Mathematical Challenges

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1. Let $\mathfrak{g}$ be a Lie algebra. Define
$$\mathfrak{A}(\mathfrak{g}) = \{\, \alpha \,:\, \mathfrak{g} \longrightarrow \mathfrak{g} \,|\, \forall\, X, Y \in \mathfrak{g} \,:\, 0 = [\alpha(X), Y] + [X, \alpha(Y)] \,\}$$
Show that $\mathfrak{A}(\mathfrak{g})$ is a Lie algebra and $X.\alpha(Y) = [X, \alpha(Y)] - \alpha([X, Y])$ defines a representation of $\mathfrak{g}$ on $\mathfrak{A}(g)$.

   Reason: Representation theory.

   Solution: Linearity is obvious for both cases, that $\mathfrak{A}(\mathfrak{g})$ is a vector space as well as that $X.\alpha$ is a linear transformation.

   (a) Let $\alpha, \beta \in \mathfrak{A}(\mathfrak{g})$.
$$\begin{aligned}[][\alpha, \beta]X, Y] &= [(\alpha\beta - \beta\alpha)X, Y] \\ &= -[\beta X, \alpha Y] + [\alpha X, \beta Y] \\ &= [X, \beta\alpha Y] - [X, \alpha\beta Y] \\ &= -[X, [\alpha, \beta]Y]\end{aligned}$$

   (b) A representation of $\mathfrak{g}$ on $\mathfrak{A}(\mathfrak{g})$ is a Lie algebra homomorphism $\varphi : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{A}(\mathfrak{g}))$ and in our case $\varphi(X)(\alpha) := [\operatorname{ad} X, \alpha]$. Therefore we have to show that $\varphi(\alpha) \in \mathfrak{A}(\mathfrak{g})$ and $\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$.
$$\begin{aligned}[](\varphi(X)(\alpha))(Y), Z] &= [[X, \alpha(Y)], Z] - [\alpha([X, Y]), Z] \\ &= -[[\alpha X, Y], Z] + [[X, Y], \alpha Z] \\ &= [[Y, Z], \alpha X] + [[Z, \alpha X], Y] - [[Y, \alpha Z], X] - [[\alpha Z, X], Y] \\ &= [[Y, Z], \alpha X] - [[\alpha Z, X], Y] - [[Y, \alpha Z], X] - [Y, [X, \alpha Z]] \\ &= -[[Z.\alpha X], Y] - [[\alpha X, Y], Z] - [[\alpha Z, X], Y] - [[Y, \alpha Z], X] - [Y, [X, \alpha Z]] \\ &= [[X, \alpha Y], Z] + [[\alpha Y, Z], X] - [Y, [X, \alpha Z]] \\ &= -[[Z, X], \alpha Y] - [Y, [X, \alpha Z]] \\ &= [Y, \alpha([X, Z])] - [Y, [X, \alpha Z]] \\ &= -[Y, (\varphi(X)(\alpha))(Z)]\end{aligned}$$
$$\begin{aligned}\varphi([X, Y])(\alpha) &= [\operatorname{ad}([X, Y]), \alpha] \\ &= [[\operatorname{ad}(X), \operatorname{ad}(Y)], \alpha] \\ &= -[[\operatorname{ad}(Y), \alpha], \operatorname{ad}(X)] - [[\alpha, \operatorname{ad}(X)], \operatorname{ad}(Y)] \\ &= [\operatorname{ad}(X), [\operatorname{ad}(Y), \alpha]] - [\operatorname{ad}(Y), [\operatorname{ad}(X), \alpha]] \\ &= [\operatorname{ad}(X), \varphi(Y)(\alpha)] - [\operatorname{ad}(Y), \varphi(X)(\alpha)] \\ &= \varphi(X)(\varphi(Y)(\alpha) - \varphi(Y)(\varphi(X)(\alpha) \\ &= ([\varphi(X), \varphi(Y)])(\alpha)\end{aligned}$$

2. Let R be a commutative ring with 1 and I an ideal. Show that R/I is an integral domain if and only if I is a prime ideal, and that R/I is a field if and only if I is a maximal ideal.

   Reason: Standard result in commutative algebra.

   Solution: R/I is an integral domain, if $\bar{a}\bar{b} = \bar{0}$ implies $\bar{a} = \bar{0}$ or $\bar{b} = \bar{0}$, i.e. $ab \in I$ implies $a \in I$ or $b \in I$ which is the definition of a prime ideal I.

   Now let $I \triangleleft R$ be a maximal ideal and $r \notin I$. Then $I + rR = R$ so we have elements $a \in I$, $s \in R$ with $1 = a + rs$ and thus $\bar{1} = \bar{0} + \bar{r}\bar{s} = \bar{r}\bar{s}$, i.e. all elements of R/I different from zero are invertible.

   If on the other hand R/I is a field, and $I \triangleleft R$ is not maximal, then there is an ideal $I \subsetneq M \triangleleft R$ and an element $m \in M - I$ such that $\bar{m} \neq \bar{0}$ is invertible. This means there is a $r \in R$ such that $\bar{r}\bar{m} = \bar{1}$ or $rm - 1 \in I \subset M$. But this means $1 \in M$ and so $R = M$, which makes I a maximal ideal.

3. Solve $x^2y'' + xy' - y = x^3$ for positive x.

   Reason: ODE.

   Solution: Let $x = e^u$ so that $u = \log(x)$. Then

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} = \frac{dy}{du}\,\frac{1}{x} \\
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{du}\,\frac{1}{x}\right) \\
&= \frac{dy}{dx}\frac{dy}{du}\,\frac{1}{x} - \frac{dy}{du}\,\frac{1}{x^2} \\
&= \frac{dy}{du}\,\frac{1}{x}\frac{dy}{du}\,\frac{1}{x} - \frac{dy}{du}\,\frac{1}{x^2} \\
&= \left(\frac{d^2y}{du^2} - \frac{dy}{du}\right)\frac{1}{x^2}
\end{aligned}$$

   Therefore,

$$x^2y'' + xy' - y = \left(\frac{d^2y}{du^2} - \frac{dy}{du}\right) + \frac{dy}{du} - y = \frac{d^2y}{du^2} - y = e^{3u}$$

   Considering the homogeneous case, we have $\lambda^2 - 1 = 0$ or $\lambda = \pm 1$. Therefore, $y_1 = e^u$ and $y_2 = e^{-u}$. Using undetermined coefficients, let

$y_p = Ae^{3u}$. Then $y' = 3Ae^{3u}$ and $y'' = 9Ae^{3u}$. Substitution gives

$$\underbrace{9Ae^{3u} - Ae^{3u}}_{8Ae^{3u}} = e^{3u}$$

so $A = \frac{1}{8}$ and the general solution is

$$\begin{aligned} y &= C_1e^u + C_2e^{-u} + \frac{1}{8}e^{3u} \\ &= C_1x + \frac{C_2}{x} + \frac{x^3}{8} \end{aligned}$$

4. Show that the Schwarzian Derivative

$$(Sf)(z) := \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

vanishes if and only if $f(z) = \dfrac{az+b}{cz+d}$ is a Möbius transformation.

Reason: Funny derivative.

Solution: Let $f(z) = \dfrac{az+b}{cz+d} = \dfrac{U(z)}{V(z)} = \dfrac{U}{V}$. Then

$$f'(z) = \frac{aV - cU}{V^2}$$

$$f''(z) = -2c\,\frac{aV - cU}{V^3}$$

and thus

$$(Sf)(z) = \left(\frac{-2c}{V}\right)' - \frac{2c^2}{V^2} = \frac{2c}{V^2}\,c - \frac{2c^2}{V^2} = 0$$

Let us now assume that $(Sf)(z) = 0$ and set $U(z) := \dfrac{f''(z)}{f'(z)}$, i.e. $2U' = U^2$ which means $U(z) = -\dfrac{2}{c_1 + z}$, i.e. $f''(z) = -\dfrac{2}{c_1+z}\,f'(z)$. The solution to this differential equation is

$$\begin{aligned} f(z) &= \frac{c_2}{c_1 + z} + c_3 \\ &= \frac{(c_2 + c_1c_3) + c_3z}{c_1 + z} \\ &= \frac{(c_2c_4 + c_1c_3c_4) + c_3c_4z}{c_1c_4 + c_4z} \\ &= \frac{az+b}{cz+d} \end{aligned}$$

5. Let $x(t)$ be the height at time t, measured positively on the downward direction. If we consider only gravity, then $\ddot{x}(t) = \dfrac{d^2x}{dt^2} = a$ is a constant, denoted g, the acceleration due to gravity. Note that $F = ma = mg$. Air resistance encountered depends on the shape of the object and other things, but under most circumstances, the most significant effect is a force opposing the motion which is proportional to a power of the velocity $v(t) = \dot{x}(t)$. So

$$\ddot{x}(t) \cdot m = m \cdot g - k\dot{x}(t)^n$$

which is a second order differential equation, but there is no x term. So it is first order in $\dot{x}$. Therefore,

$$\frac{dv}{dt} = g - \frac{k}{m}v^n$$

This is not easy to solve, so we will make the simplifying approximation that $n = 1$ (if v is small, there is not much difference between v and v^n). Therefore, we have to solve

$$\frac{dv}{dt} + \frac{k}{m}v = g$$

Reason: ODE.

Solution: The integration factor is

$$I = e^{\int \frac{k}{m}\,dt} = e^{kt/m}$$

and thus

$$\begin{aligned}\left(\frac{dv}{dt} + \frac{k}{m}\,v\right) e^{kt/m} &= ge^{kt/m} \\ e^{kt/m}v &= \frac{gm}{k}\,e^{kt/m} + C \\ v &= \frac{mg}{k} + Ce^{-kt/m}\end{aligned}$$

with an arbitrary constant C. By $v(0) = v_0$ we get $C = v_0 - \dfrac{mg}{k}$

$$\begin{aligned}\int_{x_0}^{x} dx &= \int_0^t v\,dt \\ &= \int_0^t \left(\frac{mg}{k} + \left(v_0 - \frac{mg}{k}\right) e^{-kt/m}\right) \\ &= \frac{mg}{k}\,t - \frac{m}{k}\left(v_0 - \frac{mg}{k}\right)\left(e^{-kt/m} - 1\right)\end{aligned}$$

that is

$$x(t) = x_0 + \frac{mg}{k}\,t + \frac{m}{k}\left(v_0 - \frac{mg}{k}\right)\left(1 - e^{-kt/m}\right)$$

6. Consider a land populated by foxes and rabbits, where the foxes prey upon the rabbits. Let $x(t)$ and $y(t)$ be the number of rabbits and foxes, respectively, at time t. In the absence of predators, at any time, the number of rabbits would grow at a rate proportional to the number of rabbits at that time. However, the presence of predators also causes the number of rabbits to decline in proportion to the number of encounters between a fox and a rabbit, which is proportional to the product $x(t)y(t)$. Therefore, $dx/dt = ax - bxy$ for some positive constants a and b. For the foxes, the presence of other foxes represents competition for food, so the number declines proportionally to the number of foxes but grows proportionally to the number of encounters. Therefore $dy/dt = -cy + dxy$ for some positive constants c and d. The system

$$\dot{x}(t) = \frac{dx}{dt} = ax(t) - bx(t)y(t)\ ,\ \dot{y}(t) = \frac{dy}{dt} = -cy(t) + dx(t)y(t)$$

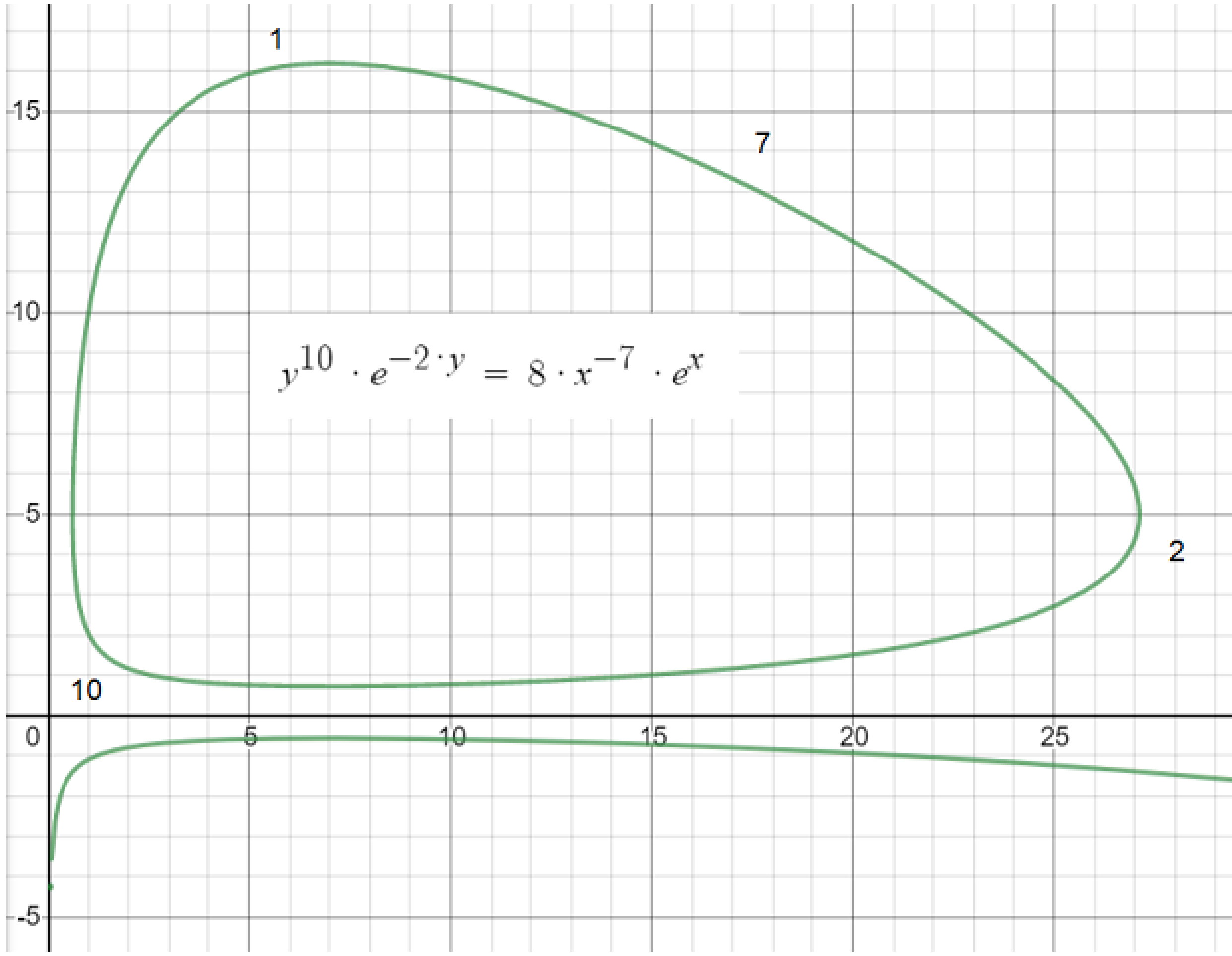
is our mathematical model. Eliminate the time parameter and find the relation between the population of foxes and the number of rabbits for parameters $a = 10\,,\ b = 2\,,\ c = 7\,,\ d = 1\,.$

Reason: Predator Prey Model.

