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Mathematical Challenges

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1. Let $S := \{ (x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2 = (2-z)^2,\, 0 \leq z \leq 2 \}$ be the surface of a cone C with a circular cross section and a peak at $(0,0,2)$. The orientation of S be such, that the normal vectors point outwards. Calculate the flux through S of the vector field

$$F\,:\,\mathbb{R}^3 \longrightarrow \mathbb{R}^3\,,\; F(x,y,z) = \begin{pmatrix} xy^2 \\ x^2y \\ (x^2+y^2)(1-z) \end{pmatrix}.$$

Reason: Flux.

Solution: In order to apply Gauß' theorem, we need to cover our cone with a disk $D := \{ (x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2 \leq 2 \}$. Then

$$\int_{S\cup D} F\cdot n\,dS = \int_S F\cdot n\,dS + \int_D F\cdot n\,dS = \int_C \operatorname{div}(F)\,dx\,dy\,dz$$

We have

$$\begin{aligned} \operatorname{div}(F) &= \partial_x(xy^2) + \partial_y(x^2y) + \partial_z((x^2+y^2)(1-z)) \\ &= y^2 + x^2 + (x^2+y^2)(-1) \\ &= 0 \end{aligned}$$

so our vector field is solenoidal and $\int_S F\cdot n\,dS = -\int_D F\cdot n\,dS$. We use the parametrization $\psi(r,\varphi) = (r\cos\varphi, r\sin\varphi, 0)$ with a normal vector $n(r,\varphi) = -(0,0,r)$ which has to point in negative z-direction.

$$\begin{aligned} \int_D F\cdot n\,dS &= \int_0^2 dr \int_0^{2\pi} d\varphi\, F(\psi(r,\varphi))\cdot n(r,\varphi) \\ &= \int_0^2 dr \int_0^{2\pi} d\varphi \begin{pmatrix} r^3\sin^2\varphi\cos\varphi \\ r^3\sin\varphi\cos^2\varphi \\ (1-0)(r^2\cos^2\varphi + r^2\sin^2\varphi) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -r \end{pmatrix} \\ &= \int_0^2 dr \int_0^{2\pi} d\varphi\,(-r^3) \\ &= -8\pi \end{aligned}$$

and $\int_S F\cdot n\,dS = 8\pi$.

2. Calculate for $|\alpha| \geq 1$

$$\int_0^\pi \log(1 - 2\alpha\cos(x) + \alpha^2)\, dx$$

   (a) without using series expansions.
   Hint: $\displaystyle\int_0^{\pi/2} \log(\sin(x))\, dx = -\frac{\pi}{2}\log(2)$

   (b) by using serious expansions.

   Reason: Feynman's Integration Trick.

   Solution:

   (a) Set $f(\alpha) = \int_0^\pi \log(1-2\alpha\cos(x)+\alpha^2)\, dx$ and assume $|\alpha| > 1$. Now we get by integration under the integral

$$\begin{aligned}
\frac{\partial f}{\partial \alpha} &= \int_0^\pi \frac{\partial}{\partial \alpha}\log(1 - 2\alpha\cos(x) + \alpha^2)\, dx \\
&= \int_0^\pi \frac{2\alpha - 2\cos(x)}{1 - 2\alpha\cos(x) + \alpha^2}\, dx \\
&= \frac{1}{\alpha}\int_0^\pi \left(\frac{2(\alpha^2 - \alpha\cos(x))}{1 - 2\alpha\cos(x) + \alpha^2} + 1 - 1\right) dx \\
&= \frac{1}{\alpha}\int_0^\pi \left(1 - \frac{1-\alpha^2}{1 - 2\alpha\cos(x) + \alpha^2}\right) dx \\
&= \frac{\pi}{\alpha} - \frac{1-\alpha^2}{\alpha}\int_0^\pi \frac{dx}{1 - 2\alpha\cos(x) + \alpha^2} \\
&= \frac{\pi}{\alpha} - \frac{1}{\alpha}\frac{1-\alpha^2}{1+\alpha^2}\int_0^\pi \frac{dx}{1 - \frac{2\alpha}{1+\alpha^2}\cos(x)}
\end{aligned}$$

   Now we use the Weierstraß substitution $u := \tan\dfrac{x}{2}$, $|x| < \pi$ with

$$\sin(x) = \frac{2u}{1+u^2}\,,\ \cos(x) = \frac{1-u^2}{1+u^2}\,,\ dx = \frac{2du}{1+u^2}$$

and calculate

$$\begin{aligned}\int_0^\pi \frac{dx}{1-\frac{2\alpha}{1+\alpha^2}\cos(x)} &= \int_0^\infty \frac{2du}{(1+u^2)(1-\frac{2\alpha}{1+\alpha^2}\frac{1-u^2}{1+u^2})}\\ &= \int_0^\infty \frac{2du}{(1+u^2)-(1-u^2)\frac{2\alpha}{1+\alpha^2}}\\ &= \int_0^\infty \frac{2du}{(1-\frac{2\alpha}{1+\alpha^2})+u^2(1+\frac{2\alpha}{1+\alpha^2})}\\ &= \int_0^\infty \frac{2(1+\alpha^2)du}{(1+\alpha^2-2\alpha)+u^2(1+\alpha^2+2\alpha)}\\ &= \int_0^\infty \frac{2(1+\alpha^2)du}{(1-\alpha)^2+u^2(1+\alpha)^2}\\ &= \frac{2(1+\alpha^2)}{(1-\alpha)^2}\int_0^\infty \frac{du}{1+\left(\frac{1+\alpha}{1-\alpha}\right)^2u^2}\end{aligned}$$

The last substitution will be

$$y=\frac{1+\alpha}{1-\alpha}\,u\,,\;dy=\frac{1+\alpha}{1-\alpha}\,du$$

and we continue

$$\begin{aligned}\int_0^\pi \frac{dx}{1-\frac{2\alpha}{1+\alpha^2}\cos(x)} &= \frac{2(1+\alpha^2)}{(1-\alpha)^2}\int_0^\infty \frac{du}{1+\left(\frac{1+\alpha}{1-\alpha}\right)^2u^2}\\ &\stackrel{(*)}{=} \frac{2(1+\alpha^2)}{(1-\alpha)^2}\frac{(1-\alpha)}{(1+\alpha)}\int_0^{-\infty}\frac{dy}{1+y^2}\\ &= \frac{2(1+\alpha^2)}{1-\alpha^2}\left[\arctan(y)\right]_0^{-\infty}\\ &= -\pi\frac{1+\alpha^2}{1-\alpha^2}\end{aligned}$$

$(*)$ Note that y and u have different signs due to our choice of $|\alpha|>1$. Combined with what we omitted for simplification we have now

$$\begin{aligned}\frac{\partial f}{\partial\alpha} &= \frac{\pi}{\alpha}-\frac{1}{\alpha}\frac{1-\alpha^2}{1+\alpha^2}\cdot\left(-\pi\frac{1+\alpha^2}{1-\alpha^2}\right)\\ &= \frac{2\pi}{\alpha}\\ &\Longrightarrow\\ f(\alpha) &= 2\pi\log(|\alpha|)+C\end{aligned}$$

In order to calculate the integration constant, and deal with the case $\alpha = 1$, we compute

$$\begin{aligned} f(1) &= \int_0^\pi \log(2-2\cos(x))\,dx \\ &= \int_0^\pi \log\left(4\sin^2\left(\frac{x}{2}\right)\right)\,dx \\ &= \pi\log(4) + 4\int_0^{\pi/2}\log(\sin(y))\,dy \\ &\overset{Hint}{=} \pi\log(4) - 4\cdot\left(\frac{\pi}{2}\log(2)\right) \\ &= 0 \end{aligned}$$

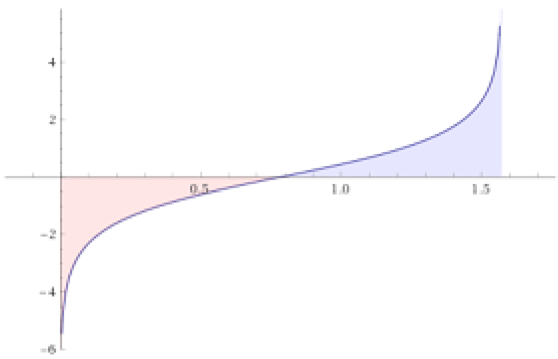
Thus we have $0 = f(1) = 2\pi\log(|1|) + C = C$ and the value of the integral is

$$\int_0^\pi \log(1-2\alpha\cos(x)+\alpha^2)\,dx = 2\pi\log(|\alpha|), \quad (|\alpha| \geq 1).$$

Proof of the hint: Let $I := \displaystyle\int_0^{\pi/2}\log(\sin(x))\,dx$.

For symmetry reasons we have

$$I = \int_0^{\pi/2}\log(\sin(x))\,dx = -\int_{\pi/2}^0 \log(\cos(x))\,dx$$



(The same sine curve in reverse order, and since the area is orientated, with a minus sign.)

Skipping the integration path yields

$$I = \int_0^{\pi/2}\log(\sin(x))\,dx = \int_0^{\pi/2}\log(\cos(x))\,dx$$

Hence

$$\begin{aligned}
2I &= \int_0^{\pi/2} \left(\log(\sin(x)) + \log(\cos(x))\right)\,dx \\
&= \int_0^{\pi/2} \log(\sin(x)\cos(x))\,dx \\
&= \int_0^{\pi/2} \log\left(\frac{\sin(2x)}{2}\right)\,dx \\
&= \int_0^{\pi/2} \log(\sin(2x))\,dx - \frac{\pi}{2}\log(2) \\
&= -\frac{\pi}{2}\log(2) + \frac{1}{2}\int_0^{\pi} \log(\sin(u))\,du \\
&= -\frac{\pi}{2}\log(2) + \frac{I}{2} + \frac{1}{2}\int_{\pi/2}^{\pi} \log(\sin(x))\,dx \\
&= -\frac{\pi}{2}\log(2) + \frac{I}{2} + \frac{1}{2}\int_0^{\pi/2} \log(\cos(x))\,dx \\
&= -\frac{\pi}{2}\log(2) + \frac{I}{2} + \frac{I}{2}
\end{aligned}$$

so $I = -\dfrac{\pi}{2}\log(2)$.

(b) We start with a value $|\alpha| \leq 1$ and $0 < x < \pi$. Then by Gradshteyn, Ryzhik 1.514

$$\log(1 - 2\alpha\cos(x) + \alpha^2) = -2\sum_{k=1}^{\infty} \frac{\cos(kx)}{k}\,\alpha^k$$

and we get

$$I(\alpha) := \int_0^{\pi} \log(1-2\alpha\cos(x)+\alpha^2)\,dx = -2\sum_{k=1}^{\infty} \frac{\alpha^k}{k}\int_0^{\pi} \cos(kx)\,dx = 0$$

This means for $|\alpha| \geq 1$

$$\begin{aligned}
\int_0^{\pi} \log(1 - 2\alpha\cos(x) + \alpha^2)\,dx &= \int_0^{\pi} \log\left(\alpha^2\left(\frac{1}{\alpha^2} - \frac{2}{\alpha}\cos(x) + 1\right)\right)\,dx \\
&= 2\pi\log(|\alpha|) + I\left(\frac{1}{\alpha}\right) \\
&= 2\pi\log(|\alpha|)
\end{aligned}$$

3. (HS-1)

   (a) Show that $\sqrt{i^i} \in \mathbb{R}$ where i is the imaginary unit $i = \sqrt{-1}$.

   (b) Which of the following equation signs is wrong and why?

   $$-1 \stackrel{(1)}{=} i \cdot i \stackrel{(2)}{=} \sqrt{-1} \cdot \sqrt{-1} \stackrel{(3)}{=} \sqrt{(-1) \cdot (-1)} \stackrel{(4)}{=} \sqrt{1} \stackrel{(5)}{=} 1$$

   (c) Calculate all solutions of $z^3 = 1$ by three different methods.

   Reason: Complex Numbers.

   Solution:

   (a) We write $i = 0+1\cdot i = \cos(\pi/2)+i\sin(\pi/2) = e^{i\pi/2}$ by Euler's identity. Now $i^i = (e^{i\pi/2})^i = e^{-\pi/2}$ and $\sqrt{i^i} = e^{-\pi/4} \approx 0.45593813 \in \mathbb{R}$.

   (b) Equation (1) is the definition of the imaginary unit and can't be wrong. Equation (2) is only another (bad) way to write the imaginary unit i, so it's misleading but not wrong. Equations (4) and (5) are ordinary real arithmetic. Hence equation (3) must be wrong:

   $$\begin{aligned}\sqrt{-1} \cdot \sqrt{-1} &\stackrel{\text{Gauß}}{=} (\cos(\pi/2) + i\sin(\pi/2)) \cdot (\cos(\pi/2) + i\sin(\pi/2)) \\ &\stackrel{\text{Euler}}{=} e^{i\pi/2} \cdot e^{i\pi/2} \\ &= e^{i\pi} \\ &= \cos(\pi) + i \cdot \sin(\pi) \\ &= -1 + i \cdot 0 \\ &= -1\end{aligned}$$

   whereas

   $$\begin{aligned}\sqrt{(-1) \cdot (-1)} &= \sqrt{(\cos(\pi) + i \cdot \sin(\pi)) \cdot (\cos(\pi) + i \cdot \sin(\pi))} \\ &= \sqrt{e^{\pi} \cdot e^{\pi}} \\ &= \sqrt{e^{2\pi}} \\ &= \sqrt{\cos(2\pi) + i \cdot \sin(2\pi)} \\ &= \sqrt{1 + i \cdot 0} \\ &= 1\end{aligned}$$

   (c) One can use Euler's formula and calculate $e^{2i\pi} = (e^{i\varphi})^3 = e^{3i\varphi}$ and get $2\pi n = 3\varphi$ or $\varphi \in \{\, 0, 2\pi/3, 4\pi/3 \,\} \subseteq [0, 2\pi)$ which corresponds to

   $$e^0 = 1 \ , \ e^{2i\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \ , \ e^{4i\pi/3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2},$$

or use the fact that $z=1$ is a solution and perform a long division:

$$z^3 : (z-1) = z^2+z+1 \Longrightarrow z_{1,2} = -\frac{1}{2} \pm \sqrt{-\frac{3}{4}}.$$

For the third method we write $z = r\cdot(\cos\varphi + i\sin\varphi)$. and calculate

$$\begin{aligned} 1 = z^3 &= r^3 \cdot (\cos\varphi + i\sin\varphi)^3 \\ &= r^3 \cdot \left(\cos^3\varphi + 3i\cos^2\varphi\sin\varphi - 3\cos\varphi\sin^2\varphi - i\sin^3\varphi\right) \\ &= r^3\left(\cos^3\varphi - 3\cos\varphi\sin^2\varphi\right) + i\cdot r^3\left(3\cos^2\varphi\sin\varphi - \sin^3\varphi\right) \end{aligned}$$

We see immediately that $\varphi = 0\,,\, r=1$ is a solution, i.e. $z=1$. The solution $\varphi=\pi\,,\, r=-1$ is identical, i.e. $z=1$. Hence we may assume $\sin\varphi \neq 0$ now, and $\cos\varphi\neq 0$. By comparison of the coefficients we get

$$\begin{aligned} 0 &= 3\cos^2\varphi - \sin^2\varphi \\ 1 &= r^3\cdot\cos\varphi\cdot\left(\cos^2\varphi - 3\sin^2\varphi\right) \end{aligned}$$

We get by substituting the first into the second equation

$$-\frac{1}{8} = r^3\cdot\cos^3\varphi \Longrightarrow -\frac{1}{2} = r\cos\varphi$$

As $1 = |z|^3 = |r|^3$ the only real solutions for r are $r=\pm 1$, i.e. $\cos\varphi = \pm\frac{1}{2}$ and $\sin\varphi = \pm\frac{\sqrt{3}}{2}$. Checking all possibilities

φ	$\pi/3$	$2\pi/3$	$4\pi/3$	$5\pi/3$
$\cos\varphi$	$1/2$	$-1/2$	$-1/2$	$1/2$
r	-1	1	1	-1
$\sin\varphi$	$\sqrt{3}/2$	$\sqrt{3}/2$	$-\sqrt{3}/2$	$-\sqrt{3}/2$
z	$-\frac{1}{2}+i\frac{\sqrt{3}}{2}$	$-\frac{1}{2}+i\frac{\sqrt{3}}{2}$	$-\frac{1}{2}-i\frac{\sqrt{3}}{2}$	$-\frac{1}{2}-i\frac{\sqrt{3}}{2}$

4. (HS-2) Which is the smallest natural number $n\in\mathbb{N}_0$ such that there are no integers $a,b\in\mathbb{Z}$ with $3a^3+b^3=n$?

   Reason: Diophantine Equation.

   Solution: We start with the observation

$$\begin{array}{lll} 0 = 3\cdot 0^3 + 0^3 & 1 = 3\cdot 0^3 + 1^3 & 2 = 3\cdot 1^3 + (-1)^3 \\ 3 = 3\cdot 1^3 + 0^3 & 4 = 3\cdot 1^3 + 1^3 & 5 = 3\cdot(-1)^3 + 2^3 \end{array}$$

and show that such an equation is impossible for $n = 6$. Assume we had a solution, then $b^3 = 6 - 3a^3 \equiv 0 \mod 3$, i.e. $3 \,|\, b$ since 3 is prime. Hence we can write $b = 3c$ for some integer c, and get $6 = 3a^3 + 27c^3$ or $2 = a^3 + 9c^3$. This means, that $a^3 \equiv 2 \mod 3$, which is only possible, if $a = 3d + 2$ for some integer d. Now we have

$$2 = (3d + 2)^3 + 9c^3 = 27d^3 + 54d^2 + 36d + 8 + 9c^3 \equiv 8 \mod 9$$

which is impossible.

5. (HS-3) Is it possible to cover an equilateral triangle with two smaller equilateral triangles without a gap? It's not required that they are of equal area, nor that they won't overlap, only that they are smaller and together have a greater area than the original triangle.

   Reason: Pigeonhole Principle.

   Solution: It is impossible. Let us assume it could be done, and let the side length of the original triangle $\triangle A$ be a. Accordingly we set the side lengths of the smaller triangles $\triangle A', \triangle A''$ resp. to $a', a'' < a$.

   Now the three corners of $\triangle A$ have to be covered by two smaller triangles. W.l.o.g. we may assume that $\triangle A'$ covers two corners by the pigeonhole principle. But this means $a' \geq a$ as maximal possible distance in $\triangle A'$. But $a > a'$, a contradiction.

6. (HS-4) Given n different integers $\{ a_1, \ldots, a_n \}$, then there exists a subset $\{ a_{j_1}, \ldots, a_{j_m} \}$ with $1 \leq j_1 < \ldots < j_m \leq n$ such that n divides $a_{j_1} + \ldots + a_{j_m}$.

   Reason: Pigeonhole Principle.

