 = x_0$$

$$\dot{x}(t) = \frac{d}{dt} x(t) = \sum_{k=0}^{\infty} \frac{d}{dt} A^k x_0 \frac{(t-t_0)^k}{k!} = \sum_{k=1}^{\infty} A^k x_0 \cdot k \cdot \frac{(t-t_0)^{k-1}}{k!}$$

$$= A \sum_{m=0}^{\infty} A^m x_0 \frac{(t-t_0)^m}{m!} = A e^{A(t-t_0)} x_0 = Ax(t)$$

We next show that
$$\left(e^{At}\right)^{-1} = e^{-At}$$
$y(t) = e^{-At}y_0$ is a solution to $\dot{y}(t) = -Ay(t)$ for $y_0 = y(0)$ and $z(t) = e^{At}z_0$ is a solution to $\dot{z}(t) = -Az(t)$ for $z_0 = z(0)$ as we just saw. Therefore
$$\begin{aligned}\frac{d}{dt}\left(e^{-At}e^{At}x_0\right) &= \left(\frac{d}{dt}e^{-At}\right)e^{At}x_0 + e^{-At}\left(\frac{d}{dt}e^{At}\right)x_0\\ &= -A\cdot e^{-At}\cdot e^{At}x_0 + e^{-At}\cdot A\cdot e^{At}x_0 = 0\end{aligned}$$
by the Leibniz rule and because $Ae^{p(A)} = e^{p(A)}A$ for any polynomial $p(s) \in \mathbb{R}[s]$. Thus $e^{-At}e^{At}x_0$ is constant in t, i.e. we have for all $t \in \mathbb{R}, x_0 \in \mathbb{R}^n$
$$e^{-At}e^{At}x_0 = e^{-A\cdot 0}e^{A\cdot 0}x_0 = x_0 \text{ and thus } \left(e^{At}\right)^{-1} = e^{-At}$$
Let $y(t)$ be another solution of the differential equation, i.e. $\dot{y}(t) = Ay(t)\,,\, y(t_0) = x_0$. Then
$$\begin{aligned}\frac{d}{dt}\left(e^{-A(t-t_0)}y(t)\right) &= \left(\frac{d}{dt}e^{-A(t-t_0)}\right)y(t) + e^{-A(t-t_0)}\frac{d}{dt}y(t)\\ &= -Ae^{-A(t-t_0)}y(t) + e^{-A(t-t_0)}Ay(t) = 0\end{aligned}$$
Hence $e^{-A(t-t_0)}y(t)$ is constant in t and thus
$$e^{-A(t-t_0)}y(t) = e^{-A(t_0-t_0)}y(t_0) = 1\,x_0 = x_0 \Longrightarrow y(t) = e^{A(t-t_0)}x_0 = x(t)$$
An important consequence is the following Corollary:

The matrix exponential function $tA \longmapsto e^{At}$ is the unique solution of the matrix differential equation
$$\dot{X}(t) = AX(t)\,,\, X(0) = 1_{\mathbb{M}(n,\mathbb{R})}\,,\; X\,:\,\mathbb{R} \longrightarrow \mathbb{R}^{n\times n}$$

6. Calculate
   (a) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4+2x^2+1}\,dx$
   (b) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin^2 t}\,dt$

Reason: Function Theory.

Solution:

(a) Let $R(z) = \dfrac{P(z)}{Q(z)}$ with $P(z) = z^2$ and

$$Q(z) = z^4 + 2z^2 + 1 = (z^2+1)^2 = (z+i)^2(z-i)^2$$

There are no zeros on the real axis and the degree of the denominator polynomial is larger than the degree of the numerator polynomial, so the integral exists.