Solution:

$$\begin{aligned} \frac{dy}{dx} &= \frac{-7y + xy}{10x - 2xy} \\ \Longrightarrow \frac{(10-2y)\,dy}{y} &= \frac{(-7+x)\,dx}{x} \\ \Longrightarrow 10\log y - 2y &= -7\log x + x + C \\ \Longrightarrow y^{10}e^{-2y} &= k\,x^{-7}e^{x} \end{aligned}$$

with a constant positive parameter $k = e^C$.



7. Five vessels contain 100 balls each. Some vessels contain only balls of $10\,g$ mass, while the other vessels contain only balls of $11\,g$ mass. How can we determine with a single weighing which results in a mass, which vessels contain balls of $10\,g$ and which contain balls of $11\,g$? (It is allowed to remove balls from the vessels.)

   Reason: Riddle about the binary representation of numbers.

   Solution: We remove 2^k balls from vessel k and weigh those 31 balls. Let the result be $a\,g$. Thus we have an equation

$$x_1 + 2x_2 + 4x_3 + 8x_4 + 16x_5 = a - 31 \cdot 10\,g \quad (x_k \in \{\,0, 1\,\})$$

   which is a unique binary representation and $x_k = 0$ are the vessels with $10\,g$ balls, $x_k = 1$ are the vessels with $11\,g$ balls.

8. Let $f \in L^1(\mathbb{R}^3)$ be rotation symmetric, i.e. $f(Rx) = f(x)$ for all $R \in \mathrm{SO}(3)$. Show that the Fourier transform $\mathcal{F}f$ is rotation symmetric, too, and calculate $\mathcal{F}f$ of $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ defined by

$$f(x) = \frac{1}{|x|}\chi_{B_1(0)}(x)$$

with the Euclidean norm $|\,.\,|$, the unit ball $B_1(0)$ around the origin, and the characteristic function χ.

Reason: Fourier transformation.

Solution: Since $\det R^\tau = 1$ we get with $y = R^\tau x$

$$\begin{aligned}
\mathcal{F}f(R\xi) &= (2\pi)^{-3/2}\int_{\mathbb{R}^3} f(x)\exp(-ix\cdot R\xi)\,d\lambda_3(x)\\
&= (2\pi)^{-3/2}\int_{\mathbb{R}^3} f(R^\tau x)\exp(-iR^\tau x\cdot \xi)\,d\lambda_3(x)\\
&= (2\pi)^{-3/2}\int_{\mathbb{R}^3} f(y)\exp(-iy\cdot \xi)\,d\lambda_3(x)\\
&= \mathcal{F}f(\xi)
\end{aligned}$$

We use spherical coordinates for the second part, i.e. the function

$$\begin{aligned}
\Phi\,:\,(0,1)\times(-\pi/2,\pi/2)\times(0,2\pi) &\longrightarrow B_1(0)\\
(r,\varphi,\theta) &\longmapsto (r\cos\varphi\cos\theta, r\cos\varphi\sin\theta, r\sin\varphi)
\end{aligned}$$

where $\det D\Phi = r^2\cos\varphi$. Then we get with $\xi = te_3$, $t \in \mathbb{R}$ and Fubini the Fourier transform

$$\begin{aligned}
\mathcal{F}f(te_3) &= (2\pi)^{-3/2} \int_{B_1(0)} |x|^{-1} \exp(-itx_3)\, d\lambda_3(x) \\
&= (2\pi)^{-3/2} \int_0^1 \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} r^2 \cos\varphi \cdot r^{-1} \exp(-itr\sin\varphi)\, d\lambda_\theta\, d\lambda_\varphi\, d\lambda_r \\
&= (2\pi)^{-1/2} \int_0^1 \int_{-\pi/2}^{\pi/2} r\cos\varphi\, \exp(-itr\sin\varphi)\, d\lambda_\varphi\, d\lambda_r \\
&\overset{u(\varphi)=r\sin\varphi}{=} (2\pi)^{-1/2} \int_0^1 \int_{-r}^{r} \exp(-itu)\, d\lambda_u\, d\lambda_r \\
&= (2\pi)^{-1/2} \int_0^1 \int_{-r}^{r} \cos(tu)\, d\lambda_u\, d\lambda_r \\
&= (2\pi)^{-1/2} \int_0^1 2 \cdot \frac{\sin(tr)}{t}\, d\lambda_r \\
&= \sqrt{\frac{2}{\pi}}\, \frac{1-\cos t}{t^2}
\end{aligned}$$

Since f is rotation symmetric we know

$$\mathcal{F}f(\xi) = \mathcal{F}f(|\xi|e_3) = \sqrt{\frac{2}{\pi}}\, \frac{1-\cos|\xi|}{|\xi|^2}$$

9. Solve $(3x^2y^2 + x^2)\, dx + (2x^3y + y^2)\, dy = 0$.

Reason: Exact forms.

Solution:

$$\omega := \underbrace{(3x^2y^2 + x^2)}_{F}\, dx + \underbrace{(2x^3y + y^2)}_{G}\, dy$$

and observe that $\dfrac{\partial G}{\partial x} = 6x^2y = \dfrac{\partial F}{\partial y}$ so there exist a g such that $\omega = dg$ and

$$\frac{\partial g}{\partial x} = 3x^2y^2 + x^2\,,\ \frac{\partial g}{\partial y} = 2x^3y + y^2$$

Integrating the first yields $g = x^3y^2 + \dfrac{1}{3}x^3 + h(y)$ which differentiated with respect to y gives

$$\frac{\partial g}{\partial y} = 2x^3y + y^2 = 2x^3y + \frac{dh}{dy}$$

which means $h(y) = \dfrac{y^3}{3} + C$ for some arbitrary constant C. Hence $g = x^3y^2 + \dfrac{x^3}{3} + \dfrac{y^3}{3} + C$. With $\omega = dg = 0$ we have $x^3y^2 + \dfrac{x^3}{3} + \dfrac{y^3}{3} + C = C'$ for some arbitrary constant C'. With $D = C' - C$ which is still an arbitrary constant, the solution is

$$x^3y^2 + \frac{x^3}{3} + \frac{y^3}{3} = D$$

10. Calculate $\lim_{x\to 0} \dfrac{\cos^2 x - 1}{\sinh^2 x}$ and $\lim_{x\to 0} \dfrac{e^x + e^{-x} - 2 - x^2}{(\cos x - 1)^2}$

    Reason: L'Hôpital.

    Solution: For $f(x) := \cos^2 x - 1$ and $g(x) := \sinh^2 x$ we have $f'(x) = -2\cos x \sin x = -\sin(2x)$ and $g'(x) = 2\sinh x \cosh x = \sinh(2x)$. So $\lim_{x\to 0} f'(x) = f'(0) = 0$ and $\lim_{x\to 0} g'(x) = g'(0) = 0$ and we cannot apply the rule of L'Hôpital. However, the functions $F(x) := \sin(2x)$ and $G(x) := \sinh(2x)$ do fulfill the conditions in a neighborhood of $x = 0$ such that

$$\lim_{x\to 0} \frac{F'(x)}{G'(x)} = \lim_{x\to 0} \frac{-2\cos(2x)}{2\cosh(2x)} = -1$$

    which means that $\dfrac{F(x)}{G(x)}$ has a limit for $x \to 0$ by L'Hôpital and we have again with L'Hôpital

$$-1 = \lim_{x\to 0} \frac{F'(x)}{G'(x)} = \lim_{x\to 0} \frac{F(x)}{G(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{\cos^2 x - 1}{\sinh^2 x}$$

    By application of L'Hôpital four times we get

$$\begin{aligned}
\lim_{x\to 0} \frac{e^x + e^{-x} - 2 - x^2}{(\cos x - 1)^2} &= \lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{-2(\cos x - 1)\sin x} \\
&= \lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{-\sin(2x) + 2\sin x} \\
&= \lim_{x\to 0} \frac{e^x + e^{-x} - 2}{-2\cos(2x) + 2\cos x} \\
&= \lim_{x\to 0} \frac{e^x - e^{-x}}{4\sin(2x) - 2\sin x} \\
&= \lim_{x\to 0} \frac{e^x + e^{-x}}{8\cos(2x) - 2\cos x} \\
&= \frac{1}{3}
\end{aligned}$$

11. (HS-1) There are two bands in front of you. The two bands are of different lengths and made of different materials. But both take exactly an hour to burn from one end to the other. The burning speed is not constant, so the tape can burn fast at the beginning, then slower and faster, or randomly. You only have a box of matches and you should measure exactly 45 minutes with the help of the tapes. You must not cut the tapes, use a watch, etc.!

    Reason: Puzzle.

    Solution: You know that each band takes an hour to burn down. So you're lighting one band at both ends, and the other band at one end. When the first one has completely burned out, 30 minutes have passed, and you also ignite the second end of the other band. It takes exactly 15 minutes now to burn it down so that a total of 45 minutes has passed!

12. (HS-2) At the end of a one round chess tournament in which all players played once against each other we have the following result:

    1.Alan 2.Bernie 3.Chuck 4.David 5.Ernest

    The ranking is unambiguous, i.e. all have different scores, and as usual, a victory gets 1 point, a draw 1/2. Bernie is the only one who didn't lose, Ernest the only one who didn't win.

    Who played whom with which result?

    Reason: Puzzle.

    Solution:
    1. Alan has beaten Chuck, David and Ernest and lost against Bernie. (3)
    2. Bernie has beaten Alan and drew against the others. (2.5)
    3. Chuck has beaten David and drew against Bernie and Ernest. (2)
    4. David drew against Bernie and won against Ernest. (1.5)
    5. Ernest drew against Bernie and Chuck. (1)

13. (HS-3) A **unit** e is an element for which there is a multiplicative inverse, i.e. there is an e' such that $e \cdot e' = e' \cdot e = 1$. Units are divisors of 1 .
    An **irreducible** element $n \neq 0$ is an element, which cannot be written as $n = a \cdot b$ unless either a or b is a unit.
    A **prime** p is an element, which is not a unit and if $p \,|\, a \cdot b$ then either $p \,|\, a$ or $p \,|\, b$.
    Show that primes are irreducible, and irreducible elements are either units or primes. Bonus: If we think about integers, which essential property do we need?

Reason: Primes and Proof Techniques.

Solution: Assume p is irreducible and $p\,|\,a\cdot b$. Then $p=q\cdot a\cdot b$ and since p is irreducible, one of the factors has to be p and the others units. If $p=a$ or $p=b$ we are done, because then $p\,|\,a$ or $p\,|\,b$. If $p=q$, then $p\cdot(a\cdot b)-p=p\cdot(a\cdot b-1)=0$ and because we have an integral domain (no zero divisors), $a\cdot b=1\,(p\neq 0)$. Hence $p\,|\,1$ and p is a unit.

If p is prime then it is unequal 0 since we have an integral domain. Let $p=a\cdot b$ then $p\,|\,a$ or $p\,|\,b$, say $a=q\cdot p$. Thus $p=a\cdot b=q\cdot p\cdot b$ and again $p\cdot(1-q\cdot b)=0$ so that $q\cdot b=1$ are units. Hence p cannot be written as $p=a\cdot b$ except one factor is a unit, in our case b.

14. (HS-4) The border collie Boy is at the end of a 1 km flock of sheep, which moves forward at a constant speed. As a control he now walks - with a greater constant speed than the herd - from the end to the top of the herd and back to his place at the end of the flock. When he arrives back, the flock of sheep has walked exactly one kilometer further. Which distance did Boy run?

    Reason: Puzzle.

    Solution: Assume the flock is moving at a speed v_f and Boy at v_b. Boy's time to the top be t_1 and on the way back t_2, x the distance of the last sheep during t_1. Then we have for the first leg:

$$x = v_f \cdot t_1 \tag{1}$$
$$1 + x = v_b \cdot t_1 \tag{2}$$

and for the second leg

$$x = v_b \cdot t_2 \tag{3}$$
$$1 - x = v_f \cdot t_2 \tag{4}$$

Thus we have $(1-x)\cdot(1+x)=v_fv_bt_1t_2=x^2$ and $x=\dfrac{1}{\sqrt{2}}$ which means that Boy ran $1+2x=1+\sqrt{2}\approx 2.414$ km .

15. (HS-5) What is the smallest limit $L>\dfrac{\pi}{6}$ such that

$$\int_{\pi/6}^{L}\frac{dx}{\sin^2 x}=\int_{\pi/6}^{L}\frac{dx}{1-\cos x}+\int_{\pi/6}^{L}6\,\frac{\cot x}{\sin x}\,dx$$

Reason: Trig Functions.

Solution:

$$
\begin{aligned}
0 &= \int_{\pi/6}^{L} \left(\frac{1}{\sin^2 x} - \frac{1}{1-\cos x} - 6\,\frac{\cot x}{\sin x} \right) dx \\
&= \int_{\pi/6}^{L} \left(\frac{1}{1-\cos^2 x} \left(1 - (1+\cos x) - 6\,\cos x\right) \right) dx \\
&= -7 \int_{\pi/6}^{L} \frac{\cos x}{\sin^2 x}\, dx \\
&= 7 \left[\frac{1}{\sin x} \right]_{\pi/6}^{L} \\
&\Longleftrightarrow \\
\sin L &= \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}
\end{aligned}
$$

and $L = \dfrac{5\pi}{6}$ is the smallest possible value L.

2 May 2019

1. (a) Let $(\mathfrak{su}(2,\mathbb{C}),\varphi,V)$ be a finite dimensional representation of the Lie algebra $\mathfrak{g}=\mathfrak{su}(2,\mathbb{C})$. Calculate $H^{\,0}(\mathfrak{g},\varphi)$ and $H^{\,1}(\mathfrak{g},\varphi)$ for the Chevalley-Eilenberg complex in the cases
   i. $(\varphi,V)=(\operatorname{ad},\mathfrak{g})$
   ii. $(\varphi,V)=(0,\mathfrak{g})$
   iii. $(\varphi,V)=(\pi,\mathbb{C}^2)$ is the natural representation on $\mathbb{C}^2$.

   (b) Consider the Heisenberg algebra $\mathfrak{g}=\mathfrak{h}=\left\{\begin{pmatrix}0&a&c\\0&0&b\\0&0&0\end{pmatrix}\,\middle|\, a,b,c\in\mathbb{R}\right\}$ and calculate $H^{\,0}(\mathfrak{h},\operatorname{ad})$ and $H^{\,1}(\mathfrak{h},\operatorname{ad})$.

Reason: Cohomology of Lie algebras.

Solution: The differentials of the cochains

$$C^{\,n}=C^{\,n}(\mathfrak{g},V)=\operatorname{Hom}(\wedge^n\mathfrak{g},V)\,,\;C^{\,-1}=\{\,0\,\}\,,\;C^{\,0}=V$$

are given by

$$d^{\,n}\,:\,C^{\,n}\longrightarrow C^{\,n+1}$$

$$\begin{aligned}d^{\,n}(\omega).(X_1\wedge\ldots\wedge X_{n+1})\\ &=\sum_i(-1)^{i+1}\varphi(X_i)\Big(\omega(X_1\wedge\ldots\wedge\hat{X}_i\wedge\ldots\wedge X_{n+1})\Big)\\ &+\sum_{i<j}(-1)^{i+j}\omega([X_i,X_j]\wedge X_1\wedge\ldots\wedge\hat{X}_i\wedge\ldots\wedge\hat{X}_j\wedge\ldots\wedge X_{n+1})\end{aligned}$$

As $d^{\,n+1}d^{\,n}=0$ we have the cocycles $Z^{\,n}=Z^{\,n}(\mathfrak{g},\mathfrak{g})=\ker d^{\,n}$, the coboundaries $B^{\,n}=B^{\,n}(\mathfrak{g},\mathfrak{g})=\operatorname{im}d^{\,n-1}$ and the cohomology groups $H^{\,n}=H^{\,n}(\mathfrak{g},\mathfrak{g})=H^{\,n}(\mathfrak{g},\operatorname{ad})=Z^{\,n}/B^{\,n}$.