   Solution: We consider the n sums $s_j := a_1 + \ldots + a_j$. If $s_i = s_j$ then $n \,|\, 0 = s_i - s_j = a_{j+1} + \ldots + a_i$ and we are done. So we may assume that all sums are pairwise different. Each of them can be written as $s_j = q_j \cdot n + r_j$ with $0 \leq r_j < n$. If one $r_j = 0$ then we are done again, so we may assume $1 \leq r_j < n$. But since we have n different s_j two remainders must be equal, say $r_i = r_j$. Thus n divides

$$a_{j+1} + \ldots + a_i = s_i - s_j = (q_i - q_j) \cdot n + (r_i - r_j) = (q_i - q_j) \cdot n$$

2 May 2020

1. Let $1 < p < 4$ and $f \in L^p((1,\infty))$ with the Lebesgue measure λ. We define $g : (1,\infty) \longrightarrow \mathbb{R}$ by
$$g(x) = \frac{1}{x}\int_x^{10x} \frac{f(t)}{t^{1/4}}\,d\lambda(t).$$
Show that there exists a constant $C = C(p)$ which depends on p but not on f such that $\|g\|_2 \leq C \cdot \|f\|_p$ so $g \in L^2((1,\infty))$.

Reason: L^p-Norms.

Solution: Let us first assume $p \neq 4/3$. By Hölder's inequality we get for a fixed $x \in \mathbb{R}_+$ with $\frac{1}{p}+\frac{1}{q}=1$ and $q \neq 4$
$$\begin{aligned}\int_x^{10x} t^{-1/4}|f(t)|d\lambda(t) &\leq \|f\|_p \left(\int_x^{10x} t^{-q/4}\,dt\right)^{1/q}\\ &= \|f\|_p\left(\left[\frac{4}{4-q}\,t^{-\frac{q}{4}+1}\right]_x^{10x}\right)^{1/q}\\ &= \|f\|_p\left(\frac{4}{4-q}\left(10^{-\frac{q}{4}+1}-1\right)\right)^{1/q}\cdot x^{-\frac{1}{4}+\frac{1}{q}}\end{aligned}$$

Now we have
$$\begin{aligned}\|g\|_2^2 &\leq \int_1^\infty \left(\frac{1}{x}\int_x^{10x}\frac{|f(t)|}{t^{1/4}}\,d\lambda(t)\right)^2 d\lambda(x)\\ &\leq \|f\|_p^2 \underbrace{\left(\frac{4}{4-q}\left(10^{-\frac{q}{4}+1}-1\right)\right)^{2/q}}_{=:C_1(p)} \underbrace{\int_1^\infty \frac{1}{x^2}\cdot x^{-\frac{1}{2}+\frac{2}{q}}\,d\lambda(x)}_{=:I(p)}\end{aligned}$$

The integral $I(p) = \displaystyle\int_1^\infty \frac{1}{x^2}\cdot x^{-\frac{1}{2}+\frac{2}{q}}\,d\lambda(x) = \int_1^\infty x^{-\frac{5}{2}+\frac{2}{q}}\,d\lambda(x)$ only depends on q and therewith on p. It is finite if and only if $-\dfrac{5}{2}+\dfrac{2}{q} < -1$, i.e. $1/q < 3/4$ or $p < 4$ and thus
$$\|g\|_2 \leq \underbrace{\sqrt{C_1(p)\cdot I(p)}}_{=:C}\cdot\|f\|_p\,.$$

In case $p = \frac{3}{4}$ or $q = 4$ we get $\displaystyle\int_x^{10x} t^{-1/4}|f(t)|d\lambda(t) \leq \|f\|_{3/4} \cdot (\log 10)^{1/4}$.

$$\begin{aligned}
\|g\|_2^2 &\leq \int_1^\infty \left(\frac{1}{x} \int_x^{10x} \frac{|f(t)|}{t^{1/4}}\, d\lambda(t) \right)^2 d\lambda(x) \\
&\leq \|f\|_{3/4}^2 \cdot (\log 10)^{1/2} \int_1^\infty \frac{dx}{x^2} \\
&= \|f\|_{3/4}^2 \cdot \underbrace{(\log 10)^{1/2}}_{=:C^2}
\end{aligned}$$

2. We define
$$\mathbb{R}^\infty = \mathbb{R}^{(\mathbb{N})} = \{\, (x_1, x_2, \ldots) \mid x_i \overset{a.a.}{=} 0 \,\}$$
and equip $\mathbb{R}^\infty$ with the Euclidean metric
$$d((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = \sqrt{\sum_{i=1}^\infty |x_i - y_i|^2}.$$
which defines a topology
$$\mathcal{S} := \{\, U \subseteq \mathbb{R}^\infty \mid \forall\, p \in U\, \exists\, \varepsilon > 0 \,:\, B_\varepsilon(p) \subseteq U \,\}$$
with the open ball $B_\varepsilon(p) = \{\, q \in \mathbb{R}^\infty \mid d(p,q) < \varepsilon \,\}$.

(a) Show that the function
$$\begin{aligned}
\alpha \,:\, (\mathbb{R}^\infty, \mathcal{S}) &\longrightarrow (\mathbb{R}, \mathcal{E}) \\
(x_1, x_2, \ldots) &\longmapsto \sum_{i=1}^\infty 2^i \cdot x_i
\end{aligned}$$
is not continuous, where $\mathcal{E}$ is the usual Euclidean topology on $\mathbb{R}$.

(b) Let B be the diagonal matrix where the diagonal entries are 2^i for $i = 1, 2, \ldots$, i.e.
$$B = \begin{bmatrix} 2 & 0 & 0 & \ldots \\ 0 & 4 & 0 & \ldots \\ 0 & 0 & 8 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$
Show that $\beta \,:\, (\mathbb{R}^\infty, \mathcal{S}) \longrightarrow (\mathbb{R}^\infty, \mathcal{S})$ defined by $\beta(x) = Bx$ is not continuous.

(c) Define a topology $\mathcal{T}$ on $\mathbb{R}^\infty$ such that the inclusion maps

$$\begin{aligned} \iota_n \,:\, (\mathbb{R}^n, \mathcal{E}) &\longrightarrow (\mathbb{R}^\infty, \mathcal{T}) \\ (x_1, \ldots, x_n) &\longmapsto (x_1, \ldots, x_n, 0, \ldots) \end{aligned}$$

are continuous for any $n \in \mathbb{N}_0$.

Reason: Topologies on $\mathbb{R}^\infty$.

Solution:

(a) We consider $U = (-1, 1) \subseteq \mathbb{R}$ so $0 \in \alpha^{-1}(U)$. Suppose there is an $\varepsilon > 0$ such that $B_\varepsilon(0) \subseteq U$. We pick $i \in \mathbb{N}$ with $2^{-i} < \varepsilon$ and the point $q = (0, \ldots, 0, 2^{-i}, 0, \ldots)$. Then $d(0, q) = 2^{-i} < \varepsilon$ and $q \in B_\varepsilon(0)$. But $\alpha(q) = 1$, so $q \notin \alpha^{-1}(U)$, i.e. $B_\varepsilon(0) \nsubseteq U$, a contradiction.

(b) $U = B_1(0) \subseteq \mathbb{R}^\infty$ is an open set and $0 \in \beta^{-1}(U)$. Suppose there is an $\varepsilon > 0$ such that $B_\varepsilon(0) \subseteq U$. We pick again $i \in \mathbb{N}$ such that $2^{-i} < \varepsilon$. For $q = (0, \ldots, 0, 2^{-i}, 0, \ldots)$ we have $d(0, q) = 2^{-i} < \varepsilon$, so $q \in B_\varepsilon(0) \subseteq U$. But $\beta(q) = (0, \ldots, 0, 1, 0, \ldots)$ so $d(0, \beta(q)) = 1$ which means that $q \notin U$, a contradiction.

(c) We define the so called weak topology on $\mathbb{R}^\infty$ as

$$\mathcal{T} := \{\, U \subseteq \mathbb{R}^\infty \,|\, \forall\, n \in \mathbb{N}_0 \,:\, U \cap \mathbb{R}^n \subseteq \mathbb{R}^n \text{ is open } \}.$$

This means: A map $f \,:\, (\mathbb{R}^\infty, \mathcal{T}) \longrightarrow (X, \mathcal{X})$ of topological spaces is continuous with respect to $\mathcal{T}$ if and only if for each $n \in \mathbb{N}_0$ the map

$$(\mathbb{R}^n, \mathcal{E}) \overset{\iota_n}{\hookrightarrow} (\mathbb{R}^\infty, \mathcal{T}) \overset{f}{\longrightarrow} (X, \mathcal{X})$$

is continuous.

3. Calculate
$$\int_{-\infty}^{+\infty} \frac{\cos(\alpha x)}{1 + x^2}\, dx \quad (\alpha \geq 0).$$

Reason: Integral.

Solution: For $\alpha = 0$ we have

$$\int_{-\infty}^{+\infty} \frac{dx}{1 + x^2} = [\arctan(x)]_{-\infty}^{+\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

so we may assume $\alpha > 0$ now, substitute $t = \alpha x, \beta = \alpha^{-1}$ and observe using integration by parts twice

$$\begin{aligned}\int_0^\infty e^{-t}\sin(\beta t)\,dt &= -e^{-t}\sin(\beta t)\Big|_0^\infty + \beta\int_0^\infty e^{-t}\cos(\beta t)\,dt\\ &= 0+\beta\left(\left[-e^{-t}\cos(\beta t)\right]_0^\infty - \beta\int_0^\infty (-e^{-t})(-\sin(\beta t))\,dt\right)\\ &= \beta - \beta^2\int_0^\infty e^{-t}\sin(\beta t)\,dt\end{aligned}$$

and thus

$$\int_0^\infty e^{-t}\sin(\beta t)\,dt = \frac{\beta}{1+\beta^2}$$

This means

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{\cos(\alpha x)}{1+x^2}\,dx &= \int_{-\infty}^{+\infty}\frac{\cos(\alpha x)}{x}\cdot\frac{x}{1+x^2}\,dx\\ &= 2\int_0^\infty\left(\frac{\cos(\alpha x)}{x}\int_0^\infty e^{-t}\sin(xt)\,dt\right)\,dx\\ &= \int_0^\infty\int_0^\infty e^{-t}\cdot\frac{2\sin(xt)\cos(\alpha x)}{x}\,dt\,dx\\ &= \int_0^\infty\int_0^\infty e^{-t}\cdot\frac{\sin(x(t+\alpha))+\sin(x(t-\alpha))}{x}\,dx\,dt\\ &= \int_0^\infty e^{-t}\cdot\left(\frac{\pi}{2}\operatorname{sgn}(t+\alpha)+\frac{\pi}{2}\operatorname{sgn}(t-\alpha)\right)\,dt\\ &= \pi\int_\alpha^\infty e^{-t}\,dt\\ &= \pi\cdot e^{-\alpha}\end{aligned}$$

4. Calculate

$$\int_0^1 \sin(\pi x)\,x^x\,(1-x)^{1-x}\,dx.$$

Hint: You may use calculators to determine residues.

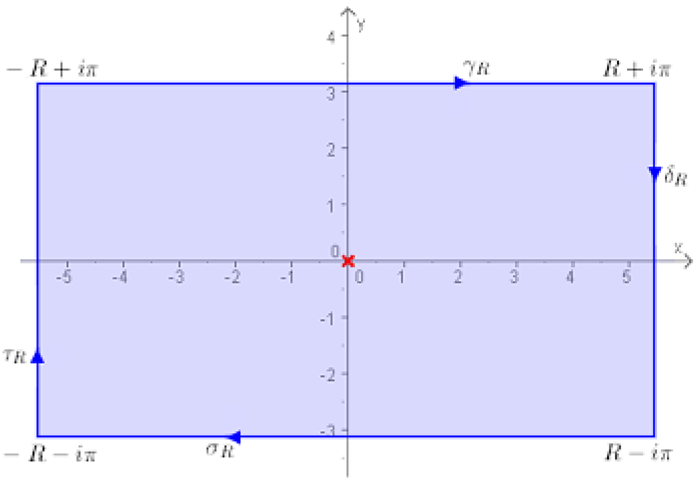
Reason: Ramanujan Integral.

Solution: We set $S := \displaystyle\int_0^1 e^{i\pi x}\,x^x\,\frac{1-x}{(1-x)^x}\,dx$ and substitute $t =$

$\log(x) - \log(1-x)$, i.e. $x = \dfrac{e^t}{e^t+1}$.

$$\begin{aligned} S &= \int_0^1 e^{i\pi x}\, e^{(\log(x))\,x}\, \frac{1-x}{e^{\log(1-x)\,x}}\, dx = \int_0^1 (1-x)\, e^{(i\pi+\log(x)-\log(1-x))x}\, dx \\ &= \int_{-\infty}^{\infty} \frac{1}{e^t+1}\, e^{(i\pi+t)\frac{e^t}{e^t+1}}\, \frac{e^t}{(e^t+1)^2}\, dt \\ &= \int_{-\infty+i\pi}^{\infty+i\pi} \frac{1}{-e^s+1}\, e^{s\frac{-e^s}{-e^s+1}}\, \frac{-e^s}{(-e^s+1)^2}\, ds \\ &= \int_{-\infty+i\pi}^{\infty+i\pi} \frac{e^t}{(e^t-1)^3}\, e^{t\,\frac{e^t}{e^t-1}}\, dt \end{aligned}$$

With $f(z) := \dfrac{e^z}{(e^z-1)^3}\, e^{z\,\frac{e^z}{e^z-1}}$ we have a meromorphic function on $D := \{\, z \in \mathbb{C} \mid -\pi \leq \Im(z) \leq \pi \,\}$.



The only singularity is at $z=0$ and we have the residue $\operatorname{res}(f,0) = -\dfrac{e}{24}$, see e.g. WolframAlpha.com with the input

```
residue of (e^(z(e^z/(e^z-1)))(e^z/(e^z-1)^3))at z=0)
```

Consider the path $\kappa_R = \gamma_R + \delta_R + \sigma_R + \tau_R$ around $z=0$ as in the graphic. Then $\oint_{\kappa_R} f(z)\,dz = -2\pi i \cdot \operatorname{res}(f(z),0) = 2i\,\dfrac{\pi e}{24}$.

$(f(z_n))_{n\in\mathbb{N}}$ converges to 0 for any sequence $(z_n)_{n\in\mathbb{N}} \subseteq D$ such that $|z_n| \longrightarrow \infty$, so

$$\lim_{R\to\infty} \int_{\delta_R} f\,dz = \lim_{R\to\infty} \int_{\tau_R} f\,dz = 0$$

The function $f(z)$ is odd, so $\int_{\sigma_R} f(z)\,dz = \int_{\gamma_R} f(z)\,dz$ and $2S = 2\lim_{R\to\infty}\int_{\gamma_R} f(z)\,dz = \lim_{R\to\infty}\oint_{\kappa_R} f(z)\,dz = 2i\,\dfrac{\pi e}{24}$ and our integral becomes:

$$\int_0^1 \sin(\pi x)\, x^x\,(1-x)^{1-x}\,dx = \Im(S) = \frac{\pi e}{24}.$$

5. The p-Prüfer group is defined as

$$G := \mathbb{C}_{p^\infty} = \{\,\exp(2n\pi i/p^m)\,|\, n\in\mathbb{Z}, m\in\mathbb{N}\,\} \cong \mathbb{Z}\left[\frac{1}{p}\right]/\mathbb{Z}$$

Show that G is isomorphic to the factor group F/R of the free Abelian group over an countably infinite basis $\{\,a_1,a_2,\ldots,a_n,\ldots\}$ with the subgroup of relations R generated by $\{\,pa_1, a_1-pa_2,\ldots,a_n-pa_{n+1},\ldots\}$, so

$$G = \langle x_1,x_2,\ldots\,|\,x_1^p=1, x_2^p=x_1, x_3^p=x_2,\ldots\rangle$$

Reason: Prüfer Group.

Solution: Let $F=\langle a_1,a_2,\ldots\rangle$ be the free Abelian group as defined. Then we consider the function

$$f\,:\,S\longrightarrow G\,,\,a_m\longmapsto \exp(2\pi i/p^m)$$

By the universal property of F there is a unique group homomorphism

$$\varphi\,:\,F\longrightarrow G\,,\,\varphi(a_m)=f(a_m)\;\forall\,m$$

Now $\exp(2n\pi i/p^m)=\exp(2\pi i/p^m)^n=\varphi(a_m)^n=\varphi(n\cdot a_m)$ shows that φ is surjective. By

$$\begin{aligned}\varphi(p\cdot a_1) &= \varphi(a_1)^p = \exp(2\pi i/p)^p = \exp(2\pi i) = 1\\ \varphi(a_n-pa_{n+1}) &= \varphi(a_n)\varphi(a_{n+1})^{-p} = \exp(2\pi i/p^n)\exp(2\pi i/p^{n+1})^{-p}\\ &= \exp(2\pi i/p^n)/\exp(2p\pi i/p^{n+1}) = 1\end{aligned}$$

we see that $R\subseteq\ker\varphi$. Let $x=n_1a_{i_1}+\ldots+n_ra_{i_r}\in F$ be in the kernel of φ, and assume $1\leq i_1\leq i_2\leq\ldots\leq i_r=:s$. Then

$$\begin{aligned}1 &= \varphi(x)=\varphi(n_1a_{i_1}+\ldots+n_ra_{i_r})=\varphi(a_{i_1})^{n_1}\cdots\varphi(a_{i_r})^{n_r}\\ &= \prod_{k=1}^{r}\exp(2\pi i/p^{i_k})^{n_k} = \prod_{k=1}^{s}\exp(2\delta_{ki_k}n_k\pi i/p^k) = \exp\left(\sum_{k=1}^{s}\frac{2\delta_{ki_k}n_k\pi i}{p^k}\right)\end{aligned}$$

So $2m\pi i = \sum_{k=1}^{s} \frac{2\delta_{ki_k} n_k \pi i}{p^k}$ or $0 = -mp^s + \sum_{k=1}^{s} \delta_{ki_k} n_k \cdot p^{s-k}$ or

$$
\begin{aligned}
n_s &= mp^s - \sum_{k=1}^{s-1} \delta_{ki_k} n_k \cdot p^{s-k} \\
x &= \sum_{k=1}^{s-1} \left(\delta_{ki_k} n_k a_k - \delta_{ki_k} n_k \cdot p^{s-k} a_s \right) + mp^s a_s \\
&= \sum_{k=1}^{s-1} \left(\delta_{ki_k} n_k (a_k - p a_{k+1}) \right) \quad \in R \\
&+ \sum_{k=1}^{s-1} \left(p\delta_{ki_k} n_k a_{k+1} - \delta_{ki_k} n_k \cdot p^{s-k} a_s \right) + mp^s a_s \\
&\stackrel{mod\ R}{\equiv} \sum_{k=1}^{s-1} \left(p\delta_{ki_k} n_k (a_{k+1} - p a_{k+2}) \right) \quad \in R \\
&+ \sum_{k=1}^{s-1} \left(p^2\delta_{ki_k} n_k a_{k+2} - \delta_{ki_k} n_k \cdot p^{s-k} a_s \right) + mp^{s-1} a_{s-1} \\
&\stackrel{mod\ R}{\equiv} \sum_{k=1}^{s-1} \left(p^2\delta_{ki_k} n_k (a_{k+2} - p a_{k+3}) \right) \quad \in R \\
&+ \sum_{k=1}^{s-1} \left(p^3\delta_{ki_k} n_k a_{k+3} - \delta_{ki_k} n_k \cdot p^{s-k} a_s \right) + mp^{s-2} a_{s-2} \\
&\stackrel{mod\ R}{\equiv} \\
&\vdots \\
&\stackrel{mod\ R}{\equiv} \sum_{k=1}^{s-1} \left(p^{s-1-k}\delta_{ki_k} n_k (a_{s-1} - p a_s) \right) \quad \in R \\
&+ \sum_{k=1}^{s-1} \left(p^{s-k}\delta_{ki_k} n_k a_s - \delta_{ki_k} n_k \cdot p^{s-k} a_s \right) + mp^{k+1} a_{k+1} \\
&\stackrel{mod\ R}{\equiv} mp^{k+1} a_{k+1} \stackrel{mod\ R}{\equiv} mp^k a_k \stackrel{mod\ R}{\equiv} mp^{k-1} a_{k-1} \\
&\stackrel{mod\ R}{\equiv} \ldots \stackrel{mod\ R}{\equiv} mpa_1 \stackrel{mod\ R}{\equiv} 0
\end{aligned}
$$

and we have shown that $R \supseteq \ker \varphi$. Hence $F/R = F/\ker \varphi \cong G$.