We want to apply the residue theorem and choose as closed curve the interval $I(r) = [-r, r]$ and the upper half circle $C(r)$ around 0 with radius $r > 1$ in the complex number plane to surround the zero at $z = i$. Hence we get from the residue theorem

$$\int_{I(r)} R(z)\,dz + \int_{C(r)} R(z)\,dz = 2\pi i \operatorname{Res}_i(R(z))$$

Since $|R(z)| \leq M \cdot |z|^{-2}$ for large values of $|z|$, we have

$$\left| \int_{C(r)} R(z)\,dz \right| \leq \pi r \cdot M r^{-2} = M\pi r^{-1} \stackrel{r\to\infty}{\longrightarrow} 0$$

and with a singularity of order two at $z = i$

$$\begin{aligned}
\int_{-\infty}^{\infty} R(z)\,dz &= \lim_{r\to\infty} \int_{I(r)} R(z)\,dz \\
&= 2\pi i \operatorname{Res}_i(R(z)) - \lim_{r\to\infty} \int_{C(r)} R(z)\,dz \\
&= 2\pi i \operatorname{Res}_i(R(z)) \\
&= 2\pi i \cdot \frac{1}{(2-1)!} \lim_{z\to i} \frac{d}{dz} \left[(z-i)^2 R(z)\right] \\
&= 2\pi i \lim_{z\to i} \frac{d}{dz} \frac{z^2}{(z+i)^2} \\
&= 2\pi i \lim_{z\to i} \frac{2z(z+i)^2 - z^2(2z+2i)}{(z+i)^4} \\
&= \frac{\pi}{2}
\end{aligned}$$

where we calculated the residue from the formula

$$\operatorname{Res}_a f = \frac{1}{(n-1)!} \lim_{z\to a} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z)\right]$$

for a n-th order singularity of f at a.

(b) With $\gamma(t)=e^{it}$ we have

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin^2 t}\,dt &= \frac{1}{4}\int_0^{2\pi} \frac{1}{1+\sin^2 t}\,dt\\ &= \frac{1}{4}\int_0^{2\pi}\left(1-\frac{1}{4}\left(e^{it}-e^{-it}\right)^2\right)^{-1} dt\\ &= \frac{1}{4}\int_\gamma \left(1-\frac{1}{4}(z^2-2+z^{-2})\right)^{-1}\frac{1}{iz}\,dz\\ &= \frac{1}{4}\int_\gamma \frac{4z}{6iz^2-iz^4-i}\,dz\\ &= i\int_\gamma \frac{z}{\underbrace{z^4-6z^2+1}_{=:z/P(z)}}\,dz\end{aligned}$$

$$P(z)=\left(z-\underbrace{\sqrt{3+2\sqrt{2}}}_{\notin \overline{B_0(1)}}\right)\left(z-\underbrace{\sqrt{3-2\sqrt{2}}}_{\in B_0(1)}\right)\left(z+\underbrace{\sqrt{3+2\sqrt{2}}}_{\notin \overline{B_0(1)}}\right)\left(z+\underbrace{\sqrt{3-2\sqrt{2}}}_{\in B_0(1)}\right)$$

By the residue theorem we get

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin^2 t}\,dt &= -2\pi \operatorname{Res}_{\sqrt{3-2\sqrt{2}}}\left(\frac{z}{P(z)}\right) - 2\pi \operatorname{Res}_{-\sqrt{3-2\sqrt{2}}}\left(\frac{z}{P(z)}\right)\\ &= -2\pi\left(\frac{\sqrt{3-2\sqrt{2}}}{P'(\sqrt{3-2\sqrt{2}})}+\frac{-\sqrt{3-2\sqrt{2}}}{P'(-\sqrt{3-2\sqrt{2}})}\right)\\ &= \frac{-4\pi}{4\sqrt{3-2\sqrt{2}}^{\,2}-12} = \frac{\pi}{2\sqrt{2}}\end{aligned}$$

where we calculated the residue from the formula

$$\operatorname{Res}_a \frac{g}{f} = \frac{g(a)}{f'(a)}$$

for a first order zero $f(a)=0$ and a holomorphic g in a.

7. Prove the following well known theorem by using topological and analytical tools only.

For every real symmetric matrix A there is a real orthogonal matrix Q such that $Q^\tau AQ$ is diagonal.