The relevant sequence is

$$\{\,0\,\}\xrightarrow{d^{\,-1}}V\xrightarrow{d^{\,0}}\operatorname{Hom}(\mathfrak{g},V)\xrightarrow{d^{\,1}}\operatorname{Hom}(\mathfrak{g}\wedge\mathfrak{g},V)\xrightarrow{d^{\,2}}\cdots$$

We want to know $H^{\,0}(\mathfrak{g},\varphi)=\ker d^{\,0}$, $H^{\,1}(\mathfrak{g},\varphi)=\ker d^{\,1}/\operatorname{im}d^{\,0}$.

(a) $H^0(\mathfrak{g},\varphi) = \{v \in V \,|\, \forall\, X \in \mathfrak{g}\,:\, \varphi(X)(v)=0\} = \begin{cases} \mathfrak{Z}(\mathfrak{g}) = \{\,\mathbf{0}\,\} & \text{if } \varphi = \operatorname{ad} \\ \mathfrak{g} = \mathfrak{su}(2,\mathbb{C}) & \text{if } \varphi = 0 \\ \{\,\mathbf{0}\,\} & \text{if } \varphi = \pi \end{cases}$

$$\begin{aligned} B^1 &= \operatorname{im} d^0 \\ &= \{\omega \in C^1 \,|\, \omega(X) = d^0(v)(X) = \varphi(X)(v) \text{ for a } v \in V\} \\ &= \begin{cases} \operatorname{ad}(\mathfrak{g}) \cong \mathfrak{su}(2,\mathbb{C}) & \text{if } \varphi = \operatorname{ad} \\ \{\,\mathbf{0}\,\} & \text{if } \varphi = 0 \\ \mathbb{C}^2 \quad (^*) & \text{if } \varphi = \pi \end{cases} \\ Z^1 &= \ker d^1 \\ &= \{\omega \in C^1 \,|\, d^1(\omega)(X,Y) = 0\} \\ &= \{\omega \in C^1 \,|\, \omega([X,Y]) = \varphi(X)\omega(Y) - \varphi(Y)\omega(X)\} \\ &= \begin{cases} \operatorname{Der}(\mathfrak{g}) = \operatorname{ad}(\mathfrak{g}) \cong \mathfrak{su}(2,\mathbb{C}) & \text{if } \varphi = \operatorname{ad} \\ \{\,\mathbf{0}\,\} & \text{if } \varphi = 0 \\ \mathbb{C}^2 \quad (^{**}) & \text{if } \varphi = \pi \end{cases} \end{aligned}$$

(*) According to the basis

$$i\sigma_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\ ,\ i\sigma_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\ ,\ i\sigma_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

of $\mathfrak{su}(2,\mathbb{C})$ we have $B^1(\mathfrak{g},\pi) = \left\{ \begin{bmatrix} iz_2 & z_2 & iz_1 \\ iz_1 & -z_1 & -iz_2 \end{bmatrix} \,\middle|\, z_1, z_2 \in \mathbb{C} \right\} \cong \mathbb{C}^2$.

(**) With the same basis as above and $\omega = \begin{bmatrix} A & B & C \\ U & V & W \end{bmatrix}$ we can solve the three equations

$$\omega([X,Y]) = \pi(X)\omega(Y) - \pi(Y)\omega(X) = X \cdot \omega(Y) - Y \cdot \omega(X)$$

by using $[\mathfrak{su}(2,\mathbb{C}), \mathfrak{su}(2,\mathbb{C})] = \mathfrak{su}(2,\mathbb{C})$ and find

$$Z^1(\mathfrak{g},\pi) = \left\{ \begin{bmatrix} A & -iA & U \\ U & iU & -A \end{bmatrix} \,\middle|\, A, U \in \mathbb{C} \right\} \cong \mathbb{C}^2$$

Therefore $H^1(\mathfrak{su}(2,\mathbb{C}),\varphi) = \{\,\mathbf{0}\,\}$ in all three cases of (φ, V).

(b) Let E_{ij} be a matrix with a 1 in $i-$th row and $j-$th column, and 0 elsewhere. Then we choose as basis $(A = E_{12}, B = E_{23}, C = E_{13})$ for $\mathfrak{h}$. As before we get

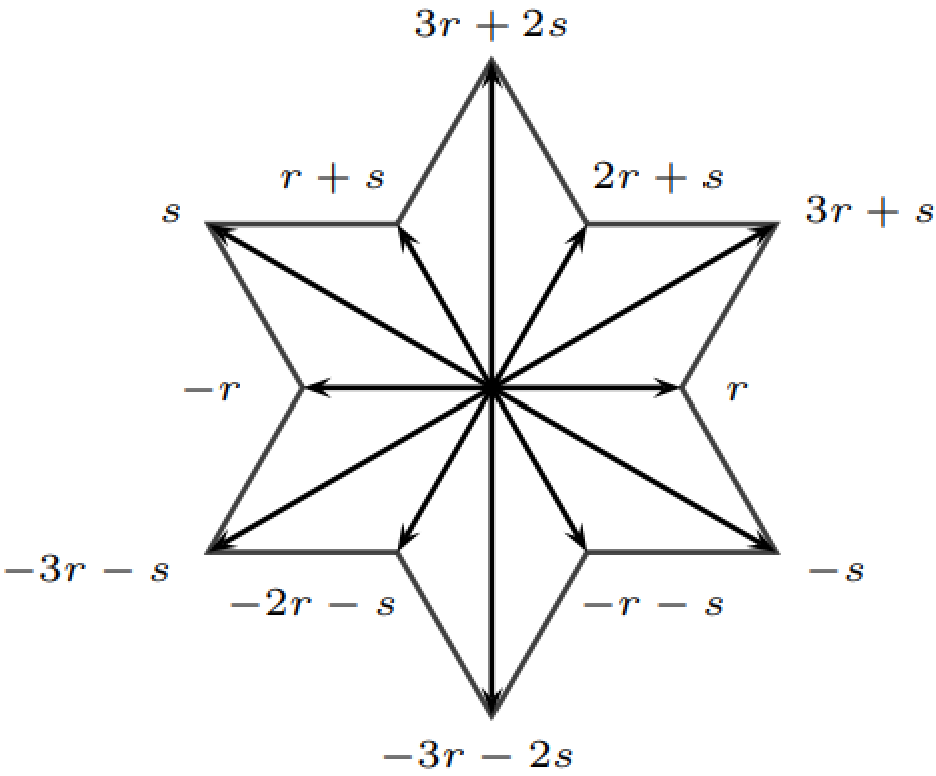
$$\begin{aligned}
H^{\,0}(\mathfrak{h},\operatorname{ad}) &= \{\, Y \in \mathfrak{h}\,|\,\forall\, X \in \mathfrak{h}\,:\,[X,Y]=0\,\} = \mathfrak{Z}(\mathfrak{h}) = \mathbb{R}\cdot C \\
B^{\,1}(\mathfrak{h},\operatorname{ad}) &= \{\,\omega \in C^{\,1}\,|\,\omega(X) = \operatorname{ad}(X)(Y) \text{ for a } Y \in \mathfrak{h}\,\} \\
&= \{\,\omega \in C^{\,1}\,|\,\omega = -\operatorname{ad}(Y) \text{ for a } Y \in \mathfrak{h}\,\} \\
&= \operatorname{ad}(\mathfrak{h}) \\
&\cong \langle \operatorname{ad} A, \operatorname{ad} B\,|\,[\operatorname{ad} A, \operatorname{ad} B] = \operatorname{ad}[A,B] = \operatorname{ad} C = 0\rangle \\
&= \langle E_{32}\,,\,-E_{31}\rangle \cong \mathbb{R}^2 \\
Z^{\,1}(\mathfrak{h},\operatorname{ad}) &= \{\omega \in C^{\,1}\,|\,\omega([X,Y]) = [X,\omega(Y)] - [Y,\omega(X)]\} \\
&= \operatorname{Der}(\mathfrak{h}) \\
&= \left\{ \omega = \begin{pmatrix} \alpha & r_{12} & 0 \\ r_{21} & \beta & 0 \\ r_{31} & r_{32} & \alpha+\beta \end{pmatrix} \,\middle|\, \alpha,\beta,r_{ij} \in \mathbb{R} \right\} \\
H^{\,1}(\mathfrak{h},\operatorname{ad}) &= Z^{\,1}(\mathfrak{h},\operatorname{ad})/B^{\,1}(\mathfrak{h},\operatorname{ad}) \\
&= \left\{ \omega = \begin{pmatrix} \alpha & r_{12} & 0 \\ r_{21} & \beta & 0 \\ 0 & 0 & \alpha+\beta \end{pmatrix} \,\middle|\, \alpha,\beta,r_{ij} \in \mathbb{R} \right\} \\
&\cong \mathfrak{gl}(\mathbb{R}^2) \\
&\cong \mathfrak{gl}\,(\mathfrak{h}/[\mathfrak{h},\mathfrak{h}])
\end{aligned}$$

This demonstrates, that there are significant differences between semisimple and solvable Lie algebras (cp. Whitehead Lemmas).

2. Show that the dihedral group D_{12} of order twelve is the finite reflection group of the root system of type G_2.

Reason: Buildings.

Solution: The roots of G_2 are $\pm\{\,r, s, r+s, 2r+s, 3r+s, 3r+2s\,\}$ (cp. https://www.physicsforums.com/insights/lie-algebras-a-walkthrough-the-structures/) which can be visualized by the following figure:



The covering transformations are generated by a rotation R of $30°$ and a reflection S at the axis $r \longleftrightarrow -r$ which is the group

$$\{\, R, S \,:\, S^2 = R^6 = 1\,,\, SRS = R^{-1}\,\} = D_{12}.$$

3. Consider the set

$$\mathcal{P}_n := \{\, \{2\}, \{4\}, \ldots, \{2n\}\,\} \subseteq \mathcal{P}(\mathbb{N})$$

and determine the $\sigma-$algebra $\mathcal{A}_\sigma(\mathcal{P}_n) \subseteq \mathcal{P}(\mathbb{N})$, and show that $\bigcup_{n\in\mathbb{N}} \mathcal{A}_\sigma(\mathcal{P}_n)$ isn't a $\sigma-$algebra.

Reason: Measure Theory.

Solution:

$$\begin{aligned}\mathcal{A}_\sigma(\mathcal{P}_n) = \{\, \emptyset, \mathbb{N}\,\} &\cup \{\, B \subseteq \mathbb{N} \,:\, B \subseteq \{2, 4, , \ldots, 2n\}\,\} \\ &\cup \{\, B \subseteq \mathbb{N} \,:\, 2k \in B \,\forall\, k > n \,\wedge\, 2k-1 \in B \,\forall\, k \in \mathbb{N}\,\}\end{aligned}$$

Assume $\bigcup_{n\in\mathbb{N}} \mathcal{A}_\sigma(\mathcal{P}_n)$ is a $\sigma-$algebra. Since for all $n \in \mathbb{N}$ we have $B_n := \{\, 2, 4, \ldots, 2n\,\} \in \mathcal{A}_\sigma(\mathcal{P}_n)$, the union $\bigcup_{k\in\mathbb{N}} B_k \in \bigcup_{n\in\mathbb{N}} \mathcal{A}_\sigma(\mathcal{P}_n)$

which contradicts $\bigcup_{k\in\mathbb{N}} B_k \notin \mathcal{A}_\sigma(\mathcal{P}_n)$ for all $n \in \mathbb{N}$.

4. Linear Operators. (Only solutions to both count!)

   (a) Show that eigenvectors to different eigenvalues of a self-adjoint linear operator are orthogonal and the eigenvalues real.

   (b) Given a real valued, bounded, continuous function $g \in C([0,1])$ with
   $$m = \inf_{t\in[0,1]} g(t)\ ,\ M = \sup_{t\in[0,1]} g(t)$$
   and an operator $T_g(f)(t) := g(t)f(t)$ on the Hilbert space $\mathcal{H} = L^2([0,1])$. Calculate the spectrum of T_g.

   Reason: Spectrum of Operators.

   Solution:
   $$\lambda\langle x,y\rangle = \langle T(x),y\rangle = \langle x,Ty\rangle = \bar{\mu}\langle x,y\rangle \Longrightarrow \langle x,y\rangle = 0$$
   $$\lambda\langle x,x\rangle = \langle T(x),x\rangle = \langle x,Tx\rangle = \bar{\lambda}\langle x,x\rangle \Longrightarrow \lambda = \bar{\lambda}$$

   From the boundaries of g we get that m, M are a lower, resp. upper bound of T_g. Hence $\sigma(T_g) \subseteq [m,M]$. According to the mean value theorem for continuous functions we know, that g takes every value in $[m,M]$ at least once, i.e for every $\mu \in [m,M]$ there is a real number $t_\mu \in [0,1]$ such that $g(t_\mu) = \mu$. Thus
   $$T_g(f)(t_\mu) = g(t_\mu)f(t_\mu) = \mu \cdot f(t_\mu)$$
   and $T - \mu$ isn't bounded invertible, hence $\mu \in \sigma(T_g)$ and $\sigma(T_g) = [m,M]$.

   TSF - operator.pdf; Beispiel V6 p.24

5. Let $\mathbb{F}$ be a field. Then for a polynomial $f \in \mathbb{F}[X_1,\ldots,X_n]$ we define $D(f) = \{\, q \in \mathbb{A}^n(\mathbb{F}) \,|\, f(q) \neq 0 \,\}$. Show that these sets build a basis of the Zariski topology on $\mathbb{A}^n(\mathbb{F})$ and decide whether finitely many of them are sufficient.

   Reason: Affine Variety.

   Solution: Recall that $\mathbb{F}[V] = \mathbb{F}[X_1,\ldots,X_n]/I(V)$ with $I(V) = \{\, f \in \mathbb{F}[X_1,\ldots,X_n] \,|\, f(p) = 0\, \forall\, p \in V \,\}$ is the coordinate ring of the affine variety $V \subseteq \mathbb{A}^n(\mathbb{F})$ and $V = V(I(V))$, i.e. V is the vanishing set of polynomials in the ideal $I(V)$.

(a) For a point $p \in \mathbb{A}^n(\mathbb{F}) - V$ outside an affine variety there is a polynomial $f \in \mathbb{F}[X_1, \ldots, X_n]$ such that $f(p) = 1$ and $f(q) = 0$ for all $q \in V$:
Since $p \notin V = V(I(V))$ there is a polynomial $g \in I(V)$ with $g(p) \neq 0$. Then $f := g(p)^{-1} \cdot f$ has the required properties.