6. Give an example of a quotient $R-$module M/N which is Artinian although neither the ring R nor the modules M, N are.

Reason: Artinian Modules.

Solution: An example is the p-Prüfer group considered as $\mathbb{Z}$-module:

$$\mathbb{C}_{p^\infty} = \{\, \exp(2n\pi i/p^m) \mid n \in \mathbb{Z}, m \in \mathbb{N}\,\} \cong_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{p}\right]/\mathbb{Z}$$

The $\mathbb{Z}$-module homomorphismus $\varphi\,:\,\mathbb{Z}[1/p] \longrightarrow \mathbb{C}_{p^\infty}\,,\, q \longmapsto \exp(2q\pi i)$ is surjective and has the kernel $\mathbb{Z}$.

(a) $\mathbb{Z}$ is not Artinian.

$$\ldots (q^n) \subsetneq (q^{n-1}) \subsetneq \ldots \subsetneq (q) \subsetneq \mathbb{Z}\,,\, q \text{ prime}$$

So $\mathbb{Z}$ is neither an Artinian ring nor an Artinian $\mathbb{Z}$-module.

(b) $\mathbb{Z}[1/p]$ is not Artinian.

The same descending series of $\mathbb{Z}$-modules as above is also an infinitely long descending series of $\mathbb{Z}[1/p]$-submodules, if we choose $q \neq p$, so $\mathbb{Z}[1/p]$ isn't an Artinian $\mathbb{Z}$-module.

(c) $\mathbb{C}_{p^\infty}$ is Artinian.

The chain of $\mathbb{Z}$-submodules $(1/p^{n-1}) \subsetneq (1/p^n)$ in $\mathbb{C}_{p^\infty}$ is infinitely long, so $\mathbb{C}_{p^\infty}$ is not Noetherian. Now let us consider a chain

$$\mathbb{C}_{p^\infty} \supsetneq M_1 \supsetneq M_2 \supsetneq M_3 \supsetneq \ldots$$

of descending $\mathbb{Z}$-submodules. We show that this can be written as

$$\mathbb{C}_{p^\infty} \supsetneq (1/n_1) \supsetneq (1/n_2) \supsetneq (1/n_3) \supsetneq \ldots$$

of descending $\mathbb{Z}$-submodules for some positive integers $n_j > 0$.

Let $0 \neq M \subseteq \mathbb{Z}[1/p]/\mathbb{Z}$ be a $\mathbb{Z}$-submodule. Then every element $m \in M$ can be written as

$$\mathbb{Z} \not\ni m = \sum_{k=1}^{n} \frac{a_k}{p^{r_k}} = \frac{c_m}{p^{r_m}} \quad (a_k, c_m \in \mathbb{Z},\ r_k, r_m \in \mathbb{N})$$

Now we can cancel all factors p of c_m. Since $M \neq 0 = \mathbb{Z}$ there is an element $m = \dfrac{c_m}{p^{r_m}}$ such that $r := r_m > 0$. If $(c_m, p) = 1$ then $(c_m, p^r) = 1$ and we can find $\alpha, \beta \in \mathbb{Z}$ such that $1 = \alpha c + \beta p^r$, i.e.

$$\frac{1}{p^r} = \alpha \cdot \frac{c}{p^r} + \beta \equiv \alpha \cdot \frac{c}{p^r} \in M \mod \mathbb{Z}\,.$$

Hence every $\mathbb{Z}$-submodule of $\mathbb{Z}[1/p]/\mathbb{Z}$ has the form $M = (1/n)$ with a positive integer $n > 0$.

Now $(1/n_k) \supsetneq (1/n_{k+1})$ implies $n_{k+1} \,|\, n_k$. Hence $n_1 > n_2 > n_3 > \ldots$ is a decreasing sequence of positive integers, which thus must terminate, i.e. $\mathbb{C}_{p^\infty}$ is an Artinian $\mathbb{Z}$-module.

7. (a) Let $u_1, \ldots, u_n$ be solutions of the one dimensional heat equation $\dfrac{du}{dt} - \dfrac{d^2u}{dx^2} = 0$ $(x \in \mathbb{R}, t > 0)$. Show that

$$u(x_1, \ldots, x_n, t) := \prod_{k=1}^{n} u_k(x_k, t)$$

is a solution of the n dimensional heat equation $\dfrac{\partial u}{\partial t} - \Delta u = 0$.

(b) Calculate a solution for

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \Delta u(x,t) = 0 & \text{for } x \in \mathbb{R}^3,\ t > 0 \\ u(x,0) = x_1^2 x_2^2 x_3 & \text{for } x = (x_1, x_2, x_3) \in \mathbb{R}^3 \end{cases}$$

Reason: Heat Equation.

Solution:

(a) We get by differentiating $u = u(x_1, \ldots, x_n, t)$

$$\begin{aligned}
\frac{\partial u}{\partial t}(x,t) &= \sum_{k=1}^{n} \frac{\partial u_k}{\partial t}(x_k, t) \prod_{j \neq k} u_j(x_j, t) \\
\frac{\partial u}{\partial x_k}(x,t) &= \frac{\partial u_k}{\partial x_k}(x_k, t) \prod_{j \neq k} u_j(x_j, t) \\
\frac{\partial^2 u}{\partial x_k^2}(x,t) &= \frac{\partial^2 u_k}{\partial x_k^2}(x_k, t) \prod_{j \neq k} u_j(x_j, t) \\
\Delta u(x,t) &= \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} = \sum_{k=1}^{n} \frac{\partial^2 u_k}{\partial x_k^2}(x_k, t) \prod_{j \neq k} u_j(x_j, t)
\end{aligned}$$

and so

$$\frac{\partial u}{\partial t} - \Delta u = \sum_{k=1}^{n} \underbrace{\left(\frac{\partial u_k}{\partial t}(x_k, t) - \frac{\partial^2 u_k}{\partial x_k^2}(x_k, t) \right)}_{=0 \text{ by assumption}} \prod_{j \neq k} u_j(x_j, t) = 0$$

(b) If $u_k(x_k,t)$ with $u_k(x_k,0)=f_k(x_k)$ are solutions of the one dimensional heat equation, then $u(x,t)=u_1(x_1,t)u_2(x_2,t)u_3(x_3,t)$ solves the given problem with $u(x,0)=f_1(x_1)f_2(x_2)f_3(x_3)$. We therefore want to find solutions to

$$\begin{cases} \dfrac{du_k}{dt}(x_k,t)-\dfrac{d^2}{dx_k^2}u(x_k,t)=0 & \text{for } x_k\in\mathbb{R},\ t>0 \\ u_k(x_k,0)=x_k^2 & \text{for } x_k\in\mathbb{R} \end{cases}$$

$$\begin{cases} \dfrac{du_3}{dt}(x_3,t)-\dfrac{d^2}{dx_3^2}u(x_3,t)=0 & \text{for } x_3\in\mathbb{R},\ t>0 \\ u_3(x_3,0)=x_3 & \text{for } x_3\in\mathbb{R} \end{cases}$$

for $k=1,2$. Let's set $u_k(x_k,t)=v_k(x_k)+w_k(t)$ so

$$0=\frac{\partial}{\partial t}u_k(x_k,t)-\frac{\partial^2}{\partial x_k^2}u(x_k,t)=w_k'(t)-v_k''(x_k)\ \text{ for } x_k\in\mathbb{R},\ t>0$$

i.e. $w_k'(t)=v_k''(x_k)=c_k\in\mathbb{R}$ is constant. Thus

$$w_k(t)=c_kt \text{ and } v_k(x_k)=\frac{1}{2}c_kx_k^2+d_kx_k+e_k$$

The initial value $u_k(x_k,0)=v_k(x_k)+w_k(t)=x_k^2$ means $c_k=2$ and $d_k=e_k=$ hence $u_k(x_k,t)=2t+x_k^2$ for $k=1,2$. The third equation leads in an analogue way to $u_3(x_3,t)=x_3$.

One (not necessarily all) solution to our initial value problem of the three dimensional heat equation is given by

$$u(x_1,x_2,x_3,t)=(2t+x_1^2)(2t+x_2^2)x_3$$

8. Prove and give an example of a solvable group which is not supersolvable.

Reason: Solvable Groups.

Solution: A_4.
The non Abelian, alternating group $A_4\subseteq S_4$ has $\frac{4!}{2}=12$ elements and is solvable:

$$\{\,(1)\,\}\lhd\{\,(1),(12)(34)\,\}\lhd V_4=\{\,(1),(12)(34),(13)(24),(14)(23)\,\}\lhd A_4$$

The first two factor groups (from the left) are isomorphic to $\mathbb{Z}_2$, i.e. of index 2, hence Abelian and normal in each other. The Klein $4-$group V_4

is also normal in A_4 because it coincides with the commutator subgroup of $A_4\,:\,V_4=[A_4,A_4]$. This means especially, that the factor group is Abelian. It is also isomorphic to $\mathbb{Z}_3$. We calculate the examples

$$[(123),(124)]=(123)(124)(132)(142)=(12)(34)\in V_4$$
$$[(123),(12)(34)]=(123)(12)(34)(132)(12)(34)=(13)(24)\in V_4$$

Assume we have a composition series $1=G_1\triangleleft G_2\triangleleft\ldots\triangleleft G_n=A_4$ where all $G_k\triangleleft A_4$ are normal in the main group, and have cyclic factor groups. Since

$$\prod_{k=1}^{n-1}|G_{k+1}/G_k|=|A_4|=12$$

we have maximal four factors, i.e. $n\leq 4$ because we included $\{1\}$. Subgroups of A_4 with 2 or 3 elements are not normal:

$$((12)(34))\,(132)\,((12)(34))=(132)(13)(24)=(124)\notin\{\,(1),(123),(132)\,\}$$
$$(123)\,((12)(34))\,(132)=(13)(24)(34)(12)=(14)(23)\notin\{\,(1),(12)(34)\,\}$$

A_4 can be written $A_4\cong V_4\rtimes_{\varphi}\mathbb{Z}_3$ with

$$\varphi\,:\,\mathbb{Z}_3\longrightarrow \operatorname{Aut}(V_4)\,,\,\varphi(z)(v)=(243)v(234)$$

A subgroup with 6 elements would necessarily be normal as of index 2. But the isomorphism shows, there is none, because $S_3\ntriangleleft A_4$. A_4 is herewith an example that the opposite of Lagrange's theorem does not hold: there is no normal subgroup of a finite group for any divider of the group order. As a consequence we must have $n=4$ and $G_3=V_4$. This leaves us with $G_2,G_3\in\{\,\mathbb{Z}_2,\mathbb{Z}_3\,\}$ neither of which are normal in A_4 which therefore cannot be supersolvable.

9. (HS-1) For which natural numbers is $1!+\ldots+n!$ a square number? $n!=1\cdot 2\cdot\ldots\cdot n$.

   Reason: Modular Arithmetic.

   Solution: $1!=1$ and $1!+2!+3!=9$ are square numbers, $1!+2!=3$ and $1!+2!+3!+4!=33$ are not. Now let $n\geq 5$. Then every number

   $$x=1!+2!+3!+4!+5!+6!+\ldots+n!=33+m$$

   where m is divisible by 10 as 2 and 5 are included factors in each term from 5 onwards. So x divided by 10 has remainder 3. However, any square number divided by 10 must have a remainder from $\{\,0,1,4,5,6,9\,\}$. Hence $n\in\{\,1,3\,\}$ are the only solutions.

10. (HS-2) Determine $\{\,(x,y)\in\mathbb{N}_0\times\mathbb{N}_0\,|\,x^3+8x^2-6x+8-y^3=0\,\}$.

    Reason: Diophantic Equation.

    Solution: Assume we have an integer solution for non negative numbers x,y to $y^3=x^3+8x^2-6x+8$. We start with the motto: Get rid of what disturbs! These are apparently the cubes. As there are no $y-$terms of minor degree, we can consider expressions $\pm(y^3-(x+g)^3)$ for small integers g and find:

$$\begin{aligned} y^3-(x+1)^3 &= x^3+8x^2-6x+8-x^3-3x^2-3x-1\\ &= 5x^2-9x+7\\ &= 5\cdot\left[\left(x-\frac{9}{10}\right)^2+\frac{59}{100}\right]\\ &> 0\\ (x+3)^3-y^3 &= x^3+9x^2+27x+27-x^3-8x^2+6x-8\\ &= x^2+33x+19\\ &> 0 \end{aligned}$$

    Thus $(x+3)^3>y^3>(x+1)^3$ or $x+1<y<x+3$ which leaves $y=x+2$ as only integer possibility. Thus we have

$$\begin{aligned} x^3+8x^2-6x+8=(x+2)^3&=x^3+6x^2+12x+8\\ 2x^2-18x&=2x(x-9)=0 \end{aligned}$$

    The only possible pairs are $(x,y)=(0,2)$ and $(x,y)=(9,11)$.
    As $9^3+8\cdot 9^2-6\cdot 9+8=729+648-54+8=1331=11^3$ and $0^3+8\cdot 0^2-6\cdot 0+8=8=2^3$ both pairs are indeed a solution.

11. (HS-3) Given two different, coprime, positive natural numbers $a,b\in\mathbb{N}$. Then there are two natural numbers $x,y\in\mathbb{N}$ such that $ax-by=1$.

    Reason: Pigeonhole Principle.

    Solution: We may assume $a,b>1$, since $(x,y)=(b+1,1),(1,a-1)$ solve the equation in case $a,b=1$ respectively, and we are done. Let $a>b$ and consider the multiples

$$m_1=a\,,\,m_2=2a\,,\,m_3=3a\,,\,\ldots\,,\,m_{b-1}=(b-1)a$$

    We write $j\cdot a=m_j=q_j\cdot b+r_j$ for $1\leq j<b$ with $0\leq r_j<b\,,\,q_j>0$.

    If $r_j=0$ for some j, then $a\,|\,q_j$, i.e. $q_j=aq_j'$, since a and b are coprime.

Thus $ja = q_j b = aq'_j b$ and $q'_j b = j < b$ which implies $q'_j = 0$. But this is impossible as otherwise we would have $ja = aq'_j b = 0$, hence $a = 0$, a contradiction.

Assume $r_j > 1$ for all j. Then we have

$$r_1, r_2, \ldots, r_{b-1} \in \{\,2, 3, \ldots, b-1\,\}$$

and two remainders have to be equal, say $r_i = r_j$. This means

$$(i-j)a = m_i - m_j = (q_i - q_j) \cdot b + (r_i - r_j) = (q_i - q_j) \cdot b$$

and by the same argument as above, all factors of a must be in $(q_i - q_j)$, $q_i - q_j = aq'$, i.e. $i - j = q'b < b$ and $q' = 0$. Then we get that either $i = j$ or $a = 0$ which is a contradiction in both cases.

We have shown that at least one remainder equals one, say $r_j = 1$. Hence $j \cdot a - q_j \cdot b = r_j = 1$ which had to be proven.

12. (HS-4) How many moves do the towers of Hanoi require to solve by an optimal strategy?

    The towers of Hanoi are three places. At the beginning there is a tower of disks on the left, the places on the right and in the middle are empty. Each disk is a bit smaller than the one below it, such that it looks like a round pyramid. The task is to move the complete tower from left to right in its original order - biggest disk at the bottom, smallest on top - where one move is the replacement of one disk at the top of a tower to the middle, to the right or to the left.

    Reason: Algorithmic Induction.

    Solution: Let n be the number of disks, i.e. the height of the tower at the beginning. We prove by induction that the solution is $2^n - 1$ moves.

    The statement is obviously true for $n = 1$. Now let us assume that the optimal strategy for n disks require $2^n - 1$ moves. What happens, if we add another biggest disk at the bottom? Since we may only move the top most disk, all towers during the game a sorted by radius, possibly upside down. The tower without the new biggest disk has to be moved twice: once to solve the problem for n disks onto the place in the middle, then to move it again to the left. The additional disk requires one additional move from left to right. Hence we get

    $$(2^n - 1) + (2^n - 1) + 1 = 2 \cdot 2^n - 2 + 1 = 2^{n+1} - 1$$

    moves total.

13. (HS-5) Among six people are always three who know each other or three who don't. Why?

    Reason: Pigeonhole Principle.

    Solution: We draw a graph of six persons and connect every knot with all others. Then we color the lines blue, if the two people representing the vertices know each other, and red if they don't. We have five edges at each vertex, so we may assume that three of them are blue at the first edge. Say we have the blue edges $\overline{AB}, \overline{AC}, \overline{AD}$. If at least one other vertex $\overline{BC}, \overline{BD}, \overline{CD}$ is blue, then we have found a blue triangle and we are done. On the other hand, if all those vertices are red, then $\triangle(BCD)$ is red and we are done again.

3 April 2020

1. Let $U \subseteq X$ be a dense subset of a normed vector space, Y a Banach space and $A \in L(U,Y)$ a linear, bounded operator. Show that there is a unique continuation $\tilde{A} \in L(X,Y)$ with $\left.\tilde{A}\right|_U = A$ and $\|\tilde{A}\| = \|A\|$.

   Reason: Operator Property.

   Solution: For an $x \in X$ we choose a sequence $(y_n)_{n\in\mathbb{N}} \subseteq U$ with $y_n \to x$. Now

   $$\|Ay_n - Ay_m\| \leq \|A\| \cdot \|y_n - y_m\| \leq \|A\|\|y_n - x\| + \|A\|\|y_m - x\| \stackrel{n,m\to\infty}{\longrightarrow} 0$$

   so $(Ay_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, which has a limit $\tilde{A}x := \lim_{n\to\infty} Ay_n$ as Y is complete. $\tilde{A}$ is linear and bounded

   $$\|\tilde{A}x\| = \|\lim_{n\to\infty} Ay_n\| \stackrel{A \text{ continuous}}{=} \lim_{n\to\infty} \|Ay_n\| \leq \|A\| \lim_{n\to\infty} \|y_n\| = \|A\|\|x\|$$

   hence continuous and $\|\tilde{A}\| \leq \|A\|$.