Hint: 'Topological and analytical tools only' forbids the words 'characteristic' and 'eigen'. You could start with Heine-Borel.

Reason: Linear Algebra by Calculus.

Solution: The groups of orthogonal real matrices $\mathrm{O}(n) \subseteq \mathbb{R}^{n^2}$ are compact subsets. They are bounded, because the columns of an orthogonal matrix $Q = (q_{ij}) \in \mathrm{O}(n)$ are unit vectors and so $|q_{ij}| < 1$ for all i, j, and they are closed, since they are solutions to the linear equations $x_{i1}x_{j1} + \ldots x_{in}x_{jn} = \delta_{ij}$. We define the sum of the squares of the off-diagonal entries $\mathrm{Od}(A) = \sum_{i \neq j} a_{ij}^2$ for any real square matrix.

Lemma: If A is a real symmetric $n \times n$ matrix that is not diagonal, i.e. $\mathrm{Od}(A) > 0$, then there exists $U \in \mathrm{O}(n)$ such that $\mathrm{Od}(U^\tau AU) < \mathrm{Od}(A)$.

The map $\varphi_A : \mathrm{O}(n) \longrightarrow \mathbb{R}^{n^2}$ defined by $\varphi_A(Q) := Q^\tau AQ$ is continuous. Its image $\varphi_A(\mathrm{O}(n))$ is thus a compact subset in $\mathbb{R}^{n^2}$. The continuous function $\mathrm{Od} : \varphi_A(\mathrm{O}(n)) \longrightarrow \mathbb{R}$ assumes therefore a minimum, say at $M = Q^\tau AQ \in \varphi_A(\mathrm{O}(n))$. This implies by the Lemma that $\mathrm{Od}(M) = 0$, hence M is diagonal which had to be proven. (If $\mathrm{Od}(M) > 0$ we first note that $M^\tau = (Q^\tau AQ)^\tau = Q^\tau A^\tau Q = Q^\tau AQ = M$ is symmetric and not diagonal, so we can apply the Lemma on M, and find an $U \in \mathrm{O}(n)$ such that $\mathrm{Od}(U^\tau MU) < \mathrm{Od}(M)$, which contradicts the minimality of M as $U^\tau MU$ is a feasible point:

$$U^\tau MU = U^\tau Q^\tau AQU = (QU)^\tau A(QU) = \varphi_A(QU) \in \varphi_A(\mathrm{O}(n)) \;,\; U, Q \in \mathrm{O}(n)$$

because $\mathrm{O}(n)$ is a group.)

Hence it remains to prove the Lemma.

Given a real symmetric matrix $A = (a_{ij})$. If A is diagonal, them we choose $U = 1 \in \mathrm{O}(n)$ and we are done, so let's assume $a_{rs} \neq 0$ for some $r \neq s$. In this case we set U to be a rotation matrix in the $(r, s)-$plane

$$U = (u_{ij}) := \begin{cases} u_{ij} = 0 & \text{if } i, j \notin \{r, s\} \wedge i \neq j \\ u_{ii} = 1 & \text{if } i \notin \{r, s\} \\ u_{rr} = u_{ss} = \cos\alpha \\ u_{rs} = -u_{sr} = \sin\alpha \end{cases}$$

which is clearly orthogonal. Let $U^\tau AU = (b_{kl}) = \left(\sum_{i,j} u_{ik}a_{ij}u_{jl}\right)$.

Then $b_{ij} = a_{ij}$ in all cases $i, j \notin \{r, s\}$.

$$\begin{aligned} b_{kr} &= \sum_i u_{ik} \sum_j a_{ij} u_{jr} = \sum_i u_{ik}(a_{ir}\cos\alpha - a_{is}\sin\alpha) \\ &= a_{kr}\cos\alpha - a_{ks}\sin\alpha \\ b_{ks} &= a_{ks}\cos\alpha + a_{kr}\sin\alpha \\ b_{kr}^2 + b_{ks}^2 &= a_{kr}^2 + a_{ks}^2 \\ b_{rl}^2 + b_{sl}^2 &= a_{rl}^2 + a_{sl}^2 \end{aligned}$$