(b) For each open set $U \subseteq \mathbb{A}^n(\mathbb{F})$ and point $p \in U$ there is a polynomial $f \in \mathbb{F}[X_1, \ldots, X_n]$ such that $p \in D(f_p) \subseteq U$:
Let $V = \mathbb{A}^n(\mathbb{F}) - U$. Then $V \neq \mathbb{A}^n(\mathbb{F})$ since $p \in U$ and so $U \neq \emptyset$. By the previous statement (a) we have a polynomial $f_p \in I(V)$ with $f_p(p) = 1$. Hence $p \in D(f_p) = \mathbb{A}^n(\mathbb{F}) - V(f_p) \subseteq \mathbb{A}^n(\mathbb{F}) - V = U$.

(c) Now let $\emptyset \neq U \subseteq \mathbb{A}^n(\mathbb{F})$ be an open set. By the previous statement (b) there are polynomials $f_p \in \mathbb{F}[X_1, \ldots, X_n]$ for every point $p \in U$ such that $p \in D(f_p) \subseteq U$. Hence $U = \bigcup_{p \in U} D(f_p)$ and

$$\bigcup_{p \in U} D(f_p) = \mathbb{A}^n(\mathbb{F}) - \bigcap_{p \in U} V(f_p) = \mathbb{A}^n(\mathbb{F}) - V(\{\, f_p \,|\, p \in U \,\})$$

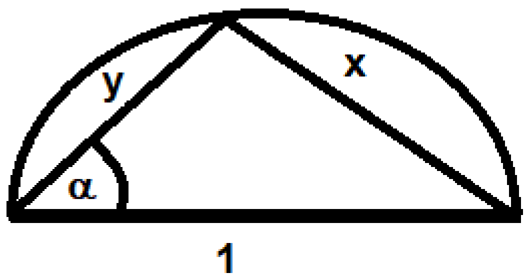
Since $\mathbb{F}[X_1, \ldots, X_n]$ is Noetherian, there are finitely many $f_1, \ldots, f_m$ with $V(\{\, f_p \,|\, p \in U \,\}) = V(f_1, \ldots, f_m)$ and $U = \bigcup_{i=1}^{m} D(f_i)$ and finitely many are sufficient.

6. Let $R := \mathbb{Q}[x, y]/\langle x^2 + y^2 - 1 \rangle$ and $\varphi \in \operatorname{Der}(R)$ a $\mathbb{Q}-$linear derivation such that $\varphi(x) = y$, $\varphi(y) = -x$. A derivation $\varphi : R \longrightarrow R$ of an algebra R is a linear function with $\varphi(p \cdot q) = \varphi(p) \cdot q + p \cdot \varphi(q)$.

(a) Determine the kernel of φ.

(b) Solve $\varphi^2 + \mathrm{id} = 0$.

(c) Since $x^2 + y^2 = 1$ we can apply Thales' theorem and identify $(x, \alpha), (y, \alpha)$ with the sides of a right triangle with hypotenuse (diameter) 1 according to an angle α. Show that



$$(x, \alpha + \beta) = (x, \alpha)(y, \beta) + (x, \beta)(y, \alpha)$$

Reason: Sine and Cosine.

Solution: $\varphi(1) = \varphi(1\cdot 1) = \varphi(1)\cdot 1 + 1\cdot\varphi(1) = 2\varphi(1)$ and thus $\varphi(1) = 0$. Since φ is $\mathbb{Q}-$linear, we get $\varphi(\lambda) = \lambda\cdot\varphi(1) = 0$ for all $\lambda\in\mathbb{Q}$, i.e. $\mathbb{Q}\subseteq\ker\varphi$. It can be shown by induction that

$$\varphi(x^n y^m) = nx^{n-1}y^{m+1} - mx^{n+1}y^{m-1}$$

and especially

$$\varphi(x^n) = nx^{n-1}y$$
$$\varphi(y^m) = -mxy^{m-1}$$

Every polynomial $p(x,y)\in R$ can be written as $p(x,y) = f(x) + y\cdot g(x)$ with $f(x), g(x)\in\mathbb{Q}[x]$. Now let

$$\begin{aligned}
0 &= \varphi(p)\\
&= \varphi(f) + \varphi(y)\cdot g + y\cdot\varphi(g)\\
&= y\sum_{i=1}^{n} f_i(ix^{i-1}) - x\cdot g(x) + y^2\cdot\sum_{j=1}^{m} g_j\cdot(jx^{j-1})\\
&= y\sum_{i=1}^{n} f_i(ix^{i-1}) - xg_0 - \sum_{j=1}^{m}\left(g_jx^{j+1} + (x^2-1)jg_jx^{j-1}\right)\\
&= y\sum_{i=1}^{n} f_i(ix^{i-1}) - xg_0 + \sum_{j=1}^{m} jg_jx^{j-1} - \sum_{j=1}^{m}(j+1)\,g_jx^{j+1}\\
&= y\sum_{i=1}^{n} f_i(ix^{i-1}) - xg_0 + g_1 + 2g_2x - mg_{m-1}x^m - (m+1)g_mx^{m+1}\\
&\quad + \sum_{j=2}^{m-1}\left((j+1)g_{j+1} - jg_{j-1}\right)x^j
\end{aligned}$$

This means $f_i = 0$ for all $i > 0$, $g_1 = g_{m-1} = g_m = 0$ and $(j+1)g_{j+1} = jg_{j-1}$ for $j = 2,\ldots,m-1$. Backwards substitution yields $g_j = 0$ for all $j\geq 0$ and $p(x,y) = f(x) + y\cdot g(x) = f_0$, i.e. $\ker\varphi\subseteq\mathbb{Q}$.

Suppose $\lambda x^n + \tau yx^m$ is the term of highest degree in a solution of $\varphi^2(p(x,y)) + p(x,y) = \varphi^2(\lambda f(x) + \tau yg(x)) + \lambda f(x) + \tau yg(x) = 0$. Then

$$\begin{aligned}
\varphi^2(\lambda\cdot x^n + \tau\cdot yx^m) &= x^n\cdot\left(-\lambda n^2\right) + x^{n-2}\cdot(\ldots)\\
&\quad + yx^m\cdot\left(-\tau\,(m+1)^2\right) + yx^{m-2}(\ldots)
\end{aligned}$$

and φ^2 cannot raise the degree. Thus we have modulo terms of lower degree from $\varphi^2(p) = -p$

$$\lambda n^2 = \lambda \text{ and } \tau(m+1)^2 = \tau$$

and $p(x,y) = \lambda x + \tau y$ are the only solutions:

$$\varphi(\lambda x + \tau y) = -\tau x + \lambda y\,,\; \varphi(-\tau x + \lambda y) = -\lambda x - \tau y$$

Since $(x, \alpha) = \sin\alpha$ and $(y, \beta) = \cos\beta$ the formula

$$(x, \alpha + \beta) = (x, \alpha)(y, \beta) + (x, \beta)(y, \alpha)$$

is simply the addition theorem of the sine function.

7. For all $a, b, c \in \mathbb{R}$ holds

$$a > 0, b > 0, c > 0 \Longleftrightarrow a + b + c > 0, ab + ac + bc > 0, abc > 0\,.$$

Reason: Vieta.

Solution: Set $p(x) = x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc$. Then $p(x) < 0$ if $x \leq 0$ so the roots a, b, c of $p(x)$ are all positive.

8. Let $a, b \in L^2\left(\left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]\right)$ given as

$$a(x) = 11\sin(x) + 8\cos(x)\;,\; b(x) = 4\sin(x) + 13\cos(x)$$

Calculate the angle $\varphi = \sphericalangle(a, b)$ between the two vectors.

Reason: Hilbert Space.

Solution: We define $f(x) = \sin(x) - 6\cos(x)$, $g(x) = 6\sin(x) + \cos(x)$ and observe, that $\{\,f, g\,\}$ is a orthogonal basis for a two dimensional subspace of $L^2\left(\left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]\right)$ with $\gamma := |f| = |g| = \sqrt{\dfrac{37\pi}{2}}$. As we are interested in an angle, we won't have to bother the length of our coordinate vectors, i.e. we do not need to normalize them. Now we have $a = -f + 2g$, $b = -2f + g$ and

$$\begin{aligned}
\cos\varphi &= \cos(\sphericalangle(a, b)) \\
&= \cos(\sphericalangle(-f + 2g, -2f + g)) \\
&= \frac{\langle -f + 2g, -2f + g\rangle}{|-f + 2g| \cdot |-2f + g|} \\
&= 2\,\frac{\langle f, f\rangle + \langle g, g\rangle}{\sqrt{(|f|^2 + 4|g|^2)} \cdot \sqrt{(4|f|^2 + |g|^2)}} \\
&= 2\,\frac{\gamma^2 + \gamma^2}{\sqrt{5\gamma^2 \cdot 5\gamma^2}} \\
&= \frac{4}{5}
\end{aligned}$$

and $\varphi \approx 80.8^\circ \approx 0.45\pi$

9. Let $\varepsilon_k := \begin{cases} 1 & , \text{ if the decimal representation of } k \text{ has no digit } 9 \\ 0 & , \text{ otherwise} \end{cases}$

   Show that $\sum_{k=1}^{\infty} \dfrac{\varepsilon_k}{k}$ converges.

   Reason: Series.

   Solution: The numbers $10^n \leq k < 10^{n+1}$ have $n+1$ digits. On the first place are 8 distinct digits unequal 9 possible, 9 such digits for the others. Thus we have $8 \cdot 9n$ numbers within the interval without the digit 9 and

$$\sum_{k=10^n}^{10^{n+1}-1} \frac{\varepsilon_k}{k} \leq \frac{8 \cdot 9n}{10^n}$$

   and the partial sums $\sum_{k=1}^{10^n-1} \dfrac{\varepsilon_k}{k} \leq \sum_{j=0}^{n-1} 8 \cdot \left(\dfrac{9}{10}\right)^n = 80$ are bounded, hence the series converges.

10. Let $x_0 \in [a,b] \subseteq \mathbb{R}$ and $f\,:\,[a,b] \longrightarrow \mathbb{R}$ continuous and differentiable on $[a,b]-\{x_0\}$. Furthermore exists the limit $c := \lim_{x \to x_0} f\,'(x)$. Then $f(x)$ is differentiable in x_0 with $f\,'(x_0) = c$.

    Proof: Let $x \in [a,b] - \{x_0\}$. According to the mean value theorem for differentiable functions there is a

$$\xi(x) \in (\min\{x,x_0\}, \max\{x,x_0\})$$

    with $f\,'(\xi(x)) = \dfrac{f(x)-f(x_0)}{x-x_0}$. Because $\lim_{x\to x_0} \min\{x,x_0\} = \lim_{x \to x_0} \max\{x,x_0\} = x_0$ we must have $\lim_{x\to x_0} \xi(x) = x_0$ and by assumption $\lim_{x\to x_0} f\,'(\xi(x)) = c$, hence $\lim_{x\to x_0} \dfrac{f(x)-f(x_0)}{x-x_0} = c$.

    What has to be regarded in this proof, and is there a way to avoid this hidden assumption?

    Reason: Axiom of choice.

    Solution: Let

$$\Lambda(x) := \left\{ \xi \in (\min\{x,x_0\}, \max\{x,x_0\}) \,:\, \frac{f(x)-f(x_0)}{x-x_0} = f\,'(\xi) \right\}$$

    The mean value theorem guarantees us that all $\Lambda(x) \neq \emptyset$, but we need more: namely a function

$$\xi \,:\, [a,b]-\{x_0\} \longrightarrow \bigcup_{x \in [a,b]-\{x_0\}} \Lambda(x)$$

i.e. we made use of the axiom of choice.

To avoid AC, let $\varepsilon > 0$. Then there is a $\delta > 0$ such that $|f\,'(x) - c| < \varepsilon$ whenever $x \in [a, b] - \{x_0\}$ with $|x - x_0| < \delta$. By the mean value theorem, $\Lambda(x) \neq \emptyset$ and we can choose $(*)$ an arbitrary element $\xi \in \Lambda(x)$ and get $|\xi - x_0| < |x - x_0| < \delta$ and thus

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - c \right| = |\, f\,'(\xi) - c\,| < \varepsilon$$

$(*)$ In this version we only used $\Lambda(x) \neq \emptyset$ for a single value x given by the mean value theorem. To select a single element from a nonempty set does not require AC. This point is given via Rolle's theorem, which again uses the existence of an extremal point in the interior of a closed interval, which again uses the theorem of Bolzano-Weierstrass, which is proven constructively via induction and the completeness of $\mathbb{R}$.

11. (HS-1) A house H and a rosary R are near a circular lake L. The Gardener walks with two watering cans from the house to the lake, fills the cans and goes to the rosary. We assume $\overline{HR} \cap L = \emptyset$. At which point S of the shore does he have to get water, so that his path length is minimal, and why?

    Reason: Reflection.

    Solution: You choose the point S on the circle, such that the tangent t to the circle in S is a mirror which bisects the angle $\sphericalangle HSR$ of his path. This is the shortest way from H to the circle and on to R.

    (a) Huygens - Fresnel principle to prove the law of reflection.

    (b) Let $H = (0, h), S = (s, 0), R = (p, q)$. Then the path length is

    $$\begin{aligned} L &= \sqrt{h^2 + s^2} + \sqrt{(p - s)^2 + q^2} \text{ and} \\ \frac{dL}{ds} &= \frac{s}{\sqrt{h^2 + s^2}} - \frac{p - s}{\sqrt{(p - s)^2 + q^2}} = \cos\alpha - \cos\beta \end{aligned}$$

    with the incident angle α and the reflected angle β. If they are equal we get the minimum which corresponds to the bisection of the normal to the tangent at S.

12. (HS-2) How long is the distance on a direct flight from London to Los Angeles and where is its most northern point? How long will it last by an assumed average speed of 494 knots over ground? We neglect the influence of weather, esp. wind.

We take the values $51^\circ\,28'\,39''\,N$, $0^\circ\,27'\,41''\,W$ for LHR in London, $33^\circ\,56'\,33''\,N$, $118^\circ\,24'\,29''\,W$ for LAX in Los Angeles, and a radius of $3,958$ miles for earth.

Reason: Spherical Trigonometry.

Solution: $9,070.546\,\text{km}$, $5,636.165\,\text{mi}$, $9\,\text{h}\;55\,\text{min}$, $61^\circ\,22'\,53''\,N$, $47^\circ\,11'\,36''\,W$

LHR: $51.4775^\circ\,\text{N}, 0.4614^\circ\,\text{W} = 0.898452\,\text{N}, 0.008053\,\text{W}$
LAX: $33.9425^\circ\,\text{N}, 118.408^\circ\,\text{W} = 0.533168\,\text{N}\;2.066609\,\text{W}$
$494\,\text{kn} = 494 \cdot 1.15078\,\text{mph} \approx 568.49\,\text{mph}$

The formula for the spherical distance is given by the spherical law of cosine as

$$D = R{\cdot}\zeta = R{\cdot}\arccos\left(\sin(\phi_A)\cdot\sin(\phi_B) + \cos(\phi_A)\cdot\cos(\phi_B)\cdot\cos(\lambda_B - \lambda_A)\right)$$

which in our case is

$$\begin{aligned} D &= 3,958 \cdot \arccos\left(\sin(0.898452)\cdot\sin(0.533168)\right. \\ &\quad \left. + \cos(0.898452)\cdot\cos(0.533168)\cdot\cos(2.066609 - 0.008053)\right)\,\text{mi} \\ &\approx 5,636.165\,\text{mi} \approx 9.9143\,\text{h} \approx 9\,\text{h}\;55\,\text{min} \end{aligned}$$

The most northern point is given with

$$\begin{aligned} \alpha_A &= \arccos\left(\frac{\cos(\phi_A)\cdot\sin(\phi_B) - \cos(\lambda_A-\lambda_B)\cdot\cos(\phi_B)\cdot\sin(\phi_A)}{\sqrt{1-\left(\cos(\lambda_A-\lambda_B)\cdot\cos(\phi_A)\cdot\cos(\phi_B) + \sin(\phi_A)\cdot\sin(\phi_B)\right)^2}}\right) \\ &\approx 0,8773446 \end{aligned}$$

by

$$\begin{aligned} P_N &= (\phi_N, \lambda_N) \\ &= \left(\arccos\left(\sin(|\alpha_A|)\cdot\cos(\phi_A)\right), \lambda_A + \operatorname{sgn}(\alpha_A)\cdot\left|\arccos\left(\frac{\tan(\phi_A)}{\tan(\phi_N)}\right)\right|\right) \\ &\approx (1.071307\,,\,0.823681) \approx (61^\circ\,22'\,53''\,N\,,\,47^\circ\,11'\,36''\,W) \end{aligned}$$

which is in SW-Greenland near Qassimiut, Ivigtut, and Kangilinnguit.