   Now let $\bar{A}$ be a second solution with the required properties. We choose again a sequence $(u_n)_{n\in\mathbb{N}} \subseteq U$ which converges to a given point $x \in X$. As $\bar{A}$ has to be continuous, we get

   $$\bar{A}x = \bar{A}(\lim_{n\to\infty} u_n) = \lim_{n\to\infty} \left.\bar{A}\right|_U u_n = \lim_{n\to\infty} Au_n = \tilde{A}x$$

   Finally we have

   $$\|A\| = \sup_{x\in U, \|x\|\leq 1} \|Ax\| = \sup_{x\in U, \|x\|\leq 1} \|\tilde{A}x\| \leq \sup_{x\in X, \|x\|\leq 1} \|\tilde{A}x\| = \|\tilde{A}\|$$

   where the inequality arises from the fact that we build the supremum over a larger set.

2. Let $X \sim \mathcal{N}(\mu,\sigma^2)$ and $Y \sim \mathcal{N}(\lambda,\sigma^2)$ be normally distributed random variables on $\mathbb{R}$ with expectation values $\mu,\lambda \in \mathbb{R}$ and standard deviation σ. We want to test the hypothesis that $\mu = \lambda$ with n independent measurements $X_1,\ldots,X_n$ and $Y_1,\ldots,Y_n$. We choose the mean distance

   $$T_n(X_1,Y_1;X_2,Y_2;\ldots;X_n,Y_n) := \frac{1}{n}\sum_{k=1}^{n}(X_k - Y_k)$$

   as estimator for the difference $\nu = \mu - \lambda$.

   (a) Does the estimator T_n have a bias and is it consistent?

(b) Let $n = 100$ and $\sigma^2 = 0.5$. We use the hypotheses $H_0 : \mu = \lambda$ and $H_1 : \mu \neq \lambda$. Determine a reasonable deterministic test φ for the error level $\alpha = 0.05$.

Reason: Normal Distribution.

Solution:

(a) Since the expectation value is linear we have

$$\begin{aligned} E(T_n) &= E\left(\frac{1}{n}\sum_{k=1}^{n}(X_k - Y_k)\right) \\ &= \frac{1}{n}\sum_{k=1}^{n}(E(X_k) - E(Y_k)) \\ &= \frac{1}{n}\sum_{k=1}^{n}(\mu - \lambda) \\ &= \mu - \lambda \end{aligned}$$

i.e. T_n is unbiased. By the strong law of large numbers

$$\frac{1}{n}\sum_{k=1}^{n}(X_k - Y_k) \xrightarrow{a.s.} E(X - Y) = \mu - \lambda$$

and T_n is consistent.

(b) Let $z := T_n(X_1, Y_1; X_2, Y_2; \ldots ; X_n, Y_n) \in \mathbb{R}$ be the result of the measurements. z is again distributed normally as the intersection of two normally distributed random variables. The expectation value is $\mu - \lambda$ and the variance

$$\begin{aligned} V(z) &= \frac{1}{n^2}\, V\left(\sum_{k=1}^{n} V(X_k - Y_k)\right) \\ &= \frac{1}{n^2}\sum_{k=1}^{n} V(X_k - Y_k) \\ &= \frac{1}{n}\, V(X_k - Y_k) \\ &= \frac{1}{n}\cdot (V(X_k) + V(Y_k)) \\ &= \frac{1}{100}\cdot\left(\frac{1}{2} + \frac{1}{2}\right) \\ &= 0.01 \end{aligned}$$

Under the null hypothesis we have $H_0 : \mu = \lambda$ and thus $z \sim \mathcal{N}(0, 0.01)$. A reasonable deterministic test rejects H_0 as soon as the measured value z differs too much from the expectation value 0. Hence we define as test function

$$\varphi : \mathbb{R} \longrightarrow \{0, 1\}$$
$$\varphi(z) := \begin{cases} 0 & , \text{ if } |z| < \varepsilon \\ 1 & , \text{ otherwise} \end{cases}$$

We must choose the critical level ε such that the first order error is at most α, i.e. $P(|z| > \varepsilon \mid H_0) \leq 0.05$. From $z \sim \mathcal{N}(0, 0.01)$ follows $10z \sim \mathcal{N}(0, 1)$ and thus

$$\begin{aligned} P(|z| > \varepsilon \mid H_0) \leq 0.05 &\Longleftrightarrow P(|z| \leq \varepsilon \mid H_0) \geq 0.95 \\ &\Longleftrightarrow P(-\varepsilon \leq z \leq \varepsilon \mid H_0) \geq 0.95 \\ &\Longleftrightarrow P(-10\varepsilon \leq 10z \leq 10\varepsilon \mid H_0) \geq 0.95 \\ &\Longleftrightarrow F_{0,1}(10\varepsilon) - F_{0,1}(-10\varepsilon) \geq 0.95 \\ &\Longleftrightarrow 2F_{0,1}(10\varepsilon) - 1 \geq 0.95 \\ &\Longleftrightarrow F_{0,1}(10\varepsilon) \geq 0.975 \end{aligned}$$

We find by looking up the table for standard normal distributions $\mathcal{N}(0, 1)$ that $0.975 = F_{0,1}(1.96)$, i.e. $10\varepsilon = 1.96$ and our error level is $\varepsilon = 0.196$.

If our measurement shows an average difference greater than 0.196, then we falsely reject the null hypothesis by at most a 5% chance.

3. (a) Solve $y''x^2 - 12y = 0$, $y(0) = 0$, $y(1) = 16$ and calculate

$$\sum_{n=1}^{\infty} \frac{1}{t_n},\; t_n := y(n) - \frac{1}{2}y'(n) + \frac{1}{8}y''(n) - \frac{1}{48}y'''(n) + \frac{1}{384}y^{(4)}(n).$$

   (b) What do we get for the initial values $y(1) = 1$, $y(-1) = -1$ and $\displaystyle\sum_{n=1}^{\infty} (y'(n) + y'''(n))$?

Reason: Riemann Zeta-Function.

Solution:

(a) We have a Cauchy-Euler equation here, so we use the transformation theorem and set $y(x) = u(\log |x|)$.

$$y''(x)x^2 = \left(u' \cdot \frac{1}{x}\right)' x^2 = \left(u'' \frac{1}{x^2} - u' \frac{1}{x^2}\right) x^2 = u'' - u' = 12y(x) = 12u$$

with the characteristic polynomial $\chi(\lambda) = \lambda^2 - \lambda - 12$ that has zeros $\lambda_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 12} \in \{-3, 4\}$. The fundamental system is thus $\{e^{-3u}, e^{4u}\}$ for the transformed version, and $\{x^{-3}, x^4\}$ for the original equation. Hence we have

$$y(x) = \alpha x^{-3} + \beta x^4 = 16x^4$$

if we apply $y(0) = 0$ and $y(1) = 16$. So

$$\begin{aligned} t_n &= y(n) - \frac{1}{2}y'(n) + \frac{1}{8}y''(n) - \frac{1}{48}y'''(n) + \frac{1}{384}y^{(4)}(n) \\ &= 16n^4 - 32n^3 + 24n^2 - 8n + 1 \\ &\Longrightarrow \end{aligned}$$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{t_n} &= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{1}{2}\left(\sum_{n=1}^{\infty} \frac{1}{n^4} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4}\right) \\ &= \frac{1}{2}\left(\zeta(4) + (1 - 2^{1-4}) \cdot \zeta(4)\right) = \frac{15}{16} \cdot \frac{\pi^4}{90} = \frac{\pi^4}{96} \approx 1.014678 \end{aligned}$$

(b) If we have the initial values $y(1) = 1$, $y(-1) = -1$ then $y(x) = \alpha x^{-3} + \beta x^4$ implies $\alpha = 1$, $\beta = 0$ and we get with $y(x) = x^{-3}$

$$\begin{aligned} \sum_{n=1}^{\infty} (y'(n) + y'''(n)) &= \sum_{n=1}^{\infty} \left(-3x^{-4} - 60x^{-6}\right) \\ &= -3 \cdot \zeta(4) - 60 \cdot \zeta(6) \\ &= -\frac{3\pi^4}{90} - \frac{60\pi^6}{945} \\ &= -\frac{\pi^4}{30} - \frac{4\pi^6}{63} \approx -64.2875534202 \end{aligned}$$

4. Solve the initial value problem $y'(x) = y(x)^2 - (2x+1)y(x) + 1 + x + x^2$ for $y(0) \in \{0, 1, 2\}$.

Reason: Differential Equation.

Solution: We are only interested in solutions, where $y(x)$ is finite and

$y'(x)$ exists. We may thus use the template $y(x) = x + \dfrac{1}{u(x)}$ and get

$$\begin{aligned} y' &= 1 - \frac{u'}{u^2} \\ &= \left(x + \frac{1}{u}\right)^2 - (2x+1)\left(x + \frac{1}{u}\right) + 1 + x + x^2 \\ &= x^2 + \frac{2x}{u} + \frac{1}{u^2} - 2x^2 - \frac{2x}{u} - x - \frac{1}{u} + 1 + x + x^2 \\ &= \frac{1}{u^2} - \frac{u}{u^2} + 1 \\ &\Leftrightarrow \\ u' &= u - 1 \\ &\Leftrightarrow \\ \frac{u'}{u-1} &= 1 \end{aligned}$$

Integrating both sides gets $\log|u-1| = x + C$ or $u(x) = 1 + e^x \cdot e^C$. Backward substitution gives us

$$y(x) = x + \frac{1}{1 + De^x}$$

Let us consider $y(0) = 0$, i.e. $y(x) = x$. It is obvious that this is a solution, too, $C = D = \infty$. For the case $y(0) = 1$ we get $y(x) = x + 1$ with $D = 0$ (or $C = -\infty$), and for $y(0) = 2$ we have $y(x) = x + \dfrac{1}{1 - \frac{1}{2}e^x}$.

5. For coprime natural numbers n, m show that

$$m^{\varphi(n)} + n^{\varphi(m)} \equiv 1 \bmod nm$$

Reason: Euler's Theorem.

Solution: Since $\gcd(n,m) = 1$ we get by Euler's theorem $m^{\varphi(n)} \equiv 1 \bmod n$ and $n^{\varphi(m)} \equiv 1 \bmod m$. Trivially true are $n^{\varphi(m)} \equiv 0 \bmod n$ and $m^{\varphi(n)} \equiv 0 \bmod m$. Thus

$$m^{\varphi(n)} + n^{\varphi(m)} \equiv 1 \bmod n \text{ and } m^{\varphi(n)} + n^{\varphi(m)} \equiv 1 \bmod m$$

Since n, m are coprime, the congruences still hold for $\operatorname{lcm}(n,m) = nm$.

6. (HS-1) Let $P(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ a monic, real polynomial of degree n, whose zeros are all negative.

Show that $\int_1^\infty \frac{dx}{P(x)}$ converges absolutely if and only if $n \geq 2$.

Reason: Integrals.

Solution: Let $n \geq 2$. Now

$$\frac{P(x)}{x^n} = 1 + a_{n-1}\frac{1}{x} + \ldots + a_1\frac{1}{x^{n-1}} + a_0\frac{1}{x^n} \longrightarrow 1$$

converges for $x \to \infty$. Multiplication by $\sqrt{x}$ yields

$$\frac{P(x)}{x^{n-\frac{1}{2}}} = \sqrt{x}\left(1 + a_{n-1}\frac{1}{x} + \ldots + a_1\frac{1}{x^{n-1}} + a_0\frac{1}{x^n}\right) \longrightarrow \infty$$

Thus there is a $x_0 > 1$, such that the numerator exceeds the denominator $P(x) > x^{n-\frac{1}{2}} \geq x_0^{n-\frac{1}{2}} > 1$ for all $x \geq x_0$ and we have

$$\begin{aligned}\int_1^\infty \left|\frac{1}{P(x)}\right| dx &= \underbrace{\int_1^{x_0} \frac{dx}{|P(x)|}}_{=:C} + \int_{x_0}^\infty \frac{dx}{|P(x)|} \\ &= C + \int_{x_0}^\infty \frac{dx}{P(x)} \\ &\leq C + \int_{x_0}^\infty x^{\frac{1}{2}-n}\,dx \\ &< \infty\end{aligned}$$

For $n = 0$ we have $P(x) = 1$ and

$$\int_1^\infty \frac{dx}{P(x)} = \int_1^\infty dx = \lim_{\zeta\to\infty}(\zeta - 1) = \infty$$

and for $n = 1$ with $P(x) = x + a_0$

$$\int_1^\infty \frac{dx}{P(x)} = \int_1^\infty \frac{1}{x+a_0}\,dx = \lim_{\zeta\to\infty}(\log|\zeta + a_0| - \log|1 + a_0|) = \infty$$

4 March 2020 - Part II

1. Let $\sum_{k=1}^{\infty} a_k$ be a given convergent series with $|a_{k+1}| \leq |a_k|$ for all k. Assume we use a computer to sum its value until the partial sum is closer than ε to the actual value of the series. Does it make sense to use $|a_n| < \varepsilon$ as a stopping criterion for the loop? Please justify your answer.

   Reason: Understanding.

   Solution: No. The smallness of the summands - even with a monotone decreasing absolute value - says nothing about the size of the remainder part, i.e. the error of the current partial sum.

   Let's consider
$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} q^k \text{ and } R_n := \sum_{k=n+1}^{\infty} q^k$$
   For different values of q we list n for which $a : n = q^n < \varepsilon := 0.001$

q	0.9	0.99	0.999	0.9999
n	66	688	6905	69075
R_n	0.0086	0.0993	0.998	9.998

   This means for our remainder
$$R_n = \sum_{k=n+1}^{\infty} q^k = \sum_{k=0}^{\infty} q^{k+n+1} = q^{n+1} \sum_{k=0}^{\infty} q^k \approx \frac{0.001}{1-q} \stackrel{q \to 1-0}{\longrightarrow} +\infty$$
   Hence we can make R_n arbitrary large, although $|a_k| < 0.001 \ (k > n)$, if we choose $q < 1$ close enough to 1.

2. "Every absolutely convergent series converges." Now why is its proof so complicated, couldn't we just say: Given an absolute convergent series $\sum_{k=1}^{\infty} |a_k|$ then we have for the sequence $R_n := \sum_{k=n+1}^{\infty} a_k$
$$|R_n| = \left| \sum_{k=n+1}^{\infty} a_k \right| \leq \sum_{k+1}^{\infty} |a_k|$$
   with the remainder of a convergent series on the right, hence a null sequence. Thus R_n is a null sequence, too, and the series is convergent.

   Reason: Proof Theory.

Solution: In order to be able to estimate R_n in the described manner, we must first be sure that it exists at all, i.e. that this series converges. But that is exactly what should be shown! Thus we just noticed that if the series converges, it converges.

3. Calculate the limit (i being the imaginary unit):

$$\lim_{n\to\infty} \operatorname{Arg}\left(\sum_{k=0}^{n} \frac{1}{k+i}\right)$$

Reason: Calculus.

Solution:

$$\sum_{k=0}^{n} \frac{1}{k+i} = \sum_{k=0}^{n} \frac{k-i}{k^2+1} = \sum_{k=0}^{n} \frac{k}{k^2+1} + i \cdot \sum_{k=0}^{n} \frac{1}{k^2+1}$$

Both real and even positive sums can be estimated by known series. The real part is unbounded, because

$$\sum_{k=0}^{n} \frac{k}{k^2+1} > \sum_{k=1}^{n-1} \frac{1}{k}$$

whereas the imaginary part is bounded by

$$\sum_{k=0}^{n} \frac{1}{k^2+1} < 1 + \sum_{k=1}^{n} \frac{1}{k^2} < 1 + \frac{\pi^2}{6}$$

Since the inverse tangent is continuous, we get

$$\begin{aligned}
\lim_{n\to\infty} \operatorname{Arg}\left(\sum_{k=0}^{n} \frac{1}{k+i}\right) &= \lim_{n\to\infty} \arctan \frac{\displaystyle\sum_{k=0}^{n} \frac{1}{k^2+1}}{\displaystyle\sum_{k=0}^{n} \frac{k}{k^2+1}} \\
&= \arctan \lim_{n\to\infty} \frac{\displaystyle\sum_{k=0}^{n} \frac{1}{k^2+1}}{\displaystyle\sum_{k=0}^{n} \frac{k}{k^2+1}} \\
&= \arctan 0 \\
&= 0
\end{aligned}$$

4. Show that there is no odd dimensional real division algebra D.

   Reason: Basic Algebra.

   Solution: Assume $\dim D = n$ is odd. Consider the left multiplication $L_a : D \longrightarrow D$, $x \longmapsto ax$ with an element $a \in D - \mathbb{R}$. It's characteristic polynomial $\chi(L_a)(x) = \det(xI_n - L_a)$ in some basis is a monic, real polynomial of odd degree and thus has a real zero z, i.e. there is an element $b \in D - \{0\}$ such that $0 = (zI_n - L_a)(b) = zb - ab = (z - a)b$. Since z is real and a is not, we have $z - a \neq 0$ and $b \neq 0$ cannot be a unit.

   Alternative Solution: A division algebra structure on R^n makes $\mathbb{R}^n \setminus \{0\}$ into a topological group, and induces a group structure on S^{n-1} via the map $x \mapsto x/|x|$. But S^m can only have a group structure for odd m by degree theory, so n must be even.

5. Let $R := \mathbb{Z}_{(5)} = \left\{ \dfrac{a}{b} \;\middle|\; 5 \nmid b \right\}$ the ring of rational numbers which don't have a factor 5 in their denominator, $M \neq \{0\}$ a finitely generated $R-$module, and $I := \left\{ \dfrac{a}{b} \in R \;\middle|\; 25 \mid a \right\}$.

   Prove that I is an ideal contained in the Jacobson radical of R and that $IM \neq M$. The Jacobson radical $J = J(R)$ is defined as the intersection of all maximal ideals.

   Reason: Nakayama's Lemma.

   Solution: We first show that $x \in J$ if and only if $1 - xy$ is a unit in R for any $y \in R$.

   If $x \in J$ and $1 - xy$ is no unit, then it belongs to some maximal ideal $K \subsetneq R$ (att.: this result uses the axiom of choice). Now $1 = k + xy \in K + JR = K + J \subseteq K$ which is absurd.
   If $x \notin J$ then $x \notin K$ for some maximal ideal $K \subsetneq R$. Hence $xR + K = R$ by the maximality of K and we can write $1 = xy + k$ for some $y \in R, k \in K$. Now $1 - xy \in K$ and thus cannot be a unit.