Now we have for the symmetric matrices A and $U^\tau AU$

$$\operatorname{Od}(A) - \operatorname{Od}(U^\tau AU) = a_{sr}^2 - b_{sr}^2 + a_{rs}^2 - b_{rs}^2 = 2(a_{rs}^2 - b_{rs}^2) = 2a_{rs}^2 > 0$$

if we can choose α in a way such that $b_{rs} = 0$.

$$\begin{aligned} b_{rs}(\alpha) &= \sum_{i,j} u_{ir} a_{ij} u_{js} \\ &= u_{rr}a_{rs}u_{ss} + u_{rr}a_{rr}u_{rs} + u_{sr}a_{sr}u_{rs} + u_{sr}a_{ss}u_{ss} \\ &= a_{rs}\left(\cos^2\alpha - \sin^2\alpha\right) + (a_{rr} - a_{ss})\cos\alpha\sin\alpha \end{aligned}$$

Now $b_{rs}(0) = a_{rs}$ and $b_{rs}(90^\circ) = -a_{rs}$. By the mean value theorem there must be a choice of α such that $b_{rs}(\alpha) = 0$, since b_{rs} depends continuously on α and $a_{rs} \neq 0$.

8. We define $e = \displaystyle\sum_{k=0}^{\infty} \frac{1}{k!}$. Prove that e^2 is irrational.

Reason: Irrationality.

Solution: Assume $e^2 = \dfrac{a}{b} \in \mathbb{Q}$. Then $be = ae^{-1}$ and with the series

$$\begin{aligned} e &= 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \ldots \\ e^{-1} &= 1 - \frac{1}{1} + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} \pm \ldots \end{aligned}$$

we have for sufficiently large even n

$$\begin{aligned} n!be &= \underbrace{n!b\left(1 + \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n!}\right)}_{:=\beta_0\in\mathbb{Z}} + \underbrace{n!b\left(\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \ldots\right)}_{:=\beta_n} \\ n!ae^{-1} &= \underbrace{n!a\left(1 - \frac{1}{1} + \frac{1}{2} \mp \ldots + \frac{(-1)^n}{n!}\right)}_{:=\alpha_0\in\mathbb{Z}} + \underbrace{n!a\left(\frac{(-1)^{n+1}}{(n+1)!} + \frac{(-1)^{n+2}}{(n+2)!} + \ldots\right)}_{:=\alpha_n} \end{aligned}$$

and

$$0 < \frac{b}{n+1} < \beta_n < \frac{b}{n+1} + \frac{b}{(n+1)^2} + \ldots = \frac{b}{n+1} \cdot \frac{1}{1-\frac{1}{n+1}} = \frac{b}{n}$$

$$\begin{aligned}
-\frac{a}{n} &\overset{(n\,\text{even})}{<} -\frac{a}{n+1} + \varepsilon = \alpha_n < -\frac{a}{n+1}\left(1 - \frac{1}{n+1} - \frac{1}{(n+1)^2} - \ldots\right) \\
&= -\frac{a}{n+1}\left(1 - \frac{1}{n+1}\sum_{k=0}^{\infty}\frac{1}{(n+1)^k}\right) \\
&= -\frac{a}{n+1}\left(1 - \frac{1}{n+1} \cdot \frac{1}{1-\frac{1}{n+1}}\right) = -\frac{a}{n+1} \cdot \left(1 - \frac{1}{n}\right) < 0
\end{aligned}$$

This means for sufficiently large even n, that $n!ae^{-1}$ is a bit smaller than the integer α_0 and $n!be$ is a bit larger than the integer β_0, hence they cannot be equal.

9. Let $p < q$ be two primes, $b \in \mathbb{N}$, and G a group with p^2q^b elements. Show that:

   (a) If there is no normal $q-$Sylow subgroup in G, then $(p,q) = (2,3)$, and there is a non trivial homomorphism from G to S_4.

   (b) G is always solvable.

   Reason: Group Theory.

