Problem: Calculate the acceleration of gravity **g** at the earth's surface using the equation for geodesic deviation [Eq. (6.87) Schutz p.163]; the Schwarzschild metric [Eq. (10.36), p.263]; the mass of the earth in geometrized units, $M_{\oplus} = 4.434\ x\ 10^{-3}\ m$ [p.187]; and the mean radius of the earth, $R_{\oplus} = 6.371\ x\ 10^{6}\ m$ [Wikipedia]. Neglect the earth's rotation: $d\phi = d\theta = 0$.

Solution: We use proper time τ as the affine parameter and the four-velocity $U^{\alpha} = dx^{\alpha}/d\tau$.

1. From Schutz, 11.7 Exer. (22):

$$\nabla_U \nabla_U \xi^r = R^r_{ttr}\, U^0 U^0\, \xi^r = \frac{2M}{r^2(r-2M)}\, \tilde{E}^2\, \xi^r$$

Note that this value is in geometrized units of m^{-1}.

2. For an infalling particle along the radius to the surface [Eq. (11.58) p.299]:

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^{\,2} - 1 + \frac{2M}{r}\quad .\quad \text{If we assume a stationary object at the earth's surface, then:}$$

$$\frac{dr}{d\tau} = 0, \qquad \tilde{E}^{\,2} = 1 - \frac{2M}{r} = \frac{r-2M}{r}, \qquad \text{and} \qquad \nabla_U \nabla_U \xi^r = \frac{2M}{r^3}\, \xi^r \to \frac{2Mc^2}{r^3} \xi^r\, .$$

Multiplying by c^2 provides SI units of acceleration: ms^{-2}.

3. Then evaluating $\nabla_U \nabla_U \xi^r\ (ms^{-2})$ using the given values: $M = M_{\oplus},\ and\ r = R_{\oplus}$.

$$\nabla_U \nabla_U\, \xi^r = \frac{2(4.434\ x\ 10^{-3}\ m)(2.998\ x\ 10^{8}\ m/s)^2}{(6.371\ x\ 10^{6}\ m)^3}\ \xi^r = 3.082\ x\ 10^{-6}\ \xi^r\ \ s^{-2}$$

4. Now ξ^r is simply the r component of the connecting vector $\vec{\xi}$ [p.162]. In the Schwarzschild metric, the reference for r is at the earth's center, so the deviation from there to the earth's surface, $\xi^r = \Delta r = 6.371\ x\ 10^{6}\ m$.

$$\nabla_U \nabla_U\, \xi^r = 3.082\ x\ 10^{-6}\ (6.371\ x\ 10^{6}) = 19.635\ \ ms^{-2}$$

Why is this twice the actual value? **g** $= 9.818\ ms^{-2}$

References:

Schutz, Bernard, *A First Course in General Relativity, 2nd Ed.,* Cambridge University Press, United Kingdom, 2009.