   R is a commutative, local ring with maximal ideal $J = \left\{ \dfrac{a}{b} \in R \;\middle|\; 5 \mid a \right\}$ which is also its Jacobson radical. It is obvious that $I \subsetneq J$ is a proper ideal. Suppose $IM = M$ and $\{ u_1, \ldots, u_n \}$ is a minimal set of generators of M. Since $M \neq \{0\}$ we have $n \geq 1$ and $u_n \neq 0$. Since $u_n \in IM$ we have an expression

$$u_n = \sum_{k=1}^{n} a_k u_k \Longrightarrow (1 - a_n) u_n = \sum_{k=1}^{n-1} a_k u_k$$

for some $a_k \in I \subseteq J$. From $a_n \in J$ we conclude that $1 - 1 \cdot a_n$ is a unit, i.e. u_n can be expressed by the other $u-k$ contradicting the minimality if the chosen system of generators.

6. Calculate
$$\lim_{n\to\infty} \frac{\sqrt{n\pi}}{2^{2n}} \cdot \binom{2n}{n}$$
   (a) without using Stirling's formula.
   (b) by using Stirling's formula, with accurate remainder terms, i.e. not simply $\sim$.

   Reason: Wallis Product.

   Solution:

   (a) Wallis Product.
$$\prod_{k=1}^{n} \frac{4k^2-1}{4k^2} = \prod_{k=1}^{n} \frac{(2k-1)\cdot(2k+1)}{2k \cdot 2k} = \frac{\frac{(2n)!}{2^n\, n!} \cdot \frac{(2n)!}{2^n\, n!} \cdot (2n+1)}{2^n\, n! \cdot 2^n\, n!}$$
$$= (2n+1)\left[\frac{(2n)!}{2^{2n}\, n!^2}\right]^2 = (2n+1)\left[\frac{1}{2^{2n}}\binom{2n}{n}\right]^2$$
   so $\lim_{n\to\infty}(2n+1)\left[\frac{1}{2^{2n}}\binom{2n}{n}\right]^2 = \prod_{k=1}^{\infty}\frac{4k^2-1}{4k^2} = \frac{2}{\pi}$ by Wallis product. Hence $\lim\limits_{n\to\infty} \frac{\sqrt{2n+1}}{2^{2n}}\binom{2n}{n} = \sqrt{\frac{2}{\pi}}$ or $\lim\limits_{n\to\infty} \frac{\sqrt{\left(n+\frac{1}{2}\right)\pi}}{2^{2n}}\binom{2n}{n} = 1$.

   Because of $\lim\limits_{n\to\infty} \frac{\sqrt{n\,\pi}}{\sqrt{\left(n+\frac{1}{2}\right)\pi}} = 1$ we also have
$$\lim_{n\to\infty} \frac{\sqrt{n\,\pi}}{2^{2n}} \cdot \binom{2n}{n} = 1$$

   (b) Stirling's Formula.

We use Robbins estimation for Stirling's formula:

$$\begin{aligned}
&\sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} e^{\frac{1}{12n}} \\
&\Longrightarrow \sqrt[12n+1]{e} < \frac{n!}{\sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}} < \sqrt[12n]{e} \\
&\Longrightarrow \frac{\sqrt[24n+1]{e}}{\sqrt[6n]{e}} < \frac{(2n)!}{\sqrt{2\pi}(2n)^{2n+\frac{1}{2}} e^{-2n}} \cdot \frac{2\pi\, n^{2n+1} e^{-2n}}{(n!)^2} < \frac{\sqrt[24n]{e}}{\sqrt[12n+1]{e^2}} \\
&\Longrightarrow e^{-\frac{18n+1}{144n^2+6n}} < \sqrt{2\pi} \cdot \sqrt{n} \cdot 2^{-2n} \cdot 2^{-\frac{1}{2}} \cdot \binom{2n}{n} < e^{-\frac{36n-1}{288n^2+24n}} \\
&\Longrightarrow 1 \leq \lim_{n\to\infty} \frac{\sqrt{\pi n}}{2^{2n}} \cdot \binom{2n}{n} \leq 1
\end{aligned}$$

7. (HS-1) Prove that if for $x \in \mathbb{R} - \{0\}$ the number $x + \dfrac{1}{x}$ is an integer, then $x^n + \dfrac{1}{x^n}$ with $n \in \mathbb{N}$ are integers, too.

   Reason: Induction.

   Solution: We have the statement for $n = 1$ by assumption. For $n = 2$ and $m := n + \frac{1}{n} \in \mathbb{Z}$ we get

   $$\left(x^2 + \frac{1}{x^2}\right) = \left(x + \frac{1}{x}\right)^2 - 2 = m^2 - 2 \in \mathbb{Z}$$

   We may assume that the statement is true for every number up to n. Hence

   $$\left(x^{n+1} + \frac{1}{x^{n+1}}\right) = \left(x^n + \frac{1}{x^n}\right) \cdot \left(x + \frac{1}{x}\right) - \left(x^{n-1} + \frac{1}{x^{n-1}}\right) \in \mathbb{Z}$$

   by induction.

8. (HS-2) We define for positive integers a, b the following sequence

   $$x_n := \begin{cases} 1 & \text{if } n = 1 \\ a x_{n-1} + b & \text{if } n > 1 \end{cases}$$

   Show that the sequence contains infinitely many numbers. which are not prime, for any choice of a, b.

   Reason: Numbers.

   Solution: The sequence $(x_n)_{n\in\mathbb{N}}$ is strictly monotone increasing, since

$a, b \geq 1$. Hence it is sufficient to show, that there is a natural number $d > 1$ with the following property: There are infinitely many indices n such that $d \,|\, x_n$.

If $(a, b) > 1$ then we are done, so we may assume $(a, b) = 1$. Let $d = x_2 = ax_1 + b = a + b > 1$ and be $x_n = q_n \cdot d + r_n$, $0 \leq r_n < d$. Then we have at least two equal remainders $r_\alpha = r_{\alpha+\beta} \in \{r_2, r_3, \ldots, r_{d+2}\}$. Now any divisor of a and $d = a + b$ would also be a divisor of a and b, which are coprime. So a and d are coprime, too. Thus

$$\begin{aligned}
d \,|\, x_{\alpha+\beta} - x_\alpha = a \cdot (x_{\alpha+\beta-1} - x_{\alpha-1}) &\Longrightarrow d \,|\, x_{\alpha+\beta-1} - x_{\alpha-1} \\
&\Longrightarrow d \,|\, (q_{\alpha+\beta-1} - q_{\alpha-1})d + (r_{\alpha+\beta-1} - r_{\alpha-1}) \\
&\Longrightarrow d \,|\, r_{\alpha+\beta-1} - r_{\alpha-1} \\
&\Longrightarrow r_{\alpha+\beta-1} = r_{\alpha-1} \\
&\vdots \\
&\Longrightarrow r_2 = r_{2+\beta}
\end{aligned}$$

$$\begin{aligned}
d \,|\, a \cdot (x_{\alpha+\beta} - x_\alpha) &= x_{\alpha+\beta+1} - x_{\alpha+1} \\
&\Longrightarrow d \,|\, r_{\alpha+\beta+1} - r_{\alpha+1} \\
&\Longrightarrow r_{\alpha+\beta+1} = r_{\alpha+1} \\
&\vdots \\
&\Longrightarrow r_{\alpha+2\beta} = r_{\alpha+\beta} = r_\alpha
\end{aligned}$$

Thus we have $r_2 = r_{2+k\cdot\beta}$ for all $k \in \mathbb{N}$. But $r_2 = 0$ per construction, so for all indices $2 + k\beta$ we have that $d \,|\, x_{2+k\beta}$.

9. (HS-3) Name a convergent series $\sum_{k=1}^{\infty} a_k$ with positive a_k, where $a_{k+1}/a_k \geq 2$ holds infinitely often.

   Reason: Attention with Intuition.

   Solution: We define

$$a_k := \begin{cases} 2^{-k} & \text{if } k \text{ is even} \\ 2^{-k+2} & \text{if } k \text{ is odd} \end{cases}$$

   Now $a_{2k+1}/a_{2k} = 2^{-2k-1+2}\, 2^{-(-2k)} = 2$. The series with the first terms written out is $\sum_{k=1}^{\infty} a_k = 2 + \frac{1}{4} + \frac{1}{2} + \frac{1}{16} + \frac{1}{8} + \frac{1}{64} + \frac{1}{32} + \frac{1}{256} + \frac{1}{128} + \ldots$ which converges to $2 + 1 = 3$.

10. (HS-4) For natural numbers $1 \leq k \leq 2n$ show that

$$\binom{2n+1}{k-1} + \binom{2n+1}{k+1} \geq 2 \cdot \frac{n+1}{n+2} \cdot \binom{2n+1}{k}$$

Reason: Polynomials.

Solution:

$$\begin{aligned}\binom{2n+1}{k+1} &= \frac{(2n+1)!}{(k+1)!(2n-k)!} = \frac{2n+1-k}{k+1}\cdot\frac{(2n+1)!}{k!(2n+1-k)!}\\ &= \frac{2n+1-k}{k+1}\cdot\binom{2n+1}{k}\end{aligned}$$

$$\begin{aligned}\binom{2n+1}{k} &= \frac{(2n+1)!}{k!(2n+1-k)!} = \frac{2n+2-k}{k}\cdot\frac{(2n+1)!}{(k-1)!(2n+2-k)!}\\ &= \frac{2n+2-k}{k}\cdot\binom{2n+1}{k-1}\end{aligned}$$

$$\binom{2n+1}{k-1}+\binom{2n+1}{k+1} = \frac{k}{2n+2-k}\cdot\binom{2n+1}{k}+\frac{2n+1-k}{k+1}\cdot\binom{2n+1}{k}$$

Hence we have to show that

$$\frac{k}{2n+2-k}+\frac{2n+1-k}{k+1}\geq 2\cdot\frac{n+1}{n+2}$$

or with $m := n-k+\frac{1}{2}$ and $-(n-\frac{1}{2})\leq m\leq n-\frac{1}{2}$

$$f(n,m) := \frac{2n-2m+1}{2n+3+2m}+\frac{2n+1+2m}{2n-2m+3}-2\cdot\frac{n+1}{n+2}\overset{(!)}{\geq} 0$$

$$\begin{aligned}f(n,m) &= 2\cdot\frac{4n^2+4m^2+8n+3}{(2n+3)^2-4m^2}-2\cdot\frac{n+1}{n+2}\\ &= 2\cdot\frac{8m^2n-2n+12m^2-3}{((2n+3)^2-4m^2)\cdot(n+2)}\\ &= 2\cdot\frac{(4m^2-1)(2n+3)}{(2n+3-2m)(2n+3+2m)(n+2)}\\ &= 2\cdot\frac{(2m+1)(2m-1)(2n+3)}{(2n+3-2m)(2n+3+2m)(n+2)}\end{aligned}$$

Since we have $-2n+1\leq \pm 2m\leq 2n-1$ our denominator is positive. But $m\neq 0$ as k,n are integers, so $4m^2-1>2$ and the numerator is also positive, which had to be shown.

11. (HS-5) The year on the Earth-like planet Trappist-1e has 365 days divided into months of $28, 30, 31$ days. How many months does its year

have and how many months with (i) 28, (ii) 30, (iii) 31 days?

Reason: Pigeonhole Principle.

Solution: We prove the following two statements:

(a) There are 12 months on Trappist-1e.

(b) The following combinations are possible: $(28, 30, 31) \in \{ (0,7,5), (1,4,7), (2,1,9) \}$.

We have to solve the equation $28a + 30b + 31c = 365$ with $a, b, c \in \mathbb{N}_0$. $c \geq 1$, since 365 is odd.

(a) With $c' := c - 1$ we have

$$\begin{aligned} 28a + 30b + 31c' = 334 & \\ & \Longrightarrow 28(a + b + c') \leq 334 \leq 31(a + b + c') \\ & \Longrightarrow 10 < \frac{334}{31} \leq a + b + c' \leq \frac{334}{28} < 12 \\ & \Longrightarrow a + b + c' = 11 \\ & \Longrightarrow a + b + c = 12 \end{aligned}$$

since $a, b, c, c' \in \mathbb{N}_0$.

(b) The sum of days can also be written as

$$\begin{aligned} 365 &= (30-2)a + 30b + (30+1)c = 30(a+b+c) - 2a + c = 360 - 2a + c \\ &\Longrightarrow 5 = c - 2a \text{ with } 0 \leq a, 1 \leq c \text{ and } a + c \leq 12 \\ &\Longrightarrow a + c = a + (5 + 2a) = 3a + 5 \leq 12 \\ &\Longrightarrow (a, c) \in \{ (0,5), (1,7), (2,9) \} \\ &\Longrightarrow (a, b, c) \in \{ (0,7,5), \underbrace{(1,4,7)}_{\text{Earth}}, (2,1,9) \} \end{aligned}$$

12. (HS-6) Decrypt the affine encrypted "Ara gtynd hdm hvcrsnd mthvtjph!"

Reason: Puzzle.

Solution: Non vitae sed scholae discimus! with $x \to 17(x-1) + 14$.

5 March 2020

1. Let $\mathfrak{g} = \operatorname{lin}_{\mathbb{R}}\{\, e_1, e_2, e_3, e_4 \,\}$ on which we define the following multiplication:

$$[e_1, e_4] = 2e_1 \ , \ [e_2, e_4] = 3e_2 - e_3 \ , \ [e_3, e_4] = e_2 + 3e_3$$

and $[e_i, e_j] = 0$ otherwise, as well as $[e_i, e_i] = 0$.

Show that

(a) $\mathfrak{g}$ is a Lie algebra.

(b) There exists an $\alpha_0 \in A(\mathfrak{g})$ where

$$A(\mathfrak{g}) := \{\, \alpha : \mathfrak{g} \stackrel{\text{linear}}{\longrightarrow} \mathfrak{g} \,|\, \forall\, X, Y \in \mathfrak{g} : [\alpha(X), Y] + [X, \alpha(Y)] = 0 \,\}$$

such that $[\operatorname{ad} X, \alpha_0] \in \mathbb{R} \cdot \alpha_0$ for all $X \in \mathfrak{g}$.

(c) The center $Z(\mathfrak{g}) = \{\, 0 \,\}$.

(d) $\mathfrak{g}$ has a one dimensional ideal.

Reason: Solvable Lie Algebra.

Solution:

(a) $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] \subseteq \operatorname{lin}_{\mathbb{R}}\{\, e_1, e_2, e_3 \,\}$ which all commute. Thus we only have to consider products $[e_4, [e_a, e_b]]$ with $a, b \leq 3$.

$$[e_4, [e_a, e_b]] + [e_a, [e_b, e_4]] + [e_b, [e_4, e_a]] \in [e_4, 0] + [e_a, \mathfrak{g}'] + [e_b, \mathfrak{g}'] = 0$$

(b) We have seen that $[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] = [\mathfrak{g}', \mathfrak{g}'] = 0$, i.e. $\mathfrak{g}$ is solvable. The setting $X.\alpha := [\operatorname{ad} X, \alpha] = \operatorname{ad} X \circ \alpha - \alpha \circ \operatorname{ad} X$ makes $A(\mathfrak{g})$ into a $\mathfrak{g}-$module. By Lie's theorem there is an invariant one dimensional submodule, spanned by α_0.

Explicitly we can choose $\alpha_0(x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4) := x_4e_1$ which is easy to verify.

(c) Let $Z = z_1e_1 + z_2e_2 + z_3e_3 + z_4e_4 \in Z(\mathfrak{g})$ be a central element. Then

$$\begin{aligned} [Z, e_4] = 0 &= 2z_1e_1 + z_2(3e_2 - e_3) + z_3(e_2 + 3e_3) \\ &\Longrightarrow z_1 = 0\,,\ 3z_2 + z_3 = 0\,,\ -z_2 + 3z_3 = 0 \\ &\Longrightarrow Z = z_4e_e \end{aligned}$$

which cannot be central unless $Z = 0$ since $0 = [e_1, Z] = 2z_4e_1$.

(d) It is clear from the multiplication table, that e_1 spans a one dimensional ideal.

2. Let $A, B \in \mathbb{M}(m, \mathbb{R})$ and $\|A\|, \|B\| \leq 1$, then

$$\left\| e^{A+B} - e^A \cdot e^B \right\| \leq 6e^2 \cdot \| [A, B] \|$$

Reason: Trotter's Estimation.

Solution: For a vector $\alpha \in \{ 0, 1 \}^n$ we define

$$S(\alpha) := \prod_{k=1}^{n} A^{1-\alpha_k} B^{\alpha_k}$$

Each of these vectors can be sorted in an ascending order in n^2 steps, e.g. with Bubble sort, until $S(\alpha_{Sort}) = A^{n-|\alpha|}B^{|\alpha|}$ where every commutation between A, B results in an additional term $[A, B] = AB - BA$:

$$S(BA) - S(AB) = BA - AB = -[A, B]$$

or with an example which needs more steps:

$$\begin{aligned}
&\begin{bmatrix} \alpha_0 = (1,0,1,0) \\ S(\alpha_0) = BABA \end{bmatrix} \longrightarrow \begin{bmatrix} \alpha_1 = (0,1,1,0) \\ S(\alpha_1) = ABBA \end{bmatrix} \\
&S(\alpha_0) - S(\alpha_1) = (BA - AB)BA = -[A, B]BA \\
&\begin{bmatrix} \alpha_1 = (0,1,1,0) \\ S(\alpha_1) = ABBA \end{bmatrix} \longrightarrow \begin{bmatrix} \alpha_2 = (0,1,0,1) \\ S(\alpha_2) = ABAB \end{bmatrix} \\
&S(\alpha_1) - S(\alpha_2) = AB(BA - AB) = -AB[A, B] \\
&\begin{bmatrix} \alpha_2 = (0,1,0,1) \\ S(\alpha_2) = ABAB \end{bmatrix} \longrightarrow \begin{bmatrix} \alpha_3 = (0,0,1,1) \\ S(\alpha_3) = AABB \end{bmatrix} \\
&S(\alpha_2) - S(\alpha_3) = A(BA - AB)B = -A[A, B]B
\end{aligned}$$

So every sorting step produces a factor $[A, B]$, which combined with the assumption $\|A\|, \|B\| \leq 1$ means $\|S(\alpha_k) - S(\alpha_{k+1})\| \leq \| [A, B] \|$ and

$$\|S(\alpha) - S(\alpha_{Sort})\| \leq \sum_{k=0}^{n^2-1} \|S(\alpha_k) - S(\alpha_{k+1})\| \leq n^2 \cdot \| [A, B] \|$$

$$
\begin{aligned}
e^{A+B} &= \sum_{n=0}^{\infty} \frac{1}{n!}(A+B)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\alpha\in\{0,1\}^n} S(\alpha) \\
e^A \cdot e^B &= \left(\sum_{n=0}^{\infty} \frac{A^n}{n!} \right) \cdot \left(\sum_{n=0}^{\infty} \frac{B^n}{n!} \right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{A^{n-k}}{(n-k)!} \cdot \frac{B^k}{k!} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} A^{n-k} B^k = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\alpha\in\{0,1\}^n} A^{n-|\alpha|} B^{|\alpha|} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\alpha\in\{0,1\}^n} S(\alpha_{Sort})
\end{aligned}
$$

With these equations we get

$$
\begin{aligned}
\parallel e^{A+B} - e^A \cdot e^B \parallel &\leq \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\alpha\in\{0,1\}^n} \parallel S(\alpha) - S(\alpha_{Sort}) \parallel \\
&\leq \sum_{n=0}^{\infty} \frac{2^n}{n!} \cdot n^2 \cdot \parallel [A,B] \parallel \\
&= 6e^2 \cdot \parallel [A,B] \parallel
\end{aligned}
$$

3. Show that for $m \times m$ matrices A, B

$$
e^{t\,(A+B)} = \lim_{n\to\infty} \left(e^{t\,\frac{A}{n}} \cdot e^{t\,\frac{B}{n}} \right)^n
$$

Reason: Trotters Formula.