   Solution:

   (a) The number of $q-$Sylow subgroups in G is a divisor of p^2 and congruent 1 modulo q. If there is no normal $q-$Sylow subgroup, then a single one is excluded and there are p or p^2 many of them. However, p many of them is also excluded, because otherwise we had $q \,|\, p-1$ which is not possible for $q > p$. There are thus p^2 many $q-$Sylow subgroups in G and $p^2 \equiv 1 \mod q$. Hence $p + q\mathbb{Z}$ is a zero of $x^2 - 1 \in \mathbb{F}_q[x]$. This polynomial has two zeros: ± 1. However, $p \not\equiv 1 \mod q$ since $q \nmid p-1$, and $p \equiv -1 \mod q$, i.e. $0 < p+1 = m \cdot q \leq q$. This means $k = 1$ and $p + 1 = q$ which is only possible for the primes $(p,q) = (2,3)$.

   There are $4 = p^2$ $3-$Sylow subgroups in this case on which G operates transitive via conjugation. If we number them, we get a group homomorphism $\varphi : G \longrightarrow S_4$ whose image operates transitive on $\{1,2,3,4\}$. Now 4 divides the cardinality of the orbits by the orbit-stabilizer theorem, so $\{1\} \subsetneq \operatorname{im}(\varphi)$ is a proper subset and φ is non trivial.

(b) If G has a normal $q-$Sylow subgroup N, then it is a $q-$group of index p^2 and G/N is a $p-$group. Since p, q are both primes, N and G/N are both solvable and G is solvable, too. If G has no normal $q-$Sylow subgroup, we define $N := \ker(\varphi) \triangleleft G$ with the homomorphism from the previous part. Since $4 \,|\, |G/N| = |\operatorname{im}(\varphi)|$, the kernel N has to be a $3-$group, which is solvable. But $\operatorname{im}(\varphi) \cong G/N$ is a subgroup of the solvable group S_4, hence itself solvable. Thus N and G/N are again solvable and therewith G.

10. Let $f(x) = 2x^5 - 6x + 6 \in \mathbb{Z}[x]$. In which of the following rings is f irreducible and why?

(a) $\mathbb{Z}[x]$

(b) $(S^{-1}\mathbb{Z})[x]$ with $S = \{2^n \,|\, n \in \mathbb{N}_0\}$

(c) $\mathbb{Q}[x]$

(d) $\mathbb{R}[x]$

(e) $\mathbb{C}[x]$

Reason: Ring Theory.

Solution: (a) $f(x) = 2\cdot(x^5 - 3x - 3)$. Both factors are non units, since the units in $\mathbb{Z}[x]$ are $\{\pm 1\}$, i.e. f is reducible.

(c) 2 is a unit in this case, so it's sufficient to consider $g(x) = x^5 - 3x - 3$. By Eisenstein's criterion with the prime $p = 3$, we find that g is irreducible over $\mathbb{Q}[x]$ and so is f.

(b) It's again sufficient to consider $g(x) = x^5 - 3x - 3$. Assume $g = pq$ over $S^{-1}\mathbb{Z}$. This factorization is also valid in $\mathbb{Q}[x] \supseteq (S^{-1}\mathbb{Z})[x]$. But g is irreducible over $\mathbb{Q}$, so one of the factors is of degree 0, say p. This means $p \in S^{-1}\mathbb{Z}$ and p divides each coefficient of g, especially the leading coefficient 1, which means p is a unit in $S^{-1}\mathbb{Z}$ and f is irreducible.

(d) $\deg f > 1$ and odd, i.e. $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to+\infty} f(x) = +\infty$. Thus f has a zero by the mean value theorem and f is reducible.

(e) $\deg f > 1$ so f is reducible by the fundamental theorem of algebra.

11. (HS-1) Given a set A of 32 pairwise distinct, positive integers less than

112. Decide right or wrong:

(a) There is a number which occurs at least five times among the differences between two numbers of A.

(b) There is a number which occurs at least six times among the differences between two numbers of A.