13. (HS-3) Trial before an American district court. The witness claims he saw a blue cab drive off after a night accident. The judge decides to test the reliability of the witness. Result: The witness recognizes the color correctly in the dark in 80% of all cases. A survey also found that

85% of taxis in the city are green and 15% are blue.
With what probability has the taxi actually been blue?

Reason: Bayes' Theorem.

Solution: 15 out of 100 taxis are blue. The witness identifies 80% as blue, which are 12 taxis (and 3 taxis falsely as green). 85 taxis are green, and the witness actually identifies 80% as green, that's 68 (and 17 as blue). In total, the witness identifies 29 taxis as blue. The probability that a taxi identified as blue by the witness is actually blue is thus $12/29 = 41.38\%$.

The probability that a taxi identified as blue by the witness is actually blue is, according to Bayes: $(0.8 \cdot 0.15)/(0.8 \cdot 0.15 + 0.2 \cdot 0.85) = 41.38\%$.

14. (HS-4) A monk climbs a mountain. He starts at 8 a.m. on 1000 m above sea level and reaches the peak at 8 p.m. at 3000 m. After a bivouac on top of the mountain, he returns to the valley the next morning and again starts at 8 a.m. and returns at 8 a.m.

   (a) If he wants to avoid being at the same time of day at the same place as the day before when he climbed upwards, which strategy must he use downwards, and why?

   (b) Assume he climbed at a rate of height $u(t)$ proportional to the square root of time, determine his path in dependence of hourly noted time.

   (c) Assume he follows the same path downwards and his height is given by $d_1(t) = \dfrac{125}{9}(t-20)^2 + 1000$ in the first three hours and $d_2(t) = -125\,t + 3500$ for the rest of his way, when will he be at the same point as the day before and at which height.

Reason: Homework.

Solution:

   (a) He has to use an alternative route downwards, because if he climbs down the way he climbed up, then he will cross a certain height at the same time as the day before; just imagine he would simultaneously climb up and down. He will have to meet himself then.

   (b) We know $u(8) = 1,000$, $u(20) = 3,000$, and $u(t) \sim \sqrt{t}$. So we can write $u(t) = \alpha\sqrt{t-\beta} + 1,000$ and get

$$\beta = 8\ ,\ \alpha = \frac{2,000}{\sqrt{12}} = \frac{1,000}{\sqrt{3}} \text{ and } u(t) = \frac{1,000}{\sqrt{3}}\sqrt{t-8} + 1,000$$

(c) After three hours he has reached the height $u(11) = 2000\,m$ upwards, and the height $d_1(11) = 9 \cdot 125\,m + 1,000\,m = 2,125\,m$ downwards. He therefore reaches the same height and location on his second leg downwards, i.e. we have to solve $u(t) = d_2(t)$ or

$$\begin{aligned} 0 &= t^2 + \left(-\frac{64}{3} - 40\right)t + \left(\frac{512}{3} + 400\right) \\ t &= \frac{92}{3} - \frac{1}{3}\sqrt{92^2 - 5136} = \frac{1}{3}\left(92 - 16\sqrt{13}\right) \\ t &\approx 11.437 \approx 11^h\ 26^m\ 13^s \\ &\text{and} \\ d_2(t) &\approx 2,070.37\,m \end{aligned}$$

15. (HS-5) I'm annoyed by my two new alarm clocks. They both are powered by the grid. One leaps two minutes an hour and the other one runs a minute an hour too fast. Yesterday I took the effort and set them to the correct time. This morning, I assume there was a power loss, one clock showed exactly $6\,a.m$ while the other one showed $7\,a.m.$ When did I set the clocks and how long did they run?

    Reason: Equation of uniform movement.

    Solution: One clock runs by $v_1(t) = \frac{29}{30}t + t_0$ and the other one by $v_2(t) = \frac{61}{60}t + t_0$. We know that $v_1(t_1) \equiv 6 \bmod 24$ and $v_2(t_1) \equiv 7 \bmod 24$. From this we get, that the clocks ran $t_1 = 20$ hours, and I set them at $t_0 \equiv 7 - \frac{61}{60} \cdot 20 \equiv 31 - \frac{61}{3} \equiv 10^h\,40^m\,(a.m.)$ the previous day.

3 April 2019

1. Find the area A enclosed by the asteroid $(x,y)=(\cos^3 t,\sin^3 t)$ for $0\leq t\leq 2\pi$.

   Solution: By symmetry we have with $dx=3\cdot\cos^2 t\cdot\sin t\,dt$

$$\begin{aligned}
A&=4\int_0^1 y\,dx\\
&=4\int_0^{\pi/2}(\sin^3 t)(3\cdot\cos^2 t\cdot\sin t)\,dt\\
&=12\int_0^{\pi/2}\left(\frac{1-\cos 2t}{2}\right)^2\left(\frac{1+\cos 2t}{2}\right)dt\\
&=\frac{3}{2}\int_0^{\pi/2}(1-2\cos 2t+\cos^2 2t)(1+\cos 2t)\,dt\\
&=\frac{3}{2}\int_0^{\pi/2}(1-\cos 2t-\cos^2 2t+\cos^3 2t)\,dt\\
&=\frac{3}{2}\left[\left(t-\frac{1}{2}\sin 2t\right)-\frac{1}{2}\left(t+\frac{1}{4}\sin 4t\right)+\frac{1}{2}\left(\sin 2t-\frac{1}{3}\sin^3 2t\right)\right]_0^{\pi/2}\\
&=\frac{3\pi}{8}\\
&\approx 1.1781
\end{aligned}$$

2. Two surface ships on maneuvers are trying to determine a submarine's course and speed to prepare for an aircraft intercept. Ship A is located at $(4,0,0)$, whereas ship B is located at $(0,5,0)$. All coordinates are given in thousands of feet. Ship A locates the submarine in the direction of the vector $2\mathbf{i}+3\mathbf{j}-(1/3)\mathbf{k}$, and ship B locates it in the direction of the vector $18\mathbf{i}-6\mathbf{j}-\mathbf{k}$. Four minutes ago, the submarine was located at $A=(2,-1,-1/3)$. The aircraft is due in 20 minutes. Assuming that the submarine moves in a straight line at a constant speed, to what position should the surface ships direct the aircraft?

   Solution: Information from ship A indicates the submarine is now on the line $L_1\,:\,(x,y,z)=(4+2t,3t,-\frac{1}{3}t)$; information from ship B indicates the submarine is now on the line $L_2\,:\,(x,y,z)=(18s,5-6s,-s)$. The current position of the sub is at the intersection of both lines at $C=(6,3,-1/3)$ with $t=1,s=1/3$. The straight line path of the submarine contains both points A and C; the line representing this path is $L\,:\,(x,y,z)=(2+4t,-1+4t,-1/3)$.

The submarine traveled the distance between A and C in 4 minutes, i.e. at a speed of $\frac{1}{4}|AC| = \frac{1}{4}\sqrt{32} = \sqrt{2}$ thousand feet per minute. In 20 minutes the submarine will move $20\sqrt{2}$ thousand feet from C along the line L.

For the rendezvous point $R \in L$ we thus have for $t > 0$

$$20\sqrt{2} = |RC| = \sqrt{(-4+4t)^2 + (-4+4t)^2} \Longrightarrow 25 = (t-1)^2 \Longrightarrow t = 6$$

and the submarine will be located at $R = (26, 23, -1/3)$ in 20 minutes.

3. Calculate the following:

   (a) $\displaystyle\int \frac{\sqrt{(x^2-1)^3}}{x}\,dx$

   (b) The arc length L of $y = -\dfrac{x^2}{8} + \log x$ for $1 \leq x \leq 2$

   Solution:

   (a) We substitute $x = \sec\varphi$, $dx = \sec\varphi\tan\varphi d\varphi$ to get

$$\begin{aligned}
\int \frac{(x^2-1)^{3/2}}{x}\,dx &= \int \frac{(\sec^2\varphi - 1)^{3/2}}{\sec\varphi}\sec\varphi\tan\varphi\,d\varphi \\
&= \int \tan^4\varphi\,d\varphi \\
&= \int \tan^2\varphi\,(\sec^2\varphi - 1)\,d\varphi \\
&= \int (\tan^2\varphi\,\sec^2\varphi - (\sec^2\varphi - 1))\,d\varphi \\
&= \frac{1}{3}\tan^3\varphi - \tan\varphi + \varphi + C \\
&= \frac{1}{3}\sqrt{(x^2-1)^3} - \sqrt{x^2-1} + \operatorname{arcsec} x + C
\end{aligned}$$

(b)

$$\begin{aligned} L &= \int_1^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\ &= \int_1^2 \sqrt{1+\left(\frac{1}{x}-\frac{1}{4}x\right)^2}\,dx \\ &= \int_1^2 \left(\frac{1}{x}+\frac{x}{4}\right)\,dx \\ &= \frac{3}{8}+\log 2 \\ &\approx 1.068 \end{aligned}$$

4. Find the similarity transformations to diagonalize the following matrices:

(a) $A = \begin{pmatrix} 1 & -\sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix}$

(b) $B = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

Solution:

(a) The characteristic polynomial of A is

$$\chi(A;\lambda) = -\lambda^3+2\lambda^2-4\lambda+8 = -(\lambda-2)(\lambda+2i)(\lambda-2i)$$

and $\frac{1}{2}A \in \mathrm{SO}(3,\mathbb{R})$ with $\det(A)=8$ and $\operatorname{tr}(A)=2$. The eigenvector for $\lambda=2$ is $(1,0,1)^\tau$ and since A is orthogonal, the eigenvectors for $\pm 2i$ are of the form $(1,a,-1)^\tau$ which yields $a=\mp i\sqrt{2}$. After normalization we get

$$S^{-1}AS = \begin{pmatrix} 2i & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2i \end{pmatrix}\,,\; S = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ -i\sqrt{2} & 0 & i\sqrt{2} \\ -1 & \sqrt{2} & -1 \end{pmatrix}$$

(b) The characteristic polynomial of B is

$$\chi(B;\lambda) = \lambda^2-2\lambda\cos\varphi+1 = (\lambda-\cos\varphi-i\sin\varphi)(\lambda-\cos\varphi+i\sin\varphi)$$

that is eigenvalues $\lambda \in \{\cos\varphi \pm i\sin\varphi\} = \{e^{\pm i\varphi}\}$ with eigenvectors $(1,\mp i)^\tau$. Normalization yields

$$S^{-1}BS = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}\,,\; S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$$

5. Suppose that $\mathbb{F}$ is a finite field with say $|\mathbb{F}| = p^m = q$ and that V is a vector space of finite dimension n over $\mathbb{F}$. Find the order of $\operatorname{GL}(V)$.
Solution: There are $|V| = q^n$ elements in V and for any fixed basis $\{ v_1, \ldots, v_n \}$ there is a unique element $\varphi \in \operatorname{GL}(V)$ that transforms it into another basis $\{ v_1, \ldots, v_n \}$ and vice versa. So how many possibilities do we have to choose such a basis? For w_1 we have $q^n - 1$ possibilities, as the zero vector cannot be chosen. For w_2 we can choose any vector, which isn't one of the q multiples of w_1. For w_3 we may choose all vectors, which are not in one of the q^2 many linear combinations of the former, etc. So all in all we have
$$|\operatorname{GL}(V)| = (q^n - 1)(q^n - q)(q^n - q^2) \cdots (q^n - q^{n-1})$$

6. (HS-4) Can the numbers $1, 2, 3, \ldots, 16$ be arranged in a row so that each two adjacent numbers add up to a square number?

   Example: $2, 7, 9, 16, \ldots$ would be a possibility for the first four numbers $(2 + 7 = 9, 7 + 9 = 16, 9 + 16 = 25)$; but then we get stuck.

   Reason: Puzzle (66).

   Solution: $16, 9, 7, 2, 14, 11, 5, 4, 12, 13, 3, 6, 10, 15, 1, 8$

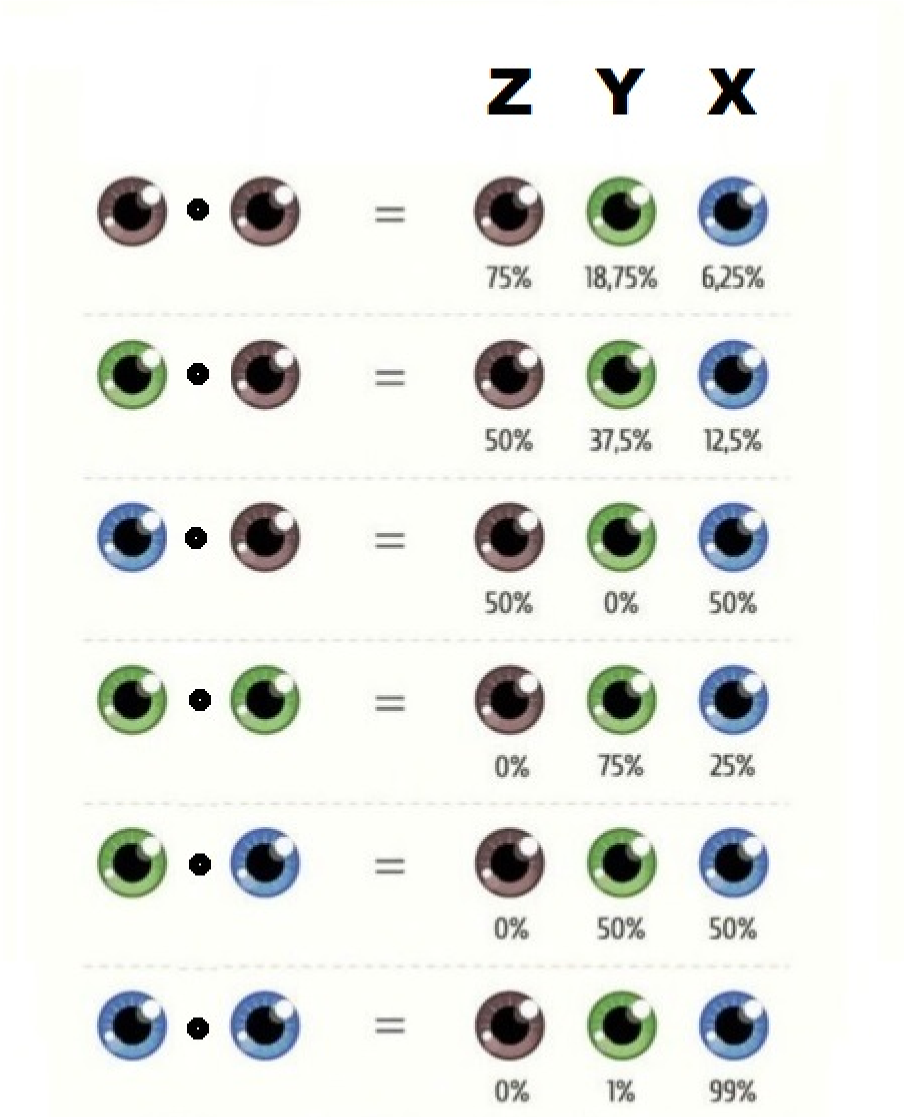
7. (HS-5) We are looking for a ten-digit number N, where the first digit indicates how many zeros occur in N, the second digit, how many ones appear in N, the third digit, how many doubles occur in N, ... and the tenth digit, how many nines appear in N.