Solution: We will use Trotter's estimation from the previous problem. Let $S := \exp\left(\frac{A}{n} + \frac{B}{n}\right)$ and $T := \exp\left(\frac{A}{n}\right) \cdot \exp\left(\frac{B}{n}\right)$. Then

$$
\begin{aligned}
\|S\| &\leq \exp\left(\frac{\|A\|}{n} + \frac{\|B\|}{n} \right) \\
\|T\| &\leq \exp\left(\frac{\|A\|}{n} \right) \cdot \exp\left(\frac{\|B\|}{n} \right) = \exp\left(\frac{\|A\|}{n} + \frac{\|B\|}{n} \right)
\end{aligned}
$$

$$\begin{aligned}
S^n - T^n &= \sum_{k=0}^{n-1} S^{n-k}T^k - \sum_{k=1}^{n} S^{n-k}T^k \\
&= \sum_{k=0}^{n-1} S^{n-1-k} \cdot S \cdot T^k - \sum_{k=0}^{n-1} S^{n-k-1} \cdot T \cdot T^k \\
&= \sum_{k=0}^{n-1} S^{n-1-k} \cdot (S-T) \cdot T^k \\
\|S^n - T^n\| &\leq \sum_{k=0}^{n-1} \|S\|^{n-1-k} \cdot \|T\|^k \cdot \|S-T\| \\
&\leq n \cdot \exp\left(\frac{\|A\|}{n} + \frac{\|B\|}{n}\right)^{n-1} \cdot \|S-T\| \\
&\leq n \cdot e^{\|A\|+\|B\|} \cdot \|S-T\|
\end{aligned}$$

Now if we choose n large enough such that $\left\|\frac{A}{n}\right\|, \left\|\frac{B}{n}\right\| \leq 1$, we know that

$$\begin{aligned}
\|S-T\| &= \left\|\exp\left(\frac{A}{n} + \frac{B}{n}\right) - \exp\left(\frac{A}{n}\right) \cdot \exp\left(\frac{B}{n}\right)\right\| \\
&\leq 6\, e^2 \cdot \left\| \left[\frac{A}{n}, \frac{B}{n}\right] \right\| = 6\, e^2 \, \frac{1}{n^2} \, \| [A,B] \|
\end{aligned}$$

$$\|S^n - T^n\| \leq n \cdot e^{\|A\|+\|B\|} \cdot \|S-T\| \leq \frac{6e^2}{n} \cdot e^{\|A\|+\|B\|} \cdot \| [A,B] \|$$

With $n \to \infty$ we get $S^n = \exp\left(\frac{A}{n} + \frac{B}{n}\right)^n = \exp(A+B) \longrightarrow T^n$ or

$$\exp(tA + tB) = \lim_{n\to\infty} \left(\exp\left(\frac{tA}{n}\right) \cdot \exp\left(\frac{tB}{n}\right)\right)^n$$

4. (HS-1) Given two integers n, m with $nm \neq 0$. Show that there is a integer expression $1 = sn + tm$ if and only if n and m are coprime, i.e. have no proper common divisor.

   Reason: Bezout's Lemma.

   Solution: Consider all numbers $\{\, x = sn + tm > 0 \,|\, s, t \in \mathbb{Z} \,\}$. If we choose the smallest one, say d, then $\mathrm{gcd(n,m)} \,|\, \mathrm{d}$.

   Division with remainder gives us $n = qd + r$ with $0 \leq r < d$. With the expression for d we get

$$n = q(sn + tm) + r \iff r = (1 - qs)n + (-qt)m$$

We had chosen d to be minimal under these conditions, so $r = 0$ is the only possibility. But then $n = qd+0$ and $d \,|\, n$. With the same argument we get $d \,|\, m$, so as d divides both given integers, $d \,|\, \mathrm{gcd(n, m)}$.

With $\mathrm{gcd(n, m) \,|\, d \,|\, gcd(n, m)}$ we have $d = \mathrm{gcd(n, m)}$ and an expression $d = sn + tm$. Finally, n and m are coprime if and only if $d = 1$.

5. (HS-2) Division of an integer by a prime number p leaves us with the possible remainders $C := \{\, 0, 1, 2, \ldots , p-1 \,\}$. We can define an addition and a multiplication on C if we wrap it around p, i.e. we identify $0 = p = 2p = 3p = \ldots$, $1 = 1 + p = 1 + 2p = 1 + 3p = \ldots$, ... This is called modular arithmetic (modulo p).

   Show that for any given numbers $a, b \in C$ the equations $a + x = b$ and $a \cdot x = b$ have a unique solution. Is this still true if we drop the requirement that p is prime?

   Remark: This problem is about proof techniques, so be as accurate as possible.

   Reason: Proof Techniques.

   Solution: We will make use of the laws of associativity, commutativity and distributivity on C which are all easy to check.

   If we set $x := q \cdot p - a + b$ such that $x \in C$, then

   $$a + x = b \Longrightarrow a + qp - a + b = qp + b \equiv b$$

   solves the equation for addition. This does not require p to be prime. If we have $a+x = b = a+y$ then $x = qp-a+b = qp-a+a+y = qp+y \equiv y$ up to multiples of p which we all identified.

   According to Bezout's Lemma, we can find integers s, t such that $1 = sa + tp$ because all numbers in C are coprime to p, if p is prime. Hence from $ax = b$ we get

   $$s \cdot b = s \cdot (a \cdot x) = (sa)x = (1 - tp)x = x - (tp)x = x - (tx)p \equiv x$$

   Now let $q \in \mathbb{Z}$ be such that $s' = s + qp \in C$. Then

   $$s'b = sb + qpb \equiv sb \equiv x \in C$$

If we have $a \cdot x = b = a \cdot y$ then

$$\begin{aligned} x &= s'b = s'(ay) = (s'a)y = (sa)y + ((qp)a)y \\ &= (1-tp)y + (qay)p = y + (qay - ty)p \\ &\equiv y \end{aligned}$$

up to multiples of p which we all identified.

The existence of a solution for $ax = b$ does not work in general, if p isn't prime. Let $p = 4$, then there is no solution for $2x = 1$.

6 February 2020

1. The function $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with $\varphi(t,r) = (t(r+2), t^2 - r)$ is injective on $U := (0,1) \times (-1,1)$. Show that $\varphi : U \longrightarrow V := \varphi(U)$ is a diffeomorphism.

   Next let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be integrable over V. Write
$$\int_V f \, d\lambda = \int_{\ldots}^{\ldots} \int_{\ldots}^{\ldots} \ldots f(\ldots, \ldots) \, dr \, dt$$
   and calculate the area of V.

   Reason: Analysis.

   Solution: We get
$$J_{(t,r)}\varphi = \begin{bmatrix} r+2 & t \\ 2t & -1 \end{bmatrix}, \; \det(D_{(t,r)}\varphi) = \det(J_{(t,r)}\varphi) = -r - 2 - 2t^2$$
   so for $(t,r) \in U$ the determinate is $\det(D_{(t,r)}\varphi) < -1 < 0$.

   The function $\varphi : U \longrightarrow V$ is obviously continuously differentiable and surjective:
$$\begin{aligned}(t(r+2), t^2 - r) = (a,b) &\Longrightarrow t = \frac{a}{r+2}, \; b = t^2 - r \\ &\Longrightarrow 0 = b(r+2)^2 - a^2 + r(r+2)^2 =: p(r)\end{aligned}$$
   If $a = 0$ we can choose $t = 0$ and $r = -b$ hence we may assume $a \neq 0$. Since $p(r)$ is a polynomial of degree 3 it has at least one real root r_0 which gives us $t_0 = \dfrac{a}{r_0 + 2}$ in case $r_0 \neq -2$. The case $r_0 = 2$ means $p(-2) = 0 = -a^2$ which we dealt with before.

   φ is injective, too: The differential of φ is everywhere in U an isomorphism, hence φ is a diffeomorphism by the inverse function theorem.
$$\begin{aligned}\int_V f \, d\lambda &= \int_U |\det(D\varphi)| \cdot (f \circ \varphi) \, d\lambda \\ &= \int_0^1 \int_{-1}^1 (2t^2 + r + 2) \cdot f(t(r+2), t^2 - r) \, dr \, dt\end{aligned}$$
   with the substitution theorem and Fubini's theorem. Especially we get
$$\operatorname{vol}(V) = \int_V 1 \, d\lambda = \int_0^1 \int_{-1}^1 2t^2 + r + 2 \, dr \, dt = \int_0^1 2(2t^2 + 2) \, dt = \frac{4}{3} + 4 = \frac{16}{3}$$

2. Calculate $\displaystyle\sum_{k=1}^{\infty}\frac{1}{\binom{2k}{k}}$

Reason: Application of the Differential Operator.

Solution: The Taylor series for $\arcsin^2 z$ and $-1<z<1$ is given by

$$\arcsin^2 z = \frac{1}{2}\sum_{k=1}^{\infty}\frac{(2z)^{2k}}{k^2\binom{2k}{k}}$$

and applying the differential operator $z\dfrac{d}{dz}$ twice yields

$$\begin{aligned}\frac{2z\arcsin z}{\sqrt{1-z^2}} &= \frac{1}{2}\sum_{k=1}^{\infty}\frac{2^{2k+1}kz^{2k}}{k^2\binom{2k}{k}}\\ \frac{2z\arcsin z}{\sqrt{1-z^2}}+\frac{2z^2}{1-z^2}+\frac{2z^3\arcsin z}{\sqrt{1-z^2}^3} &= \frac{1}{2}\sum_{k=1}^{\infty}\frac{2^{2k+2}k^2z^{2k}}{k^2\binom{2k}{k}}\\ \frac{2z^2}{1-z^2}+\frac{2z\arcsin z}{\sqrt{1-z^2}^3} &= 2\sum_{k=1}^{\infty}\frac{(2z)^{2k}}{\binom{2k}{k}}\end{aligned}$$

which is divided by 2 for $z=\dfrac{1}{2}$

$$\sum_{k=1}^{\infty}\frac{1}{\binom{2k}{k}} = \frac{1}{3}+\frac{\pi}{12}\cdot\frac{4}{3}\cdot\frac{2}{\sqrt{3}} = \frac{9+2\sqrt{3}\pi}{27}\approx 0.7364$$

3. Let a be an integer and p an odd prime which does not divide a. The left multiplication

$$\lambda_{a,p}\,:\,\mathbb{Z}_p^{\times}\longrightarrow\mathbb{Z}_p^{\times}\,;\,x\longmapsto ax \bmod p$$

is then a permutation on $\{1,\ldots,p-1\}$. Prove

$$\left(\frac{a}{p}\right)=\operatorname{sgn}(\lambda_{a,p})$$

Reason: Lemma of Zolotarev

Solution: Let $k=\operatorname{ord} a$ in $\mathbb{Z}_p^{\times}$. Then $\lambda_{a,p}$ is a product of $\dfrac{p-1}{k}$ many cycles of length k. Thus

$$\operatorname{sgn}(\lambda_{a,p})=(-1)^{(k-1)(p-1)/k}$$

If k is even, then

$$\operatorname{sgn}(\lambda_{a,p}) = (-1)^{(p-1)/k} \equiv (a^{k/2})^{(p-1)/k} \equiv a^{(p-1)/2} \bmod p$$

If k is odd, then $2k \,|\, (p-1)$ and

$$\operatorname{sgn}(\lambda_{a,p}) = 1 \equiv \left(a^k\right)^{(p-1)/2k} \equiv a^{(p-1)/2} \bmod p$$

Hence we have in both cases with Euler's criterion for the Legendre symbol of odd primes

$$\operatorname{sgn}(\lambda_{a,p}) \equiv a^{(p-1)/2} \equiv \left(\frac{a}{p}\right) \bmod p$$

4. Let $\mathcal{H}$ be a real Hilbert space and β a continuous bilinear form, $\mathcal{H}^*$ its dual space of continuous functionals on $\mathcal{H}$, and $\beta(f,f) \geq C\|f\|^2$ with $C > 0$.
   Prove that for any given continuous functional $F \in \mathcal{H}^*$ there is a unique vector $f^\dagger \in \mathcal{H}$ such that

$$F(g) = \beta(f^\dagger, g) \quad \forall\, g \in \mathcal{H}$$

Reason: Lemma of Babuška-Lax-Milgram.

Solution: The statement is a generalization of Riesz' representation theorem (or theorem of Fréchet-Riesz). If we define a continuous function $B(f)(g) := \beta(f,g)$ then Riesz' representation theorem gives us an isometric isomorphism $T : \mathcal{H}^* \longrightarrow \mathcal{H}$ such that for every $B(f) \in \mathcal{H}^*$ there is a unique $T(B(f))$ such that $\|B(f)\| = \|T(B(f))\|$ and

$$B(f)(g) = \langle T(B(f)), g \rangle_{\mathcal{H}} = \beta(f,g) \quad \forall\, g \in \mathcal{H} \quad (*)$$

or generally $f^*(g) = \langle T(f^*), g \rangle_{\mathcal{H}} \;\forall\, g \in \mathcal{H} \quad (*)$

The functionals $B(f)$ are bounded since β is continuous, i.e. $\|B\|$ is a finite real number. We get from our lower bound

$$\begin{aligned} C\|f\|^2 &\leq |\beta(f,f)| = \langle T(B(f)), f \rangle_{\mathcal{H}} \\ &\leq \|T(B(f))\| \cdot \|f\| = \|B(f)\| \cdot \|f\| \leq \|B\| \cdot \|f\|^2 \end{aligned}$$

hence $0 < \dfrac{C}{\|B\|} \leq 1$. We now define the function

$$Q(f) := f - k \cdot (T(B(f)) - T(F))$$

on $\mathcal{H}$ with a real number $k \in \mathbb{R} - \{0\}$. A vector $f^\dagger \in \mathcal{H}$ is a fixed point of Q iff $T(B(f^\dagger)) - T(F) = 0$. In general we have for all $g \in \mathcal{H}$

$$\begin{aligned} T(B(f)) - T(F) \stackrel{(**)}{=} 0 &\Longleftrightarrow F(g) \stackrel{(**)}{=} B(f)(g) = \beta(f,g) \stackrel{(*)}{=} \langle T(B(f)), g\rangle_{\mathcal{H}} \\ &\Longleftrightarrow F(g) \stackrel{(*)}{=} \langle T(F), g\rangle_{\mathcal{H}} \stackrel{(**)}{=} \langle T(B(f)), g\rangle_{\mathcal{H}} \\ &\Longleftrightarrow \langle T(B(f)) - T(F), g\rangle_{\mathcal{H}} \stackrel{(**)}{=} 0 \end{aligned}$$

again by Riesz' representation theorem and the equations above. As $g \in \mathcal{H}$ is arbitrary, we may set $g := T(B(f^\dagger)) - T(F)$ for a fixed point of Q and get $\|T(B(f^\dagger)) - T(F)\|^2 = 0$ hence

$$B(f^\dagger) = \beta(f^\dagger, -) = F$$

which has to be shown. Thus all what's left to show is, that such a unique fixed point $f^\dagger$ of Q exists, which we will prove with Banach's fixed point theorem.

$$\begin{aligned} \|Q(f) - Q(g)\|^2 &= \|f - k(T(B(f)) - T(F)) - g + k(T(B(g)) - T(F))\|^2 \\ &= \langle (f-g) - kT(B(f-g)), (f-g) - kT(B(f-g))\rangle_{\mathcal{H}} \\ &\stackrel{(*)}{=} l, \|f-g\|^2 - 2k\,\langle T(B(f-g)), f-g\rangle_{\mathcal{H}} + k^2\,\|T(B(f-g))\|^2 \\ &\stackrel{(*)}{=} \|f-g\|^2 - 2k\,\beta(f-g, f-g) + k^2\,\|B(f-g)\|^2 \\ &\leq \|f-g\|^2 - 2k\,C\,\|f-g\|^2 + k^2\,\|B\|^2\,\|f-g\|^2 \\ &= \|f-g\|^2\left(1 - 2k\,C + k^2\,\|B\|^2\right) \\ &\stackrel{\text{set } k:=C/\|B\|^2}{=} \|f-g\|^2 \left(1 - \frac{C^2}{\|B\|^2}\right) \end{aligned}$$

We have seen that $\dfrac{C}{\|B\|} \in (0,1]$ hence $q := 1 - \dfrac{C^2}{\|B\|^2} \in [0,1)$ and $\|Q(f) - Q(g)\|^2 = q\,\|f-g\|^2$ and the statement follows from Banach's fixed point theorem.

5. (HS-1)

   (a) Let $A = (-2,0)$, $B = (0,4)$ and $M = (1,3)$. What is $\alpha = \sphericalangle(AMB)$?

   (b) Let $C = (-1, 2+\sqrt{5})$, $D = (-1, 2-\sqrt{5})$ and $M = (1,3)$. What is $\beta = \sphericalangle(CMD)$?

Reason: Theorem of Thales.

Solution: We observe that A and B, as well as C and D are diametrical points on the circle $(x+1)^2+(y-2)^2=5$ and that M fulfills that equation, too. Hence by Thales' theorem, the angles at M have both to be a right angle.

6. (HS-2) Determine (with justification, but without explicit calculation) which of

   (a) 1000^{1001} and 1002^{1000}

   (b) $e^{0.000009}-e^{0.000007}+e^{0.000002}-e^{0.000001}$ and $e^{0.000008}-e^{0.000005}$

   is larger.

   Reason: Numbers.