Hint: A difference in this context is always positive, and only counted once between any two numbers of A.

Reason: Combinatorics.

Solution: There are $\binom{|A|}{2} = \dfrac{32 \cdot 31}{2} = 496$ differences to be considered. All of them are pairwise distinct, positive integers less than 112. If there was at most four occurrences of a certain difference, then we had with $4 \cdot 111 < 496$ a deficit of 52 possible differences. This proves the first part to be true.

The same argument doesn't work for six occurrences, so we need a closer look. Let the elements of A be ordered as

$$1 \leq a_1 < a_2 < \ldots < a_{31} < a_{32} < 112\,.$$

If the values among all differences would occur at most five times, then this is particularly true for the values $\{1, 2, 3, 4, 5, 6\}$ among the 31 differences $d_n = a_{n+1} - a_n$. So there is at least $31 - 5 \cdot 6 = 1$ difference with $d_n > 7$. Thus

$$d_1 + \ldots + d_{31} \geq 5 \cdot 1 + 5 \cdot 2 + \ldots + 5 \cdot 6 + 7 = 5 \cdot 21 + 7 = 112$$

On the other hand we have

$$a_{32} = a_1 + d_1 + d_2 + \ldots + d_{31} \geq a_1 + 112 > 112$$

which is a contradiction. This means that we must have at least 6 equal differences among the numbers of A.

12. (HS-2) The harmonic numbers are

$$H_n := \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\ ,\ (n \in \mathbb{N})$$

We define

$$T_n := \sum_{k=1}^{n} \frac{1}{k \cdot H_k^2} = \frac{1}{H_1^2} + \frac{1}{2 \cdot H_2^2} + \frac{1}{3 \cdot H_3^2} + \ldots + \frac{1}{n \cdot H_n^2}\ ,\ (n \in \mathbb{N})$$

Show that $T_n < 2$ for all $n \in \mathbb{N}$.

Reason: No Induction necessary.

Solution: It is $0 < H_{k-1} < H_k$ for any $k > 1$. Hence

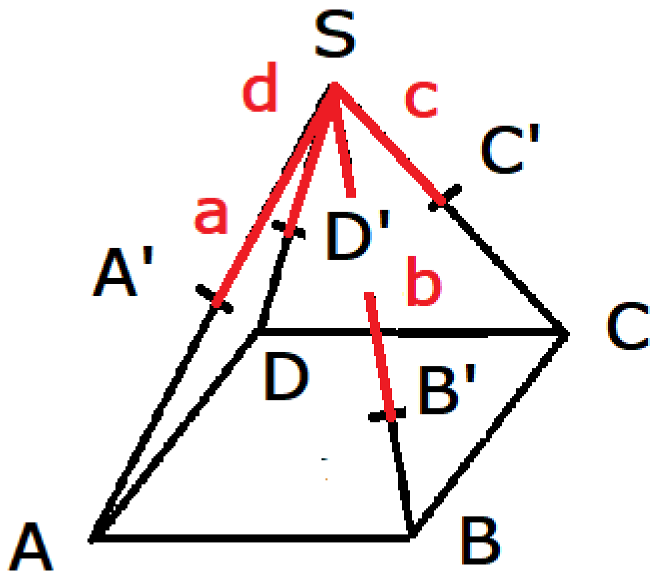
$$\frac{1}{k \cdot H_k^2} < \frac{\frac{1}{k}}{H_k \cdot H_{k-1}} = \frac{H_k - H_{k-1}}{H_k \cdot H_{k-1}} = \frac{1}{H_{k-1}} - \frac{1}{H_k}$$

and so

$$T_n = 1 + \sum_{k=2}^{n} \frac{1}{k \cdot H_k^2} < 1 + \sum_{k=2}^{n} \left(\frac{1}{H_{k-1}} - \frac{1}{H_k} \right) = 1 + \frac{1}{H_1} - \frac{1}{H_n} = 2 - \frac{1}{H_n} < 2$$

13. (HS-3) We have a four sided pyramid with summit S and a quadratic base A, B, C, D. Let A', B', C', D' be four points on the edges AS, BS, CS, DS, resp. with positive distances a, b, c, d from S, resp. Show that A', B', C', D' are coplanar if and only if

$$\frac{1}{a} + \frac{1}{c} = \frac{1}{b} + \frac{1}{d}$$



Reason: Geometry.