   Reason: Puzzle (74).

   Solution: $N = 6, 210, 001, 000$

4 March 2019

1. The graphic of eye colors shows us the probability of baby's eye color in dependency of the parents'. This yields the following multiplication table:



$$\begin{array}{lclclcl}
x \cdot x & = & 3/4\ x & + & 3/16\ y & + & 1/16\ z \\
x \cdot y & = & 1/2\ x & + & 3/8\ y & + & 1/8\ z \\
x \cdot z & = & 1/2\ x & & & + & 1/2\ z \\
y \cdot y & = & & & 3/4\ y & + & 1/4\ z \\
y \cdot z & = & & & 1/2\ y & + & 1/2\ z \\
z \cdot z & = & & & & & z
\end{array}$$

which we extend to a real, commutative, distributive, three dimensional algebra A.

(a) Is A an associative algebra?

(b) Prove, that A is a baric algebra, i.e. show that there is a non trivial algebra homomorphismus $\omega : A \longrightarrow \mathbb{R}$, the weight function.

(c) Determine a basis for $\ker \omega$ and rewrite the multiplication table according to this new basis.

(d) Prove that there is an ideal N of codimension one in A, such that $A^2 \nsubseteq N$.

(e) A algebra is called genetic, if there is a basis $\{ u_i \}$ such that the structure constants λ_{ijk} defined by

$$u_i \cdot u_j = \sum_{k=1}^{n} \lambda_{ijk} u_k$$

fulfill the following conditions:

- $\lambda_{111} = 1$
- $\lambda_{1jk} = 0$ for all $j > k$
- $\lambda_{ijk} = 0$ for all $i, j > 1$ and $k \leq \max\{i, j\}$

Prove that all genetic algebras are baric algebras.

(f) Show that A is no genetic algebra.

(g) Determine all idempotent elements of A. Is there a basis of A with idempotent elements?

Reason: Algebras, from a biological point of view.

Solution:

(a)

$$\begin{aligned}
(x\cdot x)\cdot y &= \left(\frac{3}{4}x+\frac{3}{16}y+\frac{1}{16}z\right)y\\
&= \frac{3}{8}x+\frac{9}{32}y+\frac{3}{32}z+\frac{9}{64}y+\frac{3}{64}z+\frac{1}{32}y+\frac{1}{32}z\\
&= \frac{3}{8}x+\frac{29}{64}y+\frac{11}{64}z\\
x\cdot(x\cdot y) &= x\cdot\left(\frac{1}{2}x+\frac{3}{8}y+\frac{1}{8}z\right)\\
&= \frac{3}{8}x+\frac{3}{32}y+\frac{1}{32}z+\frac{3}{16}x+\frac{9}{64}y+\frac{3}{64}z+\frac{1}{16}x+\frac{1}{16}z\\
&= \frac{5}{8}x+\frac{15}{64}y+\frac{9}{64}z
\end{aligned}$$

hence A is not associative.

(b) We define $\omega(x)=\omega(y)=\omega(z)=1$ and observe, that the sums of coefficients on the right hand sides of our multiplication table are all equal to one, i.e. ω is an algebra homomorphism. Per definition it is not the zero homomorphism.

(c) We set $a:=x-z$, $b:=y-z$, $c:=z$, such that $\ker\omega=\mathbb{R}a+\mathbb{R}b$ and $\omega(c)=1$. The new multiplication table then is

$$\begin{aligned}
a^2 &= -\frac{1}{4}\,a+\frac{3}{16}\,b & a\cdot b &= -\frac{1}{8}\,b\\
b^2 &= -\frac{1}{4}\,b & a\cdot c &= \frac{1}{2}\,a\\
c^2 &= c & b\cdot c &= \frac{1}{2}\,b
\end{aligned}$$

(d) $N:=\ker\omega$ is a proper ideal of A with codimension 1. Since $A^2=A$ we have $A^2\nsubseteq N$.

(e) Let B be a genetic algebra with basis $\{\,u_k\,\}$ and structure constants $\{\,\lambda_{ijk}\,\}$ and

$$\omega\left(\sum_{k=1}^{n}\mu_k u_k\right)=\mu_1$$

Hence

$$\begin{aligned}\omega(\mu_i)\cdot\omega(\nu_j) &= \mu_1\cdot\nu_1\\ \omega(\mu_i\cdot\nu_j) &= \omega\left(\sum_i \mu_i \sum_j \nu_j \sum_k \lambda_{ijk}\right)\\ &= \sum_i \mu_i \sum_j \nu_j\cdot\lambda_{ij1}\\ &= \sum_i \mu_i\cdot\nu_1\cdot\lambda_{i11}\\ &= \mu_1\cdot\nu_1\cdot\lambda_{111}\\ &= \mu_1\cdot\nu_1\end{aligned}$$

(f) From

$$(\alpha a+\beta b+\gamma c)^2=\left(-\frac{1}{4}\alpha^2+\alpha\gamma\right)a+\left(\frac{3}{16}\alpha^2-\frac{1}{4}\beta^2-\frac{1}{4}\alpha\beta+\beta\gamma\right)b+\gamma^2 c$$

we get for $(\alpha a+\beta b+\gamma c)^2=0$ successively $\gamma=0,\alpha=0,\beta=0$, i.e. $0\in A$ is the only element whose square vanishes. On the other hand, we have for the element $u_n\in B-\{0\}$ that $u_n^2=0$ in any genetic algebra B, hence A can't be one.

(g) With the same calculation as before, we get from $(\alpha a+\beta b+\gamma c)^2=\alpha a+\beta b+\gamma c$ the following cases:

i. $\gamma^2=\gamma\Longrightarrow\gamma\in\{0,1\}$
ii. $\alpha=0\,,\,\beta\neq 0\Longrightarrow -\frac{1}{4}\beta+\gamma=1\Longrightarrow\gamma=0\,,\,\beta=-4$
iii. $\alpha\neq 0\Longrightarrow -\frac{1}{4}\alpha+\gamma=1\Longrightarrow\gamma=0\,,\,\alpha=-4\Longrightarrow\beta^2=12$

The set of all idempotent elements of A is therefore

$$\{\,0\,,\,c\,,\,-4b\,,\,-4a\pm 2\sqrt{3}b\,\}$$

which spans the entire algebra. Note that this doesn't mean, that A is a Boolean algebra, since not every element is idempotent.

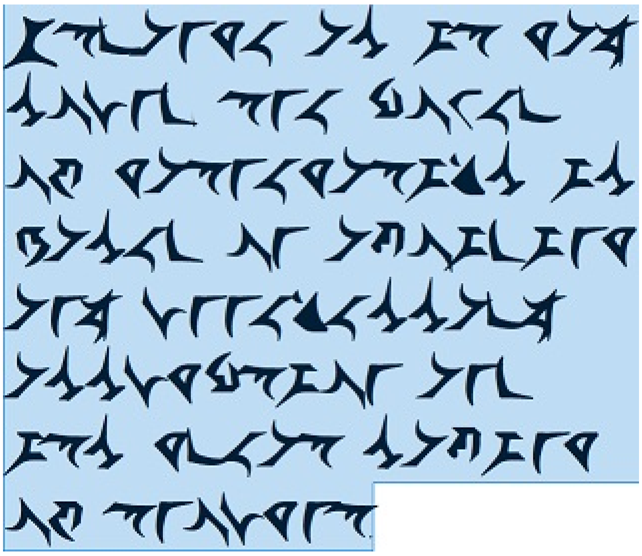
2. Prove that starting with $\frac{1}{1}$ the following binary tree

$$\begin{array}{ccc} & \frac{a}{b} & \\ \swarrow & & \searrow \\ \frac{a}{a+b} & & \frac{a+b}{b}\end{array}$$

defines a counting of all positive rational numbers without repetition and all quotients canceled out.

Reason: Calkin - Wilf counting.

Solution: We define a norm of these elements by $N(p/q) = p+q$. The parent quotient of $\frac{p}{q} \neq \frac{1}{1}$ is either $\frac{p}{q-p}$ or $\frac{p-q}{q}$. The norm of the child has strictly increased in both cases.
Assume we have uncanceled quotients of value $\frac{p}{q}$. Then there is one among them of minimal norm. If $d > 1$ is a common divisor of p, q, then d divides the two possible parent knots, too, which contradicts minimality.
Assume we had more than one knot of value $\frac{p}{q}$. Then there is one among them of minimal norm. However, its possible parent knots would occur more than once as well; again with a smaller norm. (This also follows from the previous step, as only canceled quotients can occur.)
Let $\frac{p}{q}$ be a quotient that doesn't occur. Again by minimality this quotient wouldn't have a parent knot of smaller norm.

3. Who said what here: 
   Reason: Decryption.

   Solution: "Strange as it may sound, the power of mathematics is based on avoiding any unnecessary assumption and its great saving of thought." (Ernst Mach, physicist)

4. Calculate $I := \int_0^\infty \dfrac{\sqrt{x^{e-2}}}{x^e+1}\,dx$

   Reason: Interesting Integration Trick.

   Solution: First we get rid of the inconvenient denominator, so for

$x \geq 0$ we have

$$\int_0^\infty e^{(-x^e-1)y}dy = \left[-\frac{e^{(-x^e-1)y}}{x^e+1}\right]_{y=0}^{y=\infty} = \frac{1}{x^e+1}$$

and for our integral $I = \int_0^\infty \int_0^\infty x^{\frac{e}{2}-1}e^{-x^e y}e^{-y}\,dy\,dx$. In the next step, we clean up the powers of the exponential function, that is we substitute $z = x^e y$ and get

$$\begin{aligned}
I &= \int_0^\infty \int_0^\infty e^{-z}e^{-y}x^{\frac{e}{2}-1}\frac{1}{e}y^{-1}x^{1-e}\,dz\,dx \\
&= \frac{1}{e}\int_0^\infty \int_0^\infty e^{-z}e^{-y}y^{-1}x^{-\frac{e}{2}}\,dz\,dx \\
&= \frac{1}{e}\int_0^\infty \int_0^\infty e^{-z}e^{-y}\sqrt{\frac{y}{z}}y^{-1}\,dz\,dx \\
&= \frac{1}{e}\int_0^\infty z^{-\frac{1}{2}}e^{-z}\,dz \int_0^\infty y^{-\frac{1}{2}}e^{-y}\,dy \\
&= \frac{1}{e}\,\Gamma\left(\frac{1}{2}\right)^2 \\
&= \frac{\pi}{e}
\end{aligned}$$

5. An algebra A is a vector space with a binary distributive multiplication. An example are group algebras, i.e. the distributive extension of the formal basis vectors $g \in G$ such as

   $A := \mathbb{R}[S_3] = \mathbb{R}\cdot(1)+\mathbb{R}\cdot(12)+\mathbb{R}\cdot(13)+\mathbb{R}\cdot(23)+\mathbb{R}\cdot(123)+\mathbb{R}\cdot(132)$

   (a) Find the center $Z(A) = \{\, z \in A \,|\, zv = vz \text{ for all } v \in A\,\}$ of A, and (b) determine the structure of A, i.e. its decomposition into direct factors and the corresponding isomorphisms.

   Reason: Group Algebras.

   Solution: We can identify an element $v = \sum_{\sigma\in S_3} v_\sigma \cdot \sigma \in A$ with the function $v : S_3 \longrightarrow \mathbb{R}$ given by $v(\sigma) = v_\sigma$. Multiplication can then be written as

$$(vw)(\sigma) = \sum_{\alpha\cdot\beta=\sigma} v_\alpha w_\beta = \sum_{\alpha\in S_3} v_\alpha w_{\alpha^{-1}\sigma} = \sum_{\alpha\in S_3} v_{\sigma\alpha^{-1}} w_\alpha$$

and for the function $A \supset S_3 \ni \alpha \longleftrightarrow \chi_\alpha\,:\,\sigma \longmapsto \delta_{\alpha\sigma}$

$$(\chi_\alpha \cdot v)(\sigma) = \sum_{\beta\in S_3} \chi_\alpha(\beta)v(\beta^{-1}\sigma) = v(\alpha^{-1}\sigma)$$
$$(v \cdot \chi_\alpha)(\sigma) = \sum_{\beta\in S_3} v(\sigma\beta^{-1})\chi_\alpha(\beta) = v(\sigma\alpha^{-1})$$

If $v \in Z(A)$, then $v(\alpha^{-1}\beta\alpha) = (\chi_\alpha v)(\beta\alpha) = (v\chi_\alpha)(\beta\alpha) = v(\beta\alpha\alpha^{-1}) = v(\beta)$ and vice versa if $v(\alpha^{-1}\beta\alpha) = v(\beta)$ then $[\chi_\alpha, v] = 1$ and $v \in Z(A)$. Hence v is a central element if and only if it is constant on the conjugacy classes $\beta^{S_3} = \{\,\alpha^{-1}\beta\alpha \mid \alpha \in S_3\,\}$ of S_3. Since conjugation doesn't change the cycle length, we have three conjugacy classes $\{\,(1)\,\}\,,\ \{\,(12),(13),(23)\,\}\,,\ \{\,(123),(132)\,\}$ and $|Z(A)| = 3\,$. Thus

$$Z(A) = Z(\mathbb{R}[S_3]) = \mathbb{R}\cdot(1) + \mathbb{R}\cdot((12)+(13)+(23)) + \mathbb{R}\cdot((123)+(132))$$

Group algebras and all their modules are semisimple by Maschke's theorem, and the theorem of Wedderburn and Artin states, that semisimple algebras are direct sums of full, simple matrix algebras $\mathbb{M}(n, D)$ over a division ring D which in our case are the real numbers $D = \mathbb{R}$. Since we always have the trivial module $A.m = m$, we always have the trivial component $\mathbb{M}(1,\mathbb{R})$ as direct factor of A. A comparison of dimensions yields $6 = 1+n_1^2+\ldots+n_s^2$ and so $6 = 1+1+1+1+1+1$ or $6 = 1+1+2^2$ as only possibilities. In the first case, we would have $A \cong \mathbb{R}^6$ which isn't possible, as A is non Abelian, so that

$$A \cong \mathbb{R} \times \mathbb{R} \times \mathbb{M}(2,\mathbb{R})$$

is the only decomposition into simple factors possible. This also fits to our previous result, that $Z(A) \cong \mathbb{R}^3$.