   Solution:

   (a) We show that $1000^{1001} > 1002^{1000}$:

   $$3003 = 1001 \cdot \log_{10}(1000) > 1000 \cdot \frac{\log 1002}{\log 10} = 1000 \cdot \log_{10}(1002)$$

   This is equivalent to

   $$\begin{aligned} \log_{10}(1002) = \log_{10}\left(1000 \cdot \frac{1000+2}{1000}\right) &= 3 + \log_{10}\left(1+\frac{1}{500}\right) < 3.003 \\ &\iff \\ &\log_{10}\left(1+\frac{1}{500}\right) < \frac{3}{1000} \\ &\iff \\ &1+\frac{1}{500} < \frac{10,000}{9,961} = \frac{1,000}{\underbrace{\frac{1}{1}+\ldots+\frac{1}{1}}_{=994}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}} \\ &\overset{H.M.\leq G.M.}{\leq} \sqrt[1000]{1\cdot\ldots\cdot 1\cdot 2^3\cdot 5^3} = \sqrt[1000]{1000} \approx 1.0069 \end{aligned}$$

   (b) With $a=e^{0.000001}$ we have $x=a^9-a^7+a^2-a$ and $y=a^8-a^5$ such that the difference is

   $$x-y = a^9-a^8-a^7+a^5+a^2-a = a(a-1)^2(a+1)(a^5-a^2-1)$$

By monotony of the exponential function and $1 < e < 4$ we get

$$
\begin{aligned}
1 < a < a^2 < a^5 = e^{0.000005} < 4^{0.000005} < 4^{0.5} = 2 \\
a^5 - a^2 - 1 < 2 - 1 - 1 = 0 \\
0 < a - 1 < a \\
0 < a + 1 < 2a < 4
\end{aligned}
$$

and so $x - y = \underbrace{a}_{>0} \cdot \underbrace{(a-1)^2}_{>0} \cdot \underbrace{(a+1)}_{>0} \cdot \underbrace{(a^5 - a^2 - 1)}_{<0} < 0$ and $e^{0.000009} - e^{0.000007} + e^{0.000002} - e^{0.000001} < e^{0.000008} - e^{0.000005}$

7. (HS-3) Answer the following questions:

   (a) How many knights can you place on a $n \times m$ chessboard such that no two attack each other?

   (b) In how many different ways can eight queens be placed on a chessboard, such that no queen threatens another? Two solutions are not different, if they can be achieved by a rotation or by mirroring of the board.

Reason: Puzzle. Internet Research.

Solution:

(a) Knights change color if they move. So the maximal possible number of knights on an $n \times m$ chessboard is the maximal number of squares of the same color $\lceil \frac{n \cdot m}{2} \rceil$. Exceptions are boards where n or m are small.

Case $m = 1$: In this case we can place n knights on the board.

Case $m = 2$: In this case we can place knights on blocks of four, followed by empty blocks of four. This way we get more knights on the board than we would get, if we placed all on, say black squares. This can be seen for $n = 6$, where we have eight knights by the block structure on the twelve squares, but only six black squares. If we write $n = 4k + r$ with $k \geq 0$, $r \in \{1, 2, 3, 4\}$ then the number N of possible knights is

$$N = \begin{cases} 4k + 2 \text{ if } r = 1 \\ 4k + 4 \text{ if } r > 1 \end{cases}$$

In all cases with $m \geq 3$ we are in the general case, where the squares of one color is optimal, because these are more than those we can get by building blocks. And more than that isn't possible.

(b) There are 92 solutions total, and 12 fundamental solutions, i.e. up to reflection and rotation. For a list see

`https://en.wikipedia.org/wiki/Eight_queens_puzzle#Solutions`

7 January 2020

1. Let $f\,:\,\mathbb{R}\longrightarrow\mathbb{R}$ be a smooth, $2\pi-$periodic function with square integrable derivative, and $\displaystyle\int_0^{2\pi} f(x)\,dx=0$. Prove

$$\int_0^{2\pi}\left[f(x)\right]^2\,dx \leq \int_0^{2\pi}\left[f\,'(x)\right]^2\,dx$$

For which functions does equality hold?

Reason: Wirtinger's Inequality.

Solution: The function fulfills the Dirichlet conditions, so there is a real Fourier series such that

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^\infty\left(a_n\cos(nx)+b_n\sin(nx)\right)$$

See e.g. https://en.wikipedia.org/wiki/Fourier_series. The condition about the vanishing integral implies $a_0=0$. By Parseval's equation (see challenge from November 2019) for $f(x)$ and $f\,'(x)$ we get

$$\begin{aligned}\frac{1}{\pi}\int_0^{2\pi} f^2(x)\,dx &= \sum_{n=1}^\infty\left(a_n^2+b_n^2\right)\\ &\leq \sum_{n=1}^\infty n^2\left(a_n^2+b_n^2\right)\\ &= \frac{1}{\pi}\int_0^{2\pi}(f\,')^2(x)\,dx\end{aligned}$$

and equality holds, if $a_n=b_n=0$ for all $n>1$, i.e

$$f(x)=a_1\cos(x)+b_1\sin(x)\,.$$

2. Let M be the set of all nonnegative, convex functions $f\,:\,[0,1]\longrightarrow\mathbb{R}$ with $f(0)=0$. Prove

$$\int_0^1\prod_{k=1}^n f_k(x)\,dx \geq \frac{2^n}{n+1}\prod_{k=1}^n\int_0^1 f_k(x)\,dx \quad \forall\; f_1,\ldots,f_n\in M$$

Hint: Define and use $\displaystyle\hat{f}(x)=2x\int_0^1 f(x)\,dx$.

Reason: Anderson's Inequality.

Solution:

(a) Every $f \in M$ is monotone increasing.

Assume we have points $0 \leq x_1 < x_2 \leq 1$ such that $f(x_1) > f(x_2)$. Then for some value $\lambda \in [0,1]$ and because $f(0)$

$$f((\lambda)x_2) = f(x_1) \leq \lambda f(x_2) < f(x_1)$$

which is not possible. Geometrically we would get a point $(x_1, f(x_1))$ above the secant $s(x) = \dfrac{f(x_2)}{x_2} \cdot x$ between the origin and $(x_2, f(x_2))$ when it should be below.

(b) M is multiplicatively closed: $f, g \in M \Longrightarrow f \cdot g \in M$.

We have to prove convexity. Choose $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Since f, g are both monotone (increasing), we have

$$(f(x) - f(y))\,(g(x) - g(y)) \geq 0$$

and so

$$f(x)g(x) + f(y)g(y) \geq f(x)g(y) + f(y)g(x)$$

and thus

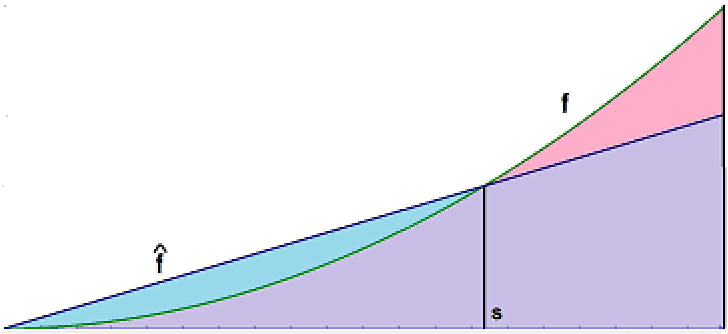
$$\begin{aligned}
(fg)(\alpha x + \beta y) &= f(\alpha x + \beta y) + g(\alpha x + \beta y) \\
&\leq (\alpha f(x) + \beta f(y))\,(\alpha g(x) + \beta g(y)) \text{ by convexity} \\
&= \alpha^2 f(x)g(x) + \alpha\beta\,\big(f(x)g(y) + f(y)g(x) + \beta^2 f(y)g(y)\big) \\
&\leq \big(\alpha^2 + \alpha\beta\big)\,f(x)g(x) + \big(\alpha\beta + \beta^2\big)\,f(y)g(y) \\
&= \alpha f(x)g(x) + \beta f(y)g(y)
\end{aligned}$$

(c) $\hat{f}(x) := 2x \displaystyle\int_0^1 f(t)\,dt \in M \;\forall\, f \in M$

since $\hat{f}(0) = 0$ and linearity in x proves convexity. Note that $\int_0^1 \hat{f}(x)\,dx = \int_0^1 f(x)\,dx$.

(d) $\displaystyle\int_0^1 g(x)f(x)\,dx \geq \int_0^1 g(x)\hat{f}(x)\,dx \;\forall\, f, g \in M$

By the equality of the areas under the function graphs of $\hat{f}(x)$ and $f(x)$ there has to be a point $0 < s < 1$ such that $f(x) \leq \hat{f}(x)$ for all $x \leq s$ and $f(x) \geq \hat{f}(x)$ for all $x \geq s$. Hence



$\int_0^s \hat{f}(x)\,dx + \int_s^1 \hat{f}(x)\,dx = \int_0^s f(x)\,dx + \int_s^1 f(x)\,dx$ and

$$\int_0^s \left(\hat{f}(x) - f(x)\right)\,dx = \int_s^1 \left(f(x) - \hat{f}(x)\right)\,dx$$

As g is monotone increasing we get

$$\begin{aligned}
\int_0^s g(x)\left(\hat{f}(x) - f(x)\right)\,dx &\leq g(s)\int_s^1 \left(\hat{f}(x) - f(x)\right)\,dx \\
&= g(s)\int_s^1 \left(f(x) - \hat{f}(x)\right)\,dx \\
&\leq \int_s^1 g(x)\left(f(x) - \hat{f}(x)\right)\,dx \\
&= -\int_s^1 g(x)\left(\hat{f}(x) - f(x)\right)\,dx
\end{aligned}$$

Thus $\displaystyle\int_0^1 g(x)\left(\hat{f}(x) - f(x)\right)\,dx \leq 0\,.$

Finally we have for any $f_1,\ldots,f_n \in M$

$$\begin{aligned}
\int_0^1 \prod_{k=1}^n f_k(x)\,dx &\geq \int_0^1 \prod_{k=1}^n \hat{f}_k(x)\,dx \\
&= 2^n \prod_{k=1}^n \left(\int_0^1 f_k(x)\,dx\right)\int_0^1 x^n\,dx \\
&= \frac{2^n}{n+1}\prod_{k=1}^n \int_0^1 f_k(x)\,dx
\end{aligned}$$

3. Consider the following differential operators on the space of smooth

functions $C^\infty(\mathbb{R})$

$$A = 2x \cdot \frac{d}{dx}\, , \, B = x^2 \cdot \frac{d}{dx}\, , \, C = -\frac{d}{dx}$$

Determine the eigenvectors and a (multiplicative) structure on lin $\operatorname{span}_{\mathbb{R}}\{\,A, B, C\,\}$.

Reason: Lie theory.

Solution: The eigenvectors are:

$$\begin{aligned} A.f = \lambda f &\Longleftrightarrow f = c\sqrt{x}^{\lambda} \\ B.g = \lambda g &\Longleftrightarrow g = ce^{-\lambda/x} \\ C.h = \lambda h &\Longleftrightarrow h = ce^{-\lambda x} \end{aligned}$$

and the structure is: $\operatorname{span}_{\mathbb{R}}\{\,A, B.C\,\} = \mathfrak{sl}(2,\mathbb{R})$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\, , \, B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\, , \, C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

4. Prove

$$\sum_{k=0}^{n} \frac{1}{\binom{n}{k}} = \frac{n+1}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^k}{k}$$

Reason: Useful Series Identity.

Solution: We define $a_n := \dfrac{2^{n+1}}{n+1} \sum_{k=0}^{n} \dfrac{1}{\binom{n}{k}}$. So

$$\begin{aligned} a_n - a_{n-1} &= \frac{2^{n+1}}{n+1}\left(1 + \sum_{k=0}^{n-1}\left(\frac{(n-k)!k!}{n!} - \frac{(n+1)(n-1-k)!k!}{2n(n-1)!}\right)\right) \\ &= \frac{2^n}{n+1}\left(2 + \sum_{k=0}^{n-1}\frac{(n-k)!k!}{n!}\left(2 - \frac{n+1}{n-k}\right)\right) \\ &= \frac{2^n}{n+1}\left(2 + \sum_{k=0}^{n-1}\frac{(n-k)!k!}{n!}\cdot\frac{n-2k-1}{n-k}\right) \\ &= \frac{2^{n+1}}{n+1} + \frac{2^n}{(n+1)!}\sum_{k=0}^{n-1}\underbrace{((n-1)-k)!\,k!\,((n-1)-2k)}_{=:b_k} \end{aligned}$$

Now we get

$$\begin{aligned} b_{(n-1)-k} &= k!\,((n-1)-k)!\,((n-1)-2((n-1)-k)) \\ &= k!\,((n-1)-k)!\,(-1)\,((n-1)-2k) = -b_k \end{aligned}$$

which implies, that $\sum_{k=0}^{n-1} b_k = 0$ and

$$\begin{aligned}
a_{n-1} &= a_{n-1} - a_{n-2} + a_{n-2} - a_{n-3} \pm \ldots + a_1 - a_0 + a_0 - \underbrace{a_{-1}}_{=0} \\
&= \sum_{k=0}^{n-1}(a_k - a_{k-1}) = \sum_{k=0}^{n-1} \frac{2^{k+1}}{k+1} + \frac{2^k}{(k+1)!}\sum_{k=0}^{n-1} b_k \\
&= \sum_{k=1}^{n} \frac{2^k}{k} \quad \text{and} \quad \sum_{k=0}^{n} \frac{1}{\binom{n}{k}} = \frac{n+1}{2^{n+1}} \cdot a_n = \frac{n+1}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^k}{k}
\end{aligned}$$

5. Calculate $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\binom{2k}{2}}$

   Reason: Easy series.

   Solution:

$$\sum_{k=1}^{\infty} \frac{1}{\binom{2k}{2}} = 2\sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} = 2\sum_{k=1}^{\infty}\left(\frac{1}{2k-1} - \frac{1}{2k}\right) = 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$

   For the last equality, we use the Taylor series for natural logarithm and plug $x = 1$ in it:

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k$$

   such that

$$\sum_{k=1}^{\infty} \frac{1}{\binom{2k}{2}} = 2\log 2$$

6. Prove for $b > 0$

$$\int_{-\infty}^{\infty} f\left(x - \frac{b}{x}\right)\, dx = \int_{-\infty}^{\infty} f(x)\, dx$$

   Reason: Integration Methods. Translation Invariance.

   Solution:

$$\begin{aligned}
I &:= \int_{-\infty}^{\infty} f\left(x - \frac{b}{x}\right) dx = \int_{-\infty}^{0} f\left(x - \frac{b}{x}\right) dx + \int_{0}^{\infty} f\left(x - \frac{b}{x}\right) dx \\
&\overset{u=-b/x}{=} \int_{0}^{\infty} f\left(-\frac{b}{u} + u\right) \frac{b}{u^2}\, du + \int_{-\infty}^{0} f\left(-\frac{b}{u} + u\right) \frac{b}{u^2}\, du \\
&= \int_{-\infty}^{\infty} f\left(u - \frac{b}{u}\right) \frac{b}{u^2}\, du = \int_{-\infty}^{\infty} f\left(x - \frac{b}{x}\right) \frac{b}{x^2}\, dx
\end{aligned}$$

Therefore

$$\begin{aligned}
2I &= \int_{-\infty}^{\infty} f\left(x-\frac{b}{x}\right)\left(1+\frac{b}{x^2}\right)\,dx\\
&= \int_{-\infty}^{0} f\left(x-\frac{b}{x}\right)\left(1+\frac{b}{x^2}\right)\,dx + \int_{0}^{\infty} f\left(x-\frac{b}{x}\right)\left(1+\frac{b}{x^2}\right)\,dx\\
&\stackrel{v=x-(b/x)}{=} \int_{-\infty}^{\infty} f(v)\,dv + \int_{-\infty}^{\infty} f(v)\,dv\\
&= 2\int_{-\infty}^{\infty} f(v)\,dv
\end{aligned}$$

which is what has to be proven.

7. Let $f\,:\,\mathbb{R}\longrightarrow\mathbb{R}$ be a $\pi-$periodic function. Prove that

$$\int_{-\infty}^{\infty} f(x)\,\frac{\sin x}{x}\,dx = \int_0^{\pi} f(x)\,dx \text{ and } \int_{-\infty}^{\infty} f(x)\,\frac{\tan x}{x}\,dx = \int_0^{\pi} f(x)\,dx$$

so the integrals exist. See **(***)** at the end of the proof.

Reason: Integration Methods. Lobachevski's Formulas.

Solution: We omit will the epsilontic around the poles, and start with the series expansion of $\csc(x)$

$$\begin{aligned}
\csc(x) &= \frac{1}{x} + 2x\sum_{k\in\mathbb{N}} \frac{(-1)^k}{x^2-k^2\pi^2}\\
&= \frac{1}{x} + \sum_{k\in\mathbb{N}} (-1)^k \frac{(x+k\pi)+(x-k\pi)}{(x-k\pi)(x+k\pi)}\\
&= \frac{1}{x} + \sum_{k\in\mathbb{N}} \left(\frac{(-1)^k}{x+k\pi} + \frac{(-1)^k}{x-k\pi}\right)\\
&\stackrel{(*)}{=} \sum_{k\in\mathbb{Z}} \frac{(-1)^k}{x+k\pi} = \frac{1}{\sin x}
\end{aligned}$$

and for the second part with the series expansion of $\cot(x)$

$$\begin{aligned}
\cot(x) &= \frac{1}{x} + \sum_{k\in\mathbb{N}} \left(\frac{1}{x+k\pi} + \frac{1}{x-k\pi}\right)\\
&\stackrel{(**)}{=} \sum_{k\in\mathbb{Z}} \frac{1}{x+k\pi} = \frac{1}{\tan x}
\end{aligned}$$

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,\frac{\sin x}{x}\,dx &= \sum_{k\in\mathbb{Z}} \int_{k\pi}^{(k+1)\pi} f(x)\,\frac{\sin x}{x}\,dx \\
&\overset{y=x-k\pi}{=} \sum_{k\in\mathbb{Z}} \int_0^{\pi} f(y)\,\frac{\sin(y+k\pi)}{y+k\pi}\,dy \\
&= \sum_{k\in\mathbb{Z}} \int_0^{\pi} f(y)\,\frac{(-1)^k\sin(y)}{y+k\pi}\,dy \\
&= \int_0^{\pi} f(y) \underbrace{\sum_{k\in\mathbb{Z}} \frac{(-1)^k}{y+k\pi}}_{\overset{(*)}{=}\csc(y)} \sin y\,dy \\
&= \int_0^{\pi} f(y)\,dy = \int_0^{\pi} f(x)\,dx
\end{aligned}
$$

Note that similar can be done with the weight function $\dfrac{\sin^2 x}{x^2}$.
For the second part we get

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,\frac{\tan x}{x}\,dx &= \sum_{k\in\mathbb{Z}} \int_{(k-1/2)\pi}^{(k+1/2)\pi} f(x)\,\frac{\tan x}{x}\,dx \\
&\overset{y=x-k\pi}{=} \sum_{k\in\mathbb{Z}} \int_{-\pi/2}^{\pi/2} f(y)\,\frac{\tan y}{y+k\pi}\,dy \\
&= \int_{-\pi/2}^{\pi/2} f(y) \underbrace{\sum_{k\in\mathbb{Z}} \frac{1}{y+k\pi}}_{\overset{(**)}{=}\cot y} \tan y\,dy \\
&\overset{z=y+\pi/2}{=} \int_0^{\pi} f(z)\,dz = \int_0^{\pi} f(x)\,dx
\end{aligned}
$$

(***) The second equation is wrong in general, e.g. choose $f(x)=1$. We used exchangeability of integral, series and implicitly limits (at the poles). Hence this example shows that conditions such as in Fubini's theorem have to be carefully checked!