Solution: Without loss of generality we assume that $\overline{AS} = \overline{BS} = \overline{CS} = \overline{DS} = 1$. For the perpendiculars $A'F', AF$ from A, A' onto the plane B, C, S resp., we know that $A'F' \parallel AF$ and get from the intercept theorem $\overline{A'F'} : \overline{AF} = \overline{A'S} : \overline{AS} = a$. For the volumes of the pyramids $A'BCS$ and $ABCS$ we thus have

$$V(A'BCS) : V(ABCS) = a$$

and accordingly

$$\begin{aligned} V(A'B'CS) : V(A'BCS) &= b \\ V(A'B'C'S) : V(A'B'CS) &= c. \end{aligned}$$

Hence

$$V(A'B'C'S) : V(ABCS) = abc$$

and similarly

$$\begin{aligned} V(A'D'C'S) : V(ADCS) &= adc \\ V(A'B'D'S) : V(ABDS) &= abd \\ V(C'B'D'S) : V(CBDS) &= cbd \end{aligned}$$

The four points A', B', C', D' are coplanar if and only if $V(A'B'C'D') = 0$. This is equivalent to

$$V(A'B'C'S) + V(A'D'C'S) = V(A'B'D'S) + V(C'B'D'S) \quad (*)$$

because the difference of both sides of the equation is exactly $V(A'B'C'D')$. All areas of the triangles ABC, ABD, ADC, CBD of the square $ABCD$ have the same value, so $V(ABCS) = V(ABDS) = V(ADCS) = V(CBDS)$ and equation $(*)$ is equivalent to

$$abc + adc = abd + cbd \Longleftrightarrow \frac{1}{d} + \frac{1}{b} = \frac{1}{c} + \frac{1}{a}$$

Another way to solve the problem is by coordinates. We can choose a coordinate system such that A, B, C, S have the coordinates

$$A = (-t, -t, h)\,,\ B = (t, -t, h)\,,\ C = (t, t, h)\,,\ D = (-t, t, h)\,,\ S = (0, 0, 0)$$

for suitable $t, h > 0$. This means with the same assumption about the normed edges of the pyramid

$$A' = (-at, -at, ah)\,,\ B' = (bt, -bt, bh)\,,\ C' = (ct, ct, ch)\,,\ D' = (-dt, dt, dh)$$

Being coplanar is equivalent to the fact that

$$\det\begin{bmatrix}1 & -at & -at & ah\\ 1 & bt & -bt & bh\\ 1 & ct & ct & ch\\ 1 & -dt & dt & dh\end{bmatrix} = 4t^2h(bcd-acd+abd-abc)=0$$

14. (HS-4) Let $f(n)=[\,2\sqrt{n}\,]-\left[\,\sqrt{n-1}+\sqrt{n+1}\,\right]$ for $n\in\mathbb{N}$. Determine all values of n such that $f(n)=1$ and all n such that $f(n)=0$.

If $r\in\mathbb{R}$ with $s\leq r<s+1$ then $[r]=\lfloor r\rfloor = s$.

Reason: Cases.

Solution: We have $f(1)=[2]-[0+\sqrt{2}]=2-1=1$ and for $n\geq 2$

$$\begin{aligned} 5&<4n\\ 4n^2-4n+1&<4n^2-4<4n^2\\ 2n-1&<2\sqrt{n^2-1}<2n\\ 4n-1&<\left(\sqrt{n-1}+\sqrt{n+1}\right)^2<4n\end{aligned}$$