The corresponding representations of S_3 on $\mathbb{R}^3$ are given by

$$\begin{aligned}
\pi_1(\sigma) &:= \mathrm{id}_{\mathbb{R}^3}\\
\pi_2(\sigma) &:= (-1)^{\operatorname{sgn}(\sigma)} \cdot \mathrm{id}_{\mathbb{R}^3}\\
\pi_3((1)) &:= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \pi_3((12)) &:= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\\
\pi_3((13)) &:= \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} & \pi_3((23)) &:= \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}\\
\pi_3((123)) &:= \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} & \pi_3((132)) &:= \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}
\end{aligned}$$

where π_3 is restricted to $\mathbb{R}^2 = \{\,(x_1,x_2,x_3) \mid x_1+x_2+x_3 = 0\,\} \subseteq \mathbb{R}^3$.

6. Let $B \subseteq \mathbb{R}^n$ be measurable and $P = (a_1, \ldots, a_n, b) \in \mathbb{R}^{n+1}$ a point with $b > 0$ and $C_B = \{\, P + t(Q - P) \,|\, Q \in B \times \{0\}_{n+1}\, ,\, t \in [0, 1] \,\}$ the cone above the basis B with the peak P . Prove the measure formula

$$\lambda^{n+1}(C_B) = \frac{b}{n+1} \cdot \lambda^n(B)$$

Reason: Integration Transformation Theorem.

Solution: Define

$$\begin{aligned} \varphi \,:\, \mathbb{R}^n \times [0, b] &\longrightarrow \mathbb{R}^n \times [0, b] \\ (x_1, \ldots, x_n, t) &\longmapsto (x_1, \ldots, x_n, 0) + \frac{t}{b}\,(a_1 - x_1, \ldots, a_n - x_n, b) \end{aligned}$$

and observe that φ is a bijection on $\mathbb{R}^n \times [0, b)$ with

$$D\varphi = \begin{bmatrix} 1 - \frac{t}{b} & 0 & \cdots & 0 & \frac{a_1}{b} \\ 0 & 1 - \frac{t}{b} & \cdots & 0 & \frac{a_2}{b} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 - \frac{t}{b} & \frac{a_n}{b} \\ 0 & 0 & \vdots & 0 & 1 \end{bmatrix}$$

with determinant $|D\varphi| = \left(1 - \dfrac{t}{b}\right)^n = \dfrac{1}{b^n}\,(b-t)^n$ and φ is a diffeomorphism on $\mathbb{R}^n \times (0, b)$ with

$$\varphi\,(B \times (0, b)) = C_B - \{\, P, B \,\}$$

which are both P, B of Lebesgue measure zero. Thus we can apply the transformation theorem for integrals

$$\lambda^n(\varphi(S)) = \int_S |\, D\varphi \,| \; d\lambda^n$$

and get

$$\begin{aligned} \lambda^{n+1}(C_B) &= \int_{B \times [0,b]} \frac{1}{b^n} |b - t|^n \, d\lambda^{n+1} \\ &= \frac{1}{b^n} \cdot \int_B 1 \, d\lambda^n \cdot \int_0^b |b - t|^n \, dt \\ &= \frac{1}{b^n} \cdot \lambda^n(B) \cdot \int_0^b u^n \, du \\ &= \frac{1}{b^n} \cdot \lambda^n(B) \cdot \frac{1}{n+1} \cdot b^{n+1} \\ &= \frac{b}{n+1} \cdot \lambda^n(B) \end{aligned}$$

7. Show that
$$x \cdot y = \frac{2xy - x - y}{xy - 1}$$
defines a one dimensional, real, local Lie group G around $0 \in \mathbb{R}$ and compute the vector field of left multiplication by an element $g \in \mathbb{R}$.

Reason: Lie Groups.

Solution: The neutral element of G is 0 and the inverse $x^{-1} = \dfrac{x}{2x} - 1$ which can be verified along with associativity by simple calculations. The operations are also well-defined on the open sets $U = \{\, x \in \mathbb{R} \, : \, |x| < 1 \,\}$ and $U_0 = \{\, x \in \mathbb{R} \, : \, |x| < \frac{1}{2} \,\}$ for the inversion. The group operations are also analytic on suited neighborhoods of 0, so G is actually a Lie group. For the left multiplication $L_g \, : \, x \longmapsto g \cdot x$ we get
$$DL_g(x_0) = \left.\frac{d}{dx}\right|_{x=x_0} L_g(x) = \frac{(g-1)^2}{(gx_0 - 1)^2}$$

8. (HS-1) Two numbers a, b are called amicable, if the sum of all proper divisors of one is the other number (1 is included). The smallest example is
$$(a, b) = (220, 284) = (1+2+4+71+142, 1+2+4+5+10+11+20+22+44+55+110)$$
Let $n \in \mathbb{N}$ and $(x, y, z) = (3 \cdot 2^n - 1\, ,\, 3 \cdot 2^{n-1} - 1\, ,\, 9 \cdot 2^{2n-1} - 1)$. Prove that if x, y, z are all odd primes, then $(a, b) = (2^n \cdot x \cdot y\, ,\, 2^n \cdot z)$ are amicable numbers.

Hint: First find a formula for the sum of all divisors $\sigma(n)$ given the prime decomposition of n.

Reason: Theorem of Thabit Ibn Qurra. (9th century, Mesopotamia)

Solution: For $n = p_1^{k_1} \cdots p_r^{k_r}$ then the sum of all divisors is
$$\sigma(n) = \prod_{i=1}^{r} \frac{p_i^{k_i+1} - 1}{p_i - 1}$$

$$\begin{aligned}
\sigma(a) - a &= \sigma(2^n \cdot x \cdot y) - 2^n \cdot x \cdot y \\
&= \left(2^{n+1} - 1\right)(x+1)(y+1) - 2^n xy \\
&= \left(2^{n+1} - 1\right)(3 \cdot 2^n)(3 \cdot 2^{n-1}) - 2^n(3 \cdot 2^n - 1)(3 \cdot 2^{n-1} - 1) \\
&= \left(2^{n+1} - 1\right) \cdot 9 \cdot 2^{2n-1} - 2^n \left(9 \cdot 2^{2n-1} - 9 \cdot 2^{n-1} + 1\right) \\
&= 2^n \cdot \left(9 \cdot 2^{2n} - 9 \cdot 2^{n-1} - 9 \cdot 2^{2n-1} + 9 \cdot 2^{n-1} - 1\right) \\
&= 2^n \cdot \left(9 \cdot 2^{2n-1} - 1\right) \\
&= 2^n \cdot z \\
&= b
\end{aligned}$$

and by an analogue calculation $\sigma(b) - b = a$.

9. (HS-2) A number is called perfect, if it equals the sum of all its divisors except itself, e.g. $6 = 1 + 2 + 3$ and $28 = 1 + 2 + 4 + 7 + 14$ are perfect. If $2^k - 1$ is a prime number, then $2^{k-1}(2^k - 1)$ is a perfect number and every even perfect number has this form.

   Reason: Mersenne Numbers.

   Solution: Let $n = 2^{k-1}(2^k - 1)$ and $p = 2^k - 1$ prime. Then

$$\begin{aligned}
\sigma(n) &= (2^k - 1) \cdot \frac{p^2 - 1}{p - 1} \\
&= (2^k - 1) \cdot \frac{2^{2k} - 2^{k+1}}{2^k - 2} \\
&= (2^k - 1) \cdot 2^k \cdot 1 \\
&= 2 \cdot 2^{k-1} \cdot (2^k - 1) \\
&= 2n
\end{aligned}$$

   and n is perfect.
   If on the other hand $n = 2^{k-1}m$ is an even perfect number, $k > 1$ and m is odd, then

$$\begin{aligned}
\sigma(n) =& (2^k - 1) \cdot \sigma(m) \\
&= 2n \\
&= 2^k m
\end{aligned}$$

and $(2^k-1)\,|\,m$, say $(2^k-1)M=m$. Hence

$$\begin{aligned}\sigma(m) &= \frac{2^k m}{2^k-1} \\ &= \frac{2^k(2^k-1)M}{2^k-1} \\ &= 2^k M \\ &\geq m+M \\ &= (2^k-1)M+M \\ &= 2^k M\end{aligned}$$

since both m, M divide m. Thus equality holds everywhere and m, M are the only divisors of m, i.e. $m=(2^k-1)M$ is prime. As $k>1$ this is only possible, if $M=1$ and $m=2^k-1$ is of the desired form.

Numbers 2^k-1 are called **Mersenne numbers** and primes 2^k-1 **Mersenne primes**, in which case k has to be prime, too. It is unclear (but suspected), whether there are infinitely many Mersenne primes. The highest known number is currently $2^{82,589,933}-1$ with $24,862,048$ digits. It is also unclear whether there are infinitely many perfect numbers.

10. (HS-3)

    (a) What is the smallest five-digit number n such that n and $2n$ together contain all 10 digits from 0 to 9?

    (b) On how many zeros does the number $1000!$ end?

    (c) For which six-digit number $ABCDEF$ do we have:
    $ABCDEF\cdot 1 = ABCDEF$
    $ABCDEF\cdot 3 = BCDEFA$
    $ABCDEF\cdot 2 = CDEFAB$
    $ABCDEF\cdot 6 = DEFABC$
    $ABCDEF\cdot 4 = EFABCD$
    $ABCDEF\cdot 5 = FABCDE$

    Reason: Number Puzzle.

    Solution:

    (a) $n = 13485$ with $2n = 26970$. Other solutions are e.g. $n = 13548, 13845$ which are bigger.

(b) We have as many zeros at the end as there are factors 5, so $1000/5 + 1000/5^2 + 1000/5^3 + \lfloor 1000/5^4 \rfloor = 200 + 40 + 8 + 1 = 249$.

(c) If σ notes the cyclic shift by one digit ($\sigma(ABCDEF) = BCDEFA$) we get with $x = ABCDEF$

$$x \cdot 10^k \equiv \sigma^k(x) \bmod 7$$

i.e. σ acts like the multiplication by 10 in $\mathbb{Z}_7$, $10 : 7 = 1.42857$, and 142857 is the solution.

5 February 2019

1. A little number theory.

   - Compute the last three digits of 3^{2405} .
   - Show that there is an integer $a \in \mathbb{Z}$ such that $64959 \mid (a^2 - 7)$.

   Reason: Practice for computer science.

   Solution:

   (a) We need the result of $3^{2405} \equiv x \bmod 1000$ to compute the last three digits. Since 3 and 1000 are coprime, we have $3^{\varphi(1000)} \equiv 1 \bmod 1000$ by Euler's theorem. Now

   $$\varphi(10^3) = \varphi(8)\varphi(125) = \varphi(2^3)\varphi(5^3) = 2^3\left(1 - \frac{1}{2}\right)5^3\left(1 - \frac{1}{5}\right) = 400$$

   Thus we have $3^{2405} = \left(3^{400}\right)^6 \cdot 3^5 \equiv 1^6 \cdot 243 \equiv 243 \bmod 1000$ as the last three digits.

   (b) It is $64959 = 59 \cdot 1101 = 59 \cdot 3 \cdot 367$.

   - $\left(\frac{7}{3}\right) = \left(\frac{1}{3}\right) = 1$
   - $\left(\frac{7}{59}\right) = -\left(\frac{59}{7}\right) = -\left(\frac{3}{7}\right) = \left(\frac{7}{3}\right) = \left(\frac{1}{3}\right) = 1$
   - $\left(\frac{7}{367}\right) = -\left(\frac{367}{7}\right) = -\left(\frac{3}{7}\right) = \left(\frac{7}{3}\right) = \left(\frac{1}{3}\right) = 1$

   Hence there are integers a_1, a_2, a_3 with $a_1^2 \equiv 7 \bmod 3$, $a_2^2 \equiv 7 \bmod 59$, $a_3^2 \equiv 7 \bmod 367$ and by the Chinese remainder theorem an integer a such that $a \equiv a_1 \bmod 3$, $a \equiv a_2 \bmod 59$, $a \equiv a_3 \bmod 367$. This is still true for the squared equations $a^2 \equiv a_1^2 \bmod 3$, $a^2 \equiv a_2^2 \bmod 59$, $a^2 \equiv a_3^2 \bmod 367$ so again by the Chinese remainder theorem $a^2 \equiv 7 \bmod 64959$.

2. Let $f(x) = \dfrac{(\cos\varphi - \sqrt{3}\sin\varphi + 1)x + 2\sqrt{3}\sin\varphi}{x^2}$
   and $g(x) = \dfrac{(\cos\varphi - \sqrt{3}\sin\varphi - 1)x + 2\sqrt{3}\sin\varphi}{x^2}$.

   For which values of φ are $f \perp g$ in $L^2([1,\infty))$?

   Reason: Thales.

   Solution: The norm in $L^2([1,\infty))$ is defined by the inner product

$||h(x)||^2 = \langle h(x), h(x)\rangle = \int_1^\infty h(x)^2dx$. We define $p(x) = x^{-1}$ and $q(x) = \sqrt{3}(2-x)x^{-2}$. Then $\{\,p, q\,\}$ define a orthonormal basis of the subspace V they span in $L^2([1,\infty))$. As f, g can be written as

$$f(x) = p(x)+p(x)\cos\varphi+q(x)\sin\varphi\,,\; g(x) = -p(x)+p(x)\cos\varphi+q(x)\sin\varphi$$

which means they point to the same point on the unit circle of V from the left and from the right intersection with the diameter, the statement follows by the theorem of Thales, i.e. all values of φ fulfill the condition.

3. (HS-1) A man wants to figure out the length of an escalator, i.e. the number of steps [N] if it was out of order. Since it wasn't out of order, he counted 60 steps if he walks with the stairs and 90 steps if he walks in the opposite direction. What is [N]?

   Solution: Let's measure velocity in steps per second and distance in steps. Let v_M be the man's velocity and v_T the escalator's. We have two equations for the distance:

   $$x(t) = (v_M + v_T)\cdot\frac{60N}{v_M} = (v_M - v_T)\cdot\frac{90N}{v_M}$$

   which is $v_M = 5v_T$ and $x(t) = 360N\cdot\dfrac{v_T}{v_M} = 72N$.

4. (HS-2) We are looking for a number with eight digits: two of each 1,2,3,4. The ones are separated by one other number, the twos by two, the threes by three, and the fours by four other numbers.

   Solution: 23421314 or backwards 41312432.

5. (HS-3) Which of you four threw the ball in my window? A says: It was E. E says: It was G. F says: It was not me. G says: E lied a.) If only one of the four lied, who threw the ball? b.) If only one person has told the truth, who was the culprit?

   Solution: If only one lied, then E was the culprit If only one told the truth, then F was the culprit.

6. (HS-4) Choose any two but different natural numbers and form their sum, their difference and product. Prove that among these three numbers at least one is divisible by 3.

   Solution: If one of the two numbers is divisible by 3, so is the product. If the two numbers divided by 3 have the same remainder, then their difference is divisible by 3. If a number divided by 3 leaves the remainder 1, the other the remainder 2, then their sum is a multiple of 3.

7. (HS-5) Prove that the remainder in dividing any prime by 30 is either 1 or prime again. Is this also true when dividing a prime number by 60?