8. (a) If $\varphi : G \longrightarrow H$ is a homomorphism of finite groups, then $\operatorname{ord}(\varphi(g)) \mid \operatorname{ord}(g)$ for all elements $g \in G$.
   (b) Determine all group homomorphisms $\varphi : \mathbb{Z}_4 \longrightarrow \operatorname{Sym}(3)$ and $\psi : \operatorname{Sym}(3) \longrightarrow \mathbb{Z}_4$.

Reason: Basic Group Theory.

Solution:

(a) Let $n := \operatorname{ord}(g)$ and $k := \operatorname{ord}(g)$. Since $\varphi(g)^n = \varphi(g^n) = \varphi(e) = e$ so $k \leq n$. We can write $n = s \cdot k + r$ with non negative integers $s, 0 \leq r < k$ by division with remainder. Now

$$\varphi(g)^r = \varphi(g)^{n-sk} = \varphi(g)^n \cdot \left(\varphi(g)^k\right)^{-s} = e \cdot e^{-s} = s$$

By minimality of $k > r$ this is only possible, if $r = 0$, i.e. $n = s \cdot k$ and $k \mid n$.

(b) Every homomorphism $\varphi : \mathbb{Z}_4 \longrightarrow \operatorname{Sym}(3)$ is uniquely determined by its image $\varphi([1])$, since $[1] = 1 + 4\mathbb{Z}$ generates $\mathbb{Z}_4$. As $\varphi([1]) \mid \operatorname{ord}([1]) = 4$ the only possible elements of Sym(3) are $X = \{\, (1), (12), (13), (23) \,\}$. For every element $x \in X$ we define $\varphi_x([0]) = (1)$ and $\varphi_x([1]) = x$. Given that φ_x needs to be a homomorphism, we get

$$\begin{aligned}\varphi_x([2]) &= \varphi_x([1]) + \varphi_x([1]) = x^2 = (1)\\ \varphi_x([3]) &= \varphi_x([1]) + \varphi_x([1]) + \varphi_x([1]) = x^3 = x\end{aligned}$$

so every element of X defines a group homomorphism φ_x.

Let on the other hand be $\psi : \operatorname{Sym}(3) \longrightarrow \mathbb{Z}_4$ be a group homomorphism. Then $\operatorname{ord}(123) = 3$ and $\mathbb{Z}_4$ doesn't have an element of order 3. So $\psi(123) = \psi(132) = [0]$.
All transpositions are conjugates of each other:

$$(12)(23)(12) = (13)\,,\ (13)(12)(13) = (23)\,,\ (23)(13)(23) = (12)$$

and $\mathbb{Z}_4$ is Abelian, hence $\psi(12) = \psi(13) = \psi(23)$. Furthermore $\operatorname{ord}(\psi(\tau)) \mid \operatorname{ord}(\tau) = 2$ so $\psi(\tau) \in \{\, [0], [2] \,\}$ for all transpositions τ. There are therefore two possibilities for ψ, the homomorphism which transforms every element onto [0], or ψ given by $\psi(1) = \psi(123) = \psi(132) = [0]\,, \psi(12) = \psi(13) = \psi(23) = [2]$ which is induced by the signum-function, the sign of a permutation.

9. Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}_{\geq 0}$ be nonnegative real numbers. The elementary symmetric polynomials are

$$\sigma_k(\mathbf{a}) = \sum_{1 \leq j_1 < \ldots < j_k \leq n} a_{j_1} a_{j_2} \ldots a_{j_k}$$

and

$$S_k(\mathbf{a}) = \frac{1}{\binom{n}{k}} \cdot \sigma_k(\mathbf{a})$$

the corresponding elementary symmetric mean value. Prove

(a) $S_1(\mathbf{a}) \geq \sqrt{S_2(\mathbf{a})} \geq \sqrt[3]{S_3(\mathbf{a})} \geq \ldots \geq \sqrt[n]{S_n(\mathbf{a})}$
(b) $S_m(\mathbf{a})^2 \geq S_{m+1}(\mathbf{a}) \cdot S_{m-1}(\mathbf{a})$ for $m = 1, \ldots, n-1$

Reason: MacLaurin's and Newton's Inequality.

Solution:

$$F(x) := (x + a_1) \cdot \ldots \cdot (x + a_n) = \sum_{k=0}^{n} \sigma_k(\mathbf{a}) x^{n-k} = \sum_{k=0}^{n} \binom{n}{k} S_k(\mathbf{a}) x^{n-k}$$

$$F'(x) = \sum_{k=0}^{n} (n-k) \binom{n}{k} S_k(\mathbf{a}) x^{n-k-1} = n \sum_{k=0}^{n-1} \binom{n-1}{k} S_k(\mathbf{a}) x^{n-k-1}$$

Say $a_k < a_{k+1}$. Since $F(-a_k) = 0 = F(-a_{k+1})$ we have by Rolle's theorem a $-b_k \in]-a_{k+1}, -a_k[$ such that $F'(-b_k) = 0$. The same is true in case $a_k = a_{k+1}$. Hence we can write

$$F'(x) = n(x - (-b_1)) \cdot \ldots \cdot (x - (b_n)) = (x + b_1) \cdot \ldots \cdot (x + b_{n-1})$$

with also nonnegative numbers $\mathbf{b} = (b_1, \ldots, b_{n-1})$ and thus

$$F'(x) = n \sum_{k=0}^{n-1} \sigma_k(\mathbf{b}) x^{n-1-k} = n \sum_{k=0}^{n-1} \binom{n-1}{k} S_k(\mathbf{b}) x^{n-1-k}$$

By comparison of the two polynomials $F'(x)$ we get $S_k(\mathbf{a}) = S_k(\mathbf{b})$ for all $k \leq n-1$. Every further derivative has also all nonpositive zeros by induction and

$$\begin{aligned}
F^{(n-m)}(x) &= \frac{n!}{m!} (X + r_1^{(m_1)}) \cdot \ldots \cdot (X + r_m^{(m_m)}) \\
&= \frac{n!}{m!} \sum_{k=0}^{m} \binom{m}{k} S_k(\mathbf{r}^{(m)}) x^{m-k} \\
&= \left(\frac{d}{dx} \right)^{n-m} \sum_{k=0}^{n} \binom{n}{k} S_k(\mathbf{a}) x^{n-k} \\
&= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} S_k(\mathbf{a}) \frac{(n-k)!}{(m-k)!} x^{m-k} \\
&= \frac{n!}{m!} \sum_{k=0}^{m} \binom{m}{k} S_k(\mathbf{a}) x^{m-k}
\end{aligned}$$

which again shows by comparison that $S_k(\mathbf{a}) = S_k(\mathbf{r}^{(m)})$ for all $k \leq m$. Especially we get

$$S_m(\mathbf{r}^{(m)}) = S_m(\mathbf{a}) = r_1^{(m)} \cdot \ldots \cdot r_m^{(m)}$$

and ($\widehat{a}$ here means "without a")

$$\begin{aligned}\widehat{r_1^{(m)}}r_2^{(m)}\cdots r_m^{(m)} + r_1^{(m)}\widehat{r_2^{(m)}}\cdots r_m^{(m)} + \ldots + r_1^{(m)}r_2^{(m)}\cdots \widehat{r_m^{(m)}}\\ &= r_1^{(m)}r_2^{(m)}\cdots r_m^{(m)}\cdot\left(\frac{1}{r_1^{(m)}}+\ldots+\frac{1}{r_m^{(m)}}\right)\\ &= m\cdot S_{m-1}(\mathbf{r}^{(m)})\\ &= m\cdot S_{m-1}(\mathbf{a})\end{aligned}$$

$$r_1^{(m)}\cdots r_m^{(m)}\sum_{1\leq i<j\leq m}\frac{1}{r_i^{(m)}r_j^{(m)}} = \frac{m(m-1)}{2}\cdot S_{m-2}(\mathbf{a})$$

(a) McLaurin's inequality. Since the arithmetic mean is greater than the geometric mean, we have

$$\begin{aligned}\frac{\widehat{r_1^{(m)}}r_2^{(m)}\cdots r_m^{(m)} + r_1^{(m)}\widehat{r_2^{(m)}}\cdots r_m^{(m)} + \ldots + r_1^{(m)}r_2^{(m)}\cdots \widehat{r_m^{(m)}}}{m}\\ &\geq \sqrt[m]{\left(r_1^{(m)}\cdot\ldots\cdot r_m^{(m)}\right)^{m-1}}\end{aligned}$$

hence for all $m \leq n$

$$S_{m-1}(\mathbf{a}) \geq \sqrt[m]{S_m(\mathbf{a})^{m-1}} \Longrightarrow \sqrt[m-1]{S_{m-1}(\mathbf{a})} \geq \sqrt[m]{S_m(\mathbf{a})}$$

(b) Newton's inequality.

$$S_{m-1}(\mathbf{a})^2 \geq S_m(\mathbf{a}) \cdot S_{m-2}(\mathbf{a})$$
$$\Longleftrightarrow$$
$$\left(r_1^{(m)} \cdots r_m^{(m)}\right)^2 \cdot \frac{1}{m^2} \cdot \left(\sum_{k=1}^{m} \frac{1}{r_k^{(m)}}\right)^2 \geq$$
$$\left(r_1^{(m)} \cdots r_m^{(m)}\right)^2 \cdot \frac{2}{m(m-1)} \cdot \sum_{1\leq i<j\leq m} \frac{1}{r_i^{(m)} r_j^{(m)}}$$
$$\Longleftrightarrow$$
$$(m-1)\left(\sum_{k=1}^{m} \frac{1}{r_k^{(m)}}\right)^2 \geq 2m \sum_{1\leq i<j\leq m} \frac{1}{r_i^{(m)} r_j^{(m)}}$$
$$\Longleftrightarrow$$
$$(m-1)\left(\sum_{k=1}^{m} \frac{1}{r_k^{(m)}}\right)^2 \geq m \cdot \left(\sum_{k=1}^{m} \frac{1}{r_k^{(m)}}\right)^2 - m \cdot \sum_{k=1}^{m} \frac{1}{(r_k^{(m)})^2}$$
$$\Longleftrightarrow$$
$$m \cdot \sum_{k=1}^{m} \frac{1}{(r_k^{(m)})^2} \geq \left(\sum_{k=1}^{m} \frac{1}{r_k^{(m)}}\right)^2$$
$$\Longleftrightarrow$$
$$\frac{1}{m}\sum_{k=1}^{m} \frac{1}{(r_k^{(m)})^2} \geq \left(\frac{1}{m}\sum_{k=1}^{m} \frac{1}{r_k^{(m)}}\right)^2$$
$$\Longleftrightarrow$$
$$\sqrt{\frac{1}{m}\sum_{k=1}^{m} \frac{1}{(r_k^{(m)})^2}} \geq \frac{1}{m}\sum_{k=1}^{m} \frac{1}{r_k^{(m)}}$$
$$\Longleftrightarrow$$
$$M_{1/m}^1 \leq M_{1/m}^2$$

since the arithmetic mean is less or equal than the quadratic mean by the generalized or weighted Hölder mean inequality (see problem 1 above).

10. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers, not all zero.

Prove
$$\left(\sum_{n\in\mathbb{N}} a_n\right)^4 < \pi^2 \sum_{n\in\mathbb{N}} a_n^2 \cdot \sum_{n\in\mathbb{N}} n^2 a_n^2$$
Reason: Carlson's Inequality.

Solution: We quote two different proofs by Hardy.

(a) Consider the Fourier series $f(x)=\sum_{k=1}^\infty a_k \cos kx$ and its derivative $f'(x)=-\sum_{k=1}^\infty a_k \sin kx$. By Parseval's equation (see no. 1. in the November challenge) we get

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \frac{2}{\pi}\int_0^{\pi} |f(x)|^2\,dx = \sum_{k=1}^\infty a_k^2 =: S$$

and

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f'(x)|^2\,dx = \frac{2}{\pi}\int_0^{\pi} |f'(x)|^2\,dx = \sum_{k=1}^\infty k^2 a_k^2 =: T$$

Now $\int_0^\pi f(x)\,dx = \sum_{k=1}^\infty a_k \left[\dfrac{\sin kx}{k}\right]_0^\pi = 0$ so there is a $0<\xi<\pi$ with $f(\xi)=0$ and

$$\begin{aligned}\left(\sum_{k=1}^\infty a_k\right)^2 &= f^2(0)-f^2(\xi) = 2\int_\xi^0 f(x)f'(x)\,dx\\ &\stackrel{\text{Cauchy-Schwarz}}{<} 2\sqrt{\int_0^\pi f^2(x)\,dx}\cdot\sqrt{\int_0^\pi f'^2(x)\,dx}\\ &= 2\cdot\sqrt{\frac{\pi}{2}S}\cdot\sqrt{\frac{\pi}{2}T} = \pi\sqrt{ST}\end{aligned}$$

and squaring completes the proof.

(b) Let $\alpha,\beta>0$ and $S=\sum_{k=1}^\infty a_k^2$, $T=\sum_{k=1}^\infty k^2a_k^2$. By the Cauchy-

Schwarz inequality we get

$$\begin{aligned}\left(\sum_{k=1}^{\infty} a_k\right)^2 &= \left(\sum_{k=1}^{\infty} a_k \cdot \sqrt{\alpha+\beta k^2} \cdot \frac{1}{\sqrt{\alpha+\beta k^2}}\right)^2 \\ &\leq \sum_{k=1}^{\infty} a_k^2 \cdot (\alpha+\beta k^2) \cdot \sum_{k=1}^{\infty} \frac{1}{\alpha+\beta k^2} \\ &= (\alpha S+\beta T) \sum_{k=1}^{\infty} \frac{1}{\alpha+\beta k^2} < (\alpha S+\beta T) \int_0^{\infty} \frac{dx}{\alpha+\beta x^2} \\ &= (\alpha S+\beta T) \cdot \frac{\pi}{2} \cdot \frac{1}{\sqrt{\alpha\beta}} = \frac{\pi}{2}\left(\sqrt{\frac{\alpha}{\beta}} \cdot S + \sqrt{\frac{\beta}{\alpha}} \cdot T\right)\end{aligned}$$

With $\alpha := T$, $\beta := S$ we have

$$\left(\sum_{k=1}^{\infty} a_k\right)^2 < \frac{\pi}{2}(\sqrt{ST}+\sqrt{ST}) = \pi\sqrt{ST}$$

and squaring completes the proof.

11. (HS-1) On how many ways can 2020 be written as a sum of consecutive natural numbers (greater than zero)?

Reason: Prime numbers.

Solution:

$$\begin{aligned}2020 &= 2^3 \cdot 5 \cdot 101 \\ &= n + (n+1) + \ldots + (n+k) \\ &= n(k+1) + \frac{k}{2}(k+1) \\ 4040 &= (k+1)(2n+k)\end{aligned}$$

If $101 \,|\, (k+1)$ then $k \geq 100$ and $2n+k \geq 102$ but 4040 doesn't have two such great divisors. Hence $(k+1, 2n+k)$ can only be one of the pairs

$$\{\,(2,2020),(4,1010),(8,505),(5,808),(10,404),(20,202),(40,101)\,\}$$

But if both components were even, then n wouldn't be a natural number, so we are left with 3 possibilities:

$$\begin{aligned}2020 &= 249 + 250 + \ldots + 256 \\ 2020 &= 402 + 403 + \ldots + 406 \\ 2020 &= 31 + 32 + \ldots + 70\end{aligned}$$

12. (HS-2) A binary operation on a set S is a mapping, which maps a pair from $S \times S$ to S. E.g. addition is a binary operation on integers. Find two different binary operations for $S = \{A, B, C, D\}$ which have a neutral element, $A \circ X = X$, and can be inverted: for all $X \in S$ there is a $Y \in S$ with $X \circ Y = A$, and are associative: $X \circ (Y \circ Z) = (X \circ Y) \circ Z$.

    Reason: Group Theory.

    Solution: There are two groups of order 4: $\mathbb{Z}_4$ and $V_4 = \mathbb{Z}_2^2$. Their Abelian multiplications are given by

$\circ$	A	B	C	D
A	A	B	C	D
B	B	C	D	A
C	C	D	A	B
D	D	A	B	C

$\circ$	A	B	C	D
A	A	B	C	D
B	B	A	D	C
C	C	D	A	B
D	D	C	B	A

13. (HS-3) Find all six digit numbers with the following property: If we move the first (highest) digit at the end, we will get three times the original number.

    Reason: Puzzle.

    Solution: Set $n = 100,000\,a + b$. Then $3n = 10\,b + a$ and so

$$300,000\,a + 3\,b = 10\,b + a \Longleftrightarrow 299,999\,a = 7\,b \Longleftrightarrow b = 42,857\,a$$

    Therefore $n = 142,857\,a$ and $3n = 428,571\,a$ which means $a \in \{1, 2\}$ and the only six digit numbers are $142,857$ and $285,714$.

14. (HS-4) The Pell sequence named after the English mathematician John Pell is defined by

$$P(n) = \begin{cases} 0\,,\; n = 0 \\ 1\,,\; n = 1 \\ P(n-2) + 2P(n-1)\,,\; n > 1 \end{cases}$$

    Calculate the limit $\delta_s := \lim_{n\to\infty} \dfrac{P(n)}{P(n-1)}$.

    Reason: Silver Ratio.

Solution:

$$\begin{aligned}\delta_s &= \lim_{n\to\infty} \frac{P(n-2)+2P(n-1)}{P(n-1)} \\ &= \lim_{n\to\infty} \frac{P(n-2)}{P(n-1)} + 2 \\ &= 2 + \delta_s^{-1}\end{aligned}$$

and thus $0 = \delta_s^2 - 2\delta_s - 1$ which has the solutions $1 \pm \sqrt{2}$. Since $P(n) > P(n-1)$ only $\delta_s = 1 + \sqrt{2}$ is a possible solution.

Two quantities are in the **silver ratio** if the ratio of the sum of the smaller and twice the larger of those quantities, to the larger quantity, is the same as the ratio of the larger one to the smaller one. This leads to

$$\frac{2a+b}{a} = \frac{a}{b} = \delta_s = [2; 2, 2, 2, \ldots]$$

where the last representation is the continued fraction of the silver ratio.

15. (HS-5) Consider the graph of $f(x) = 1/x$ with $x \geq 1$ and let it rotate around the $x-$axis. This solid of revolution looks like an infinitely long trumpet. Calculate its volume V and its surface A.
If we fill it with paint, pour it out again, then we have painted it from inside. Explain this apparent contradiction to the surface you computed.

Reason: Gabriel's Horn (Wikipedia).

Solution:

$$\begin{aligned}V &= \pi \int_1^\infty \frac{1}{x^2}\,dx = \pi \\ A &= 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx > 2\pi \int_1^\infty \frac{1}{x}\,dx = \infty\end{aligned}$$

We filled in paint, poured it out again, and thus have painted an infinitely large inner surface with a finite amount of paint!
The solution is, that in reality paint has a certain thickness, so we didn't need to "fill" the entire horn, only a finite part of it.