Assume there is an integer g such that $\sqrt{4n-1}<g\leq\sqrt{n-1}+\sqrt{n+1}$, then $4n-1<g^2\leq\left(\sqrt{n-1}+\sqrt{n+1}\right)^2<4n$ which is not possible. Therefore $\left[\,\sqrt{n-1}+\sqrt{n+1}\,\right]=[\,\sqrt{4n-1}\,]$ and

$$f(n)=[\,\sqrt{4n}\,]-[\,\sqrt{4n-1}\,]\quad(*)$$

If $n=m^2$ is a square number, then $\sqrt{4n}=2m\in\mathbb{N}$ and

$$\begin{aligned} 2&\leq 2\sqrt{4n}\\ 4n-2\sqrt{4n}+1&\leq 4n-1\\ \sqrt{4n}-1&\leq\sqrt{4n-1}<\sqrt{4n}\\ [\,\sqrt{4n-1}\,]&=\sqrt{4n}-1=[\,\sqrt{4n}\,]-1\end{aligned}$$

hence $f(n)=1$ by $(*)$.

If n is not a square number, then there is no integer g such that $4n-1<g^2\leq 4n$ or $\sqrt{4n-1}<g\leq\sqrt{4n}$, which means $[\,\sqrt{4n}\,]=[\,\sqrt{4n-1}\,]$ and $f(n)=0$.

We have shown that among all positive integers n

- exactly all positive square numbers fulfill $f(n)=1$, and

- all positive non square numbers fulfill $f(n) = 0$.

15. (HS-5) Determine all pairs of non-negative integers (m, n) such that $2^m - 5^n = 7$.

    Reason: Modular Arithmetic.

    Solution: Among the numbers 2^m for integers $0 \le m \le 5$ are exactly the numbers 2^3 and 2^5 of the required form $5^n + 7$ with an integer $n \ge 0$, namely
    $$2^3 - 5^0 = 2^5 - 5^2 = 7.$$

    We will show that there are no other solutions than $(3, 0), (5, 2)$.

    Assume there were solutions (m, n) with $n \ge 0, m \ge 6$ such that $2^m = 5^n + 7$. Then $2^6 = 64 \mid (5^n + 7)$ and $5^n \equiv 57 \mod 64$. Possible remainders are periodically

n	0	1	2	3	4	5	6	7	8	...
$5^n \mod 64$	1	5	25	61	49	53	9	45	33	...
n	...	9	10	11	12	13	14	15	16	...
$5^n \mod 64$	...	37	<u>57</u>	29	17	21	41	13	1	...

so $n = 16a + 10$ for some $a \in \mathbb{N}_0$.

Let's consider now the possible remainders modulo 17 which also have a periodicity of 16. Here we find

n	0	1	2	3	4	5	6	7	8	...
$5^n \mod 17$	1	5	8	6	13	14	2	10	16	...
$5^n + 7 \mod 17$	8	12	15	13	3	4	9	0	6	...
n	...	9	<u>10</u>	11	12	13	14	15	16	...
$5^n \mod 17$	...	12	9	11	4	3	15	7	1	...
$5^n + 7 \mod 17$	...	2	<u>16</u>	1	11	10	5	14	8	...

In order for $2^m \equiv 5^n + 7 \mod 17$ to hold, the remainders must be the same. For the left hand side we get the remainders

m	0	1	2	3	4	5	6	7	8	...
$2^m \mod 17$	1	2	4	8	<u>16</u>	15	13	9	1	...

with periodicity 8, which all occurred as remainders of $5^n + 7$. We already know that $n = 16a + 10$, so only the entries for $n = 10, 26, 42, \ldots$

are relevant. Thus $2^m \equiv 5^n + 7 \equiv 16 \mod 17$ and $m = 8b + 4$ for some $b \in \mathbb{N}_0$. This means particularly that m, n are even, say $m = 2c, n = 2d$.

$$2^m \equiv 2^{2c} \equiv 4^c \equiv 1^c \equiv 1 \mod 3$$
$$5^n + 7 \equiv 5^{2d} + 7 \equiv 25^d + 7 \equiv 1^d + 1 \equiv 2 \mod 3$$

which cannot be simultaneously the case.