   Solution: Every prime number p can be written as $p = 30q + r$, q and r are natural numbers with $1 \leq r \leq 29$. For all numbers r divisible by $2, 3$, or 5 then $p = 30q + r$ is not prime. Therefore only $1, 7, 11, 13, 17, 19, 23, 29$ are possible remainders.
   Now let $p = 60q + r$ with $1 \leq r \leq 59$. Since the prime 109 is in the form $109 = 60 \cdot 1 + 49$ and 49 is not prime, the statement does not hold for 60.

6 January 2019

1. Given the surface

$$f(t, \varphi) = ((1 + t^2)\cos\varphi\ ,\ (1 + t^2)\sin\varphi\ ,\ t) \quad (t \in \mathbb{R}\ ,\ 0 \leq \varphi \leq 2\pi)$$

   (a) Compute the first fundamental form of this surface.

   (b) Compute the second fundamental form and the Gauss curvature of this surface.

   (c) Compute the geodesic curvature κ_g and the normal curvature κ_n of the circular latitude at $t = 1$.

   Only solutions to all three parts will be accepted.

   Reason: Curvatures.

   Solution:

   (a) $f_t = (2t\cos\varphi, 2t\sin\varphi, 1)\ ,\ f_\varphi = (-(1 + t^2)\sin\varphi, (1 + t^2)\cos\varphi, 0)$

$$\begin{aligned} I(af_t + bf_\varphi, cf_t + df_\varphi) &= \langle (a, b)^\tau, (c, d)^\tau \rangle \\ &= (a, b) \begin{bmatrix} A & B \\ B & C \end{bmatrix} (c, d)^\tau \\ &= ac\langle f_t, f_t \rangle + (ad + bc)\langle f_t, f_\varphi \rangle + bd\langle f_\varphi, f_\varphi \rangle \\ &= ac \cdot A + (ad + bc) \cdot B + bd \cdot C \\ &= ac(1 + 4t^2) + (ad + bc)(0) + bd((1 + t^2)^2) \end{aligned}$$

and $g(t,\varphi) = I = \begin{bmatrix} 1+4t^2 & 0 \\ 0 & (1+t^2)^2 \end{bmatrix}$

(b) $\vec{n} = \dfrac{f_t \times f_\varphi}{||f_t \times f_\varphi||} = \dfrac{1}{\sqrt{1+4t^2}} \cdot (-\cos\varphi, -\sin\varphi, 2t)^\tau$

$$\begin{aligned} f_{tt} &= (2\cos\varphi, 2\sin\varphi, 0)^\tau \\ f_{t\varphi} &= (-2t\sin\varphi, 2t\cos\varphi, 0)^\tau \\ f_{\varphi\varphi} &= (-(1+t^2)\cos\varphi, -(1+t^2)\sin\varphi, 0)^\tau \end{aligned}$$

$$h(t,\varphi) = II = \begin{bmatrix} f_{tt}\cdot\vec{n} & f_{t\varphi}\cdot\vec{n} \\ f_{\varphi t}\cdot\vec{n} & f_{\varphi\varphi}\cdot\vec{n} \end{bmatrix} = \frac{1}{\sqrt{1+4t^2}} \begin{bmatrix} -2 & 0 \\ 0 & 1+t^2 \end{bmatrix}$$

and $\kappa_G(t,\varphi) = \dfrac{\det h}{\det g} = \dfrac{-2}{(1+4t^2)^2(1+t^2)}$

(c) The circular latitude at $t=1$ is $c(\varphi) = f(1,\varphi) = (2\cos\varphi, 2\sin\varphi, 1)^\tau$ which is a circle with radius $R=2$ and so its curvature κ_R is

$$\kappa(\varphi) = \frac{||c'(\varphi)\times c''(\varphi)||}{||c'(\varphi)||^3} = \frac{4}{8} = \frac{1}{2} = \frac{1}{R}$$

For the normal curvature we get

$$\kappa_n = \frac{h(c'(\varphi), c'(\varphi))}{g(c'(\varphi), c'(\varphi))} = \frac{(0,1)\, II_{t=1}(0,1)^\tau}{(0,1)\, I_{t=1}(0,1)^\tau} = \frac{2}{\sqrt{5}}\cdot\frac{1}{4} = \frac{1}{2\sqrt{5}}$$

and the geodesic curvature is

$$\kappa_g = \sqrt{\kappa_R^2 - \kappa_n^2} = \sqrt{\frac{1}{4} - \frac{1}{20}} = \frac{1}{\sqrt{5}}$$

2. Three pirates are stranded on an island and find that there are only a few monkeys besides drinking water and coconuts. After collecting coconuts for a whole day, they want share them the next morning. At night, one of the pirates awakes and hides his third of the coconuts. But since an odd number of nuts is left, he gives one to a monkey. The second pirate awakens shortly afterwards and hides his third of the remaining coconuts. Again an odd number of coconuts remains, so he gives one to a monkey. The third does the same thing a short time later and gives a leftover nut to a monkey. The next morning they divided the few remaining coconuts among each other. Now the question: How many coconuts did the three pirates at least collect the

day before and how are they distributed on each?

Reason: Riddle.

Solution: We have to solve $N_{k+1} = \frac{2}{3}N_k - 1$ for $k = 1, 2, 3$ and $N_3 = 3R$ where N_0 is the number of coconuts collected and R the remaining share for each in the morning. Solving this recursion results in

$$3R = \frac{2^3}{3^3}\left(N_0 - \sum_{k=0}^{2}(\frac{2}{3})^k\right) \Longleftrightarrow N_0 = \frac{1}{8}(81 \cdot R + 57) \in \mathbb{Z}$$

The smallest solution to $81R + 57 \equiv R + 1 \equiv 0 \bmod 8$ is $R = 7$ which yields $N_0 = 78$. The first pirate receives 33, the second 24, the third 18, and the monkeys 3 coconuts.

3. A cyclist drives along a railway track. Every 30 minutes, he is overtaken by a train and every 20 minutes he is met by a train. At which frequency do the trains travel on this connection?

   Reason: Riddle.

   Solution: Imagine she rides one hour in one direction and one hour in the other. Then she meets three trains in the first hour and is overtaken by two in the second hour. So the frequency is thus 5 trains per direction in 120 minutes, i.e. every 24 minutes a train.

4. The Heisenberg group $H = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \, : \, a,b,c \in \mathbb{Z}^3 \right\}$ operates discontinuously on $\mathbb{R}^3$ by

$$h(p) = h(x,y,z) = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ z+ay+c \end{bmatrix}$$

   Show that the Heisenberg manifold $\mathbb{R}^3/H$ is orientable.

   Reason: Manifolds.

   Solution: A manifold is orientable, if and only if there is an atlas, such that for all charts $(U,\varphi), (V,\psi)$ with a nonempty intersection and all points p in the domain of $\varphi \circ \psi^{-1}$

$$\det(D_p(\varphi \circ \psi^{-1})) > 0$$

Therefore we get with the trivial atlas $\psi = id_{\mathbb{R}^3}$ on $\mathbb{R}^3$:

$$\det(D_p(\varphi\circ\psi^{-1})) = \det(D_p(\psi\circ h\circ\psi^{-1}(\psi(p))) = \det D_p(h) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{bmatrix} = 1$$

5. Solve $x'(t) = \dfrac{2t+2x(t)}{3t+x(t)}$, $x(2) = 0$.

   Reason: Initial Value Problem.

   Solution: With $y(t) = \dfrac{x(t)}{t}$ we get $x' = \dfrac{2+2y}{3+y}$ and $ty' = x' - y$. Hence

$$\begin{aligned} ty' &= \frac{2+2y}{3+y} - y = \frac{-y^2-y+2}{3+y} = t \cdot \frac{dy}{dt} \\ \frac{dt}{t} &= \frac{-3-y}{y^2+y-2} \cdot dy = -\frac{4}{3} \cdot \frac{1}{y-1}\, dy + \frac{1}{3} \cdot \frac{1}{y+2}\, dy \end{aligned}$$

   and $\log|t| = \frac{1}{3}\log|y+2| - \frac{4}{3}\log|y-1| + C$ or

$$t^3 = C \cdot \frac{y+2}{(y-1)^4} \Longleftrightarrow (x-t)^4 = C \cdot (x+2t) \text{ and } C = 4$$

6. Show that $T : C([1,2]) \longrightarrow C([1,2])$ defined by

$$T(y)(t) := 1 + \int_1^t \frac{y(s)}{2s}\, ds$$

   has at least one fixed point and determine them.

   Reason: Fixed points.

   Solution:

$$\begin{aligned} |T(y)(t) - T(z)(t)| &\leq \int_1^t \frac{2s}{|y(s)-z(s)|}\, ds \leq \int_1^t \frac{||y-z||_\infty}{2} \\ &\leq \frac{t-1}{2}||y-z||_\infty \leq \frac{1}{2}||y-z||_\infty \end{aligned}$$

   Differentiation of $Ty = y$ yields $y'(t) = \dfrac{y(t)}{2t}$ or $y(t) = C \cdot \sqrt{t}$. Hence $C \cdot \sqrt{t} = y = Ty = 1 + \dfrac{C}{2}\displaystyle\int_1^t \frac{1}{\sqrt{s}}\, ds = 1 + C \cdot (\sqrt{t} - 1)$ and thus $C = 1$. The only fixed point of T is $y(t) = \sqrt{t}$.

7. Compute $\exp(tA)$ where $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ and determine the behavior of $\det(\exp(tA))$ for $t \to \pm\infty$.

   Reason: Matrix Exponentiation.

   Solution: With $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = E_{13} - E_{31}$ we have $A = 1 + B$. Since $[1, B] = 0$ we get

$$\exp(tA) = \exp(t \cdot 1)\exp(tB) = e^t \cdot \exp(tE_{13} - tE_{31})$$

$$\begin{aligned}
(tE_{13} - tE_{31})^0 &= t^0(E_{11} + E_{22} + E_{33}) \\
(tE_{13} - tE_{31})^1 &= t^1(E_{13} - tE_{31}) \\
(tE_{13} - tE_{31})^2 &= t^2(-E_{11} - E_{33}) \\
(tE_{13} - tE_{31})^3 &= t^3(-E_{13} + E_{31}) \\
(tE_{13} - tE_{31})^4 &= t^4(E_{11} + E_{33}) \\
(tE_{13} - tE_{31})^5 &= t^5(E_{13} - E_{31}) \\
&\cdots
\end{aligned}$$

   which is cyclic of order four in the matrix component and $n \longmapsto t^n$ for the factor. If we now add the separate positions divided by $n!$

$$\begin{aligned}
(1,1) &: 1 - \frac{t^2}{2!} + \frac{t^4}{4!} \mp \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} = \cos t \\
(1,3) &: t - \frac{t^3}{3!} + \frac{t^5}{5!} \mp \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!} = \sin t \\
(2,2) &: 1 \\
(3,1) &: -t + \frac{t^3}{3!} - \frac{t^5}{5!} \pm \ldots = -\sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!} = -\sin t \\
(3,3) &: 1 - \frac{t^2}{2!} + \frac{t^4}{4!} \mp \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} = \cos t
\end{aligned}$$

   then $\exp(tB) = \begin{bmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{bmatrix}$ and $\exp(tA) = \begin{bmatrix} e^t \cos t & 0 & e^t \sin t \\ 0 & e^t & 0 \\ -e^t \sin t & 0 & e^t \cos t \end{bmatrix}$

   Thus $\det(\exp(tA)) = e^{\operatorname{tr}(tA)} = e^{3t}$ from which the behavior towards $\pm\infty$ is obvious.

8. Let G be a group generated by $\sigma, \varepsilon, \delta$ with $\sigma^7 = \varepsilon^{11} = \delta^{13} = 1$.

   (a) Show that there is no transitive operation of G on a set with 8 elements.

   (b) Is there are group G with the above properties, that operates transitively on a set with 12 elements?

   Reason: Groups.

   Solution:

   (a) Assume G operates transitively on $M = \{1, 2, \ldots, 8\}$ via $\varphi : G \longrightarrow S_8$. As the order of $\varphi(\varepsilon)$ is a common divisor of 11 and $|S_8| = 8!$, both numbers are coprime and thus $\varphi = 1$. The same argument applies to $\varphi(\delta)$ hence $\varphi(\sigma)$ generates $\varphi(G)$, which is a cyclic group of order 1 or 7. By the orbit-stabilizer theorem and a transitive operation we would have $8 \,|\, |\varphi(\sigma)| = |\varphi(G)| \in \{1, 7\}$ which is impossible.

   (b) Let $\sigma = (1\,2\,3\,4\,5\,6\,7)$ and $\varepsilon = (2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,12)$. Both cycles generate a subgroup $H \leq S_{12}$ which operates transitively on $M = \{1, 2, \ldots 12\}$. Now $(h, z).m := h.m$ is a transitive operation of

   $$G := H \times \mathbb{Z}/13\,\mathbb{Z}$$

   on M, too, and G is generated by $(\sigma, 0)$, $(\varepsilon, 0)$, $(1, 1 + 13\,\mathbb{Z})$.

9. Let R, S be rings and $\varphi : R \longrightarrow S$ a ring epimorphism. Further let $J \subseteq S$ be an ideal.

   (a) Define an ideal $I \subseteq R$ such that $R/I \cong S/J$.

   (b) Is the preimage of the center of S equal to the center of R?

   Reason: Rings.

   Solution:

   (a) Let $\pi : S \twoheadrightarrow S/J$ be the canonical projection. Then $\pi \circ \varphi : R \longrightarrow S/J$ is also surjective and $I := \varphi^{-1}(J) = \ker \pi \circ \varphi$. The statement follows by the homomorphism theorem.

   (b) No. Let $S = \{0\}$. Then $\ker \varphi = R$ which is the center of R if and only if R is commutative. So every non commutative ring provides a counterexample, e.g. a matrix ring.

10. A lie algebra $\mathfrak{g}$ is called reductive, if $\mathfrak{g} = \mathfrak{Z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$ is the direct sum of its center and its derived algebra. (This is an important class of Lie algebras, as they are exactly those whose representations split into a direct sum of irreducible representations. Semisimple and in particular the simple, classical matrix Lie algebras are reductive.)

    Show that the Lie algebra $\mathfrak{gl}(V)$ of all endomorphisms of a finite dimensional complex vector space is reductive.

    Reason: Lie algebras.

    Solution: $\mathfrak{Z}(\mathfrak{g}) = \mathbb{C} \cdot 1$ by Schur's Lemma and we can write every matrix $X \in \mathfrak{gl}(V)$ as $X = c \cdot 1 + S_X$ where $S_X \in \mathfrak{sl}(V)$, the simple Lie algebra of all endomorphisms of V with zero trace. For dimensional reasons, we get

    $$\mathfrak{gl}(V) = \mathbb{C} \cdot 1 \oplus \mathfrak{sl}(V) = \mathfrak{Z}(\mathfrak{g}) \oplus \mathfrak{sl}(V)$$

    Since $\mathfrak{sl}(V)$ is a simple Lie algebra, we have

    $$[\mathfrak{gl}(V), \mathfrak{gl}(V)] = [\mathfrak{sl}(V), \mathfrak{sl}(V)] = \mathfrak{sl}(V)$$

    and thus

    $$\mathfrak{gl}(V) = \mathfrak{Z}(\mathfrak{g}) \oplus [\mathfrak{gl}(V), \mathfrak{gl}(V)]$$