Paul A. Tipler and Gene Mosca, *PHYSICS For Scientists and Engineers*, Sixth Edition
W. H. Freeman & Co., New York, NY, 2008.

Chapter #15 - Problem #71

••• Consider a taut string that has a mass per unit length μ_1 carrying transverse wave pulses of the form $y = f(x - v_1 t)$ that are incident upon a point P where the string connects to a second string with mass per unit length μ_2. Derive $1 = r^2 + \left(\dfrac{v_1}{v_2}\right)\tau^2$ by equating the power incident on point P to the power reflected at P plus the power transmitted at P.

The following is essentially the erroneous solution from the Instructor's Solutions Manual. However, I have also included some of my own commentary in blue. I will also point out where I believe the errors occur in red. In addition, there are actually several obvious typographical errors in their solution. I am correcting those without any comment.

The glaring error I refer to in my original post occurs on the very first line of their solution where they write

$$P_{\rm I} + P_{\rm R} = P_{\rm T} \ , \tag{1}$$

where $P_{\rm I}$ is the power in the incident pulse, $P_{\rm R}$ is the power in the reflected pulse, and $P_{\rm T}$ is the power in the transmitted pulse. This is not only clearly wrong when we're talking about power, but violates the very statement of the problem.

They next write that:
The power transmitted in the direction of increasing x is given by:

$$P = -T\frac{\partial y}{\partial x}\frac{\partial y}{\partial t} \ .$$

This comes from considering that power is given by $P = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}$, where the force in the y-direction is given by $F_y = T\sin\theta$, and for small deformations of the string, $\sin\theta \approx \tan\theta \approx \dfrac{\partial y}{\partial x}$, and the velocity of a point on the string is given by $\dfrac{\partial y}{\partial t}$. Also, T is the tension in the string.

While I have no objection to their expression for the force in the y-direction, they use $F_y = -T\dfrac{\partial y}{\partial x}$ everywhere, whereas I believe that for the backward-traveling reflected pulse, the proper expression should be $F_y = +T\dfrac{\partial y}{\partial x}$.

They then write:
Substituting in Equation (1) yields:

$$\left(-T\frac{\partial y_{\rm I}}{\partial x}\frac{\partial y_{\rm I}}{\partial t}\right) + \left(-T\frac{\partial y_{\rm R}}{\partial x}\frac{\partial y_{\rm R}}{\partial t}\right) = \left(-T\frac{\partial y_{\rm T}}{\partial x}\frac{\partial y_{\rm T}}{\partial t}\right)$$

or upon simplification,

$$\frac{\partial y_{\rm I}}{\partial x}\frac{\partial y_{\rm I}}{\partial t} + \frac{\partial y_{\rm R}}{\partial x}\frac{\partial y_{\rm R}}{\partial t} = \frac{\partial y_{\rm T}}{\partial x}\frac{\partial y_{\rm T}}{\partial t} \ . \tag{2}$$

Because the incident pulse is given by

$$y_{\rm I} = f(x - v_1 t) \ ,$$

the reflected and transmitted pulses are given by

$$y_{\rm R} = r f(-x - v_1 t) \ ,$$

and

$$y_{\rm T} = \tau f\left(\frac{v_1}{v_2}\left[x - v_2 t\right]\right) \ .$$

Now this I totally don't understand. While the functions they give for the reflected and transmitted pulses may indeed be solutions of the wave equation, why would they change the form of the argument? For the reflected pulse, they multiply the argument by a minus-sign, and in the transmitted pulse, they multiply the argument by $\frac{v_1}{v_2}$. What's the motivation for that? What gives them the right to do it? Shouldn't the proper reflected and transmitted pulses just simply be given by $y_{\rm R} = rf(x + v_1 t)$, and $y_{\rm T} = \tau f(x - v_2 t)$? Anyhow, they go on to say:

Evaluating $\frac{\partial y}{\partial x}$ and $\frac{\partial y}{\partial t}$ using the chain rule yields:

$$\frac{\partial y}{\partial x} = \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x}$$

and

$$\frac{\partial y}{\partial t} = \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial t}$$

where η is the argument of the wave function.

For the transmitted pulse:

$$\frac{\partial y_{\rm T}}{\partial x} = \tau\frac{\partial f}{\partial \eta}\frac{\partial}{\partial x}\left(\frac{v_1}{v_2}\left[x - v_2 t\right]\right) = \tau\frac{\partial f}{\partial \eta}\frac{v_1}{v_2} \ ,$$

and

$$\frac{\partial y_{\rm T}}{\partial t} = \tau\frac{\partial f}{\partial \eta}\frac{\partial}{\partial t}\left(\frac{v_1}{v_2}\left[x - v_2 t\right]\right) = \tau\frac{\partial f}{\partial \eta}\left(-v_1\right) = -\tau v_1\frac{\partial f}{\partial \eta} \ .$$

For the reflected pulse:

$$\frac{\partial y_{\rm R}}{\partial x} = r\frac{\partial f}{\partial \eta}\frac{\partial}{\partial x}\left(-x - v_1 t\right) = -r\frac{\partial f}{\partial \eta} \ ,$$

and

$$\frac{\partial y_{\rm R}}{\partial t} = r\frac{\partial f}{\partial \eta}\frac{\partial}{\partial t}\left(-x - v_1 t\right) = r\frac{\partial f}{\partial \eta}\left(-v_1\right) = -rv_1\frac{\partial f}{\partial \eta} \ .$$

For the incident pulse:

$$\frac{\partial y_{\rm I}}{\partial x} = \frac{\partial f}{\partial \eta}\frac{\partial}{\partial x}\left(x - v_1 t\right) = \frac{\partial f}{\partial \eta} \ ,$$

and

$$\frac{\partial y_{\rm I}}{\partial t} = \frac{\partial f}{\partial \eta}\frac{\partial}{\partial t}\left(x - v_1 t\right) = \frac{\partial f}{\partial \eta}\left(-v_1\right) = -v_1\frac{\partial f}{\partial \eta} \ .$$

Substitute in Equation (2) to obtain:

$$\left(\frac{\partial f}{\partial \eta}\right)\left(-v_1\frac{\partial f}{\partial \eta}\right) + \left(-r\frac{\partial f}{\partial \eta}\right)\left(-rv_1\frac{\partial f}{\partial \eta}\right) = \left(\tau\frac{\partial f}{\partial \eta}\frac{v_1}{v_2}\right)\left(-\tau v_1\frac{\partial f}{\partial \eta}\right) \ .$$

Simplifying and rearranging terms yields:

$$\boxed{1 = r^2 + \frac{v_1}{v_2}\tau^2} \ .$$

And here's my last objection. To get the last equation, he divided the previous equation through by $\left(\frac{\partial f}{\partial \eta}\right)^2$. But all of those derivatives are not the same. The way he wrote them, they may look the same, and they may all have the same funcitonal form, but they're not all being evaluated at the same value of η. In other words, for the incident pulse, the derivative should really be $\frac{\partial y}{\partial \eta_{\rm I}}$, where $\eta_{\rm I} = x - v_1 t$. For the reflected pulse, the derivative should be $\frac{\partial y}{\partial \eta_{\rm R}}$, where $\eta_{\rm R} = -x - v_1 t$, and for the transmitted pulse, the derivative should be $\frac{\partial y}{\partial \eta_{\rm T}}$, where $\eta_{\rm T} = \frac{v_1}{v_2}\left[x - v_2 t\right]$. So if we're evaluating these derivatives at the same time, t, and the same point, x, we're evaluating them at different values of η, and therefore, they don't all have the same value and cannot be canceled.