Potential due to an Infinite Line of Charge

In § {Potential due to a Finite Line of Charge}, we found the electrostatic potential due to a finite line of charge. The answer

$$V(r,0,0)=\frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{L+\sqrt{r^2+L^2}}{-L+\sqrt{r^2+L^2}}\right) \tag{1}$$

was an unilluminating, complicated expression involving the logarithm of a fraction. Suppose, however, that the voltmeter probe were placed quite close to the charge. Then, to a fairly good approximation, the charge would look like an infinite line. Perhaps the expression for the electrostatic potential due to an infinite line is simpler and more meaningful. In principle, we should be able to get this expression by taking the limit of (1) as L goes to infinity.

$$\begin{aligned}V(r,0,0)&=\lim_{L\to\infty}\frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{L+\sqrt{r^2+L^2}}{-L+\sqrt{r^2+L^2}}\right)\\&=\lim_{L\to\infty}\frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{1+\sqrt{\frac{r^2}{L^2}+1}}{-1+\sqrt{\frac{r^2}{L^2}+1}}\right)\end{aligned} \tag{2}$$

The denominator in this last expression goes to zero in the limit, which means that the potential goes to infinity. What has happened? Remember that we assumed that the ground probe was at infinity when we wrote our original integral expression for the potential, namely the first equation in § {Electrostatic and Gravitational Potentials}. Therefore, as we let the line charge become infinitely long, in the limit, it reaches the ground probe. So, of course, the potential difference between the ground probe and the active probe is infinite. One of the probes is touching the charge. This problem will occur whenever the (idealized) source extends all the way to infinity.

What is the resolution? We must move the ground probe somewhere else. Where else? Anywhere that's not touching the charge density. Let's put the ground probe at $(r_0,0,0)$. Now, we want to calculate the difference in potential between the active probe and the ground probe. Do we need to start all over again? No, we can use the expression from § {Potential due to a Finite Line of Charge}, namely (1) above, if we are careful about the order in which we do various mathematical operations.

Notice that, even though we have written (1) as if it were the expression for $V(r,0,0)$, it is really the expression for the potential difference between the two probes, i.e. $V(r,0,0)-V(\infty,0,0)$. The potential difference that we want, i.e. $V(r,0,0)-V(r_0,0,0)$ can be found by subtracting two expressions like (1), one evaluated at r and one evaluated at r_0.

$$\begin{aligned}V(r,0,0)-\ &V(r_0,0,0)\\&=[V(r,0,0)-V(\infty,0,0)]-[V(r_0,0,0)-V(\infty,0,0)]\\&=\frac{\lambda}{4\pi\epsilon_0}\left[\ln\left(\frac{L+\sqrt{r^2+L^2}}{-L+\sqrt{r^2+L^2}}\right)-\ln\left(\frac{L+\sqrt{r_0^2+L^2}}{-L+\sqrt{r_0^2+L^2}}\right)\right]\\&=\frac{\lambda}{4\pi\epsilon_0}\ln\left[\left(\frac{L+\sqrt{r^2+L^2}}{-L+\sqrt{r^2+L^2}}\right)\left(\frac{-L+\sqrt{r_0^2+L^2}}{L+\sqrt{r_0^2+L^2}}\right)\right]\end{aligned} \tag{3}$$

Notice that each of the terms in the third line is separately infinite in the limit that $L\to\infty$. In effect, we are trying to subtract infinity from infinity and still get a sensible answer. We can do this by doing the subtraction *before* we take the limit, This process for trying to subtract infinity from infinity by first putting in a “cut-off,” in this case, the length of the source L, so that the subtraction makes sense and then taking a limit, is a process that is used often in advanced particle physics. In those cases, the process is called "renormalization."

$$\begin{aligned}
V(r,0,0) - \ & V(r_0,0,0) \\
&= \frac{\lambda}{4\pi\epsilon_0}\ln\left[\left(\frac{1+\sqrt{\frac{r^2}{L^2}+1}}{-1+\sqrt{\frac{r^2}{L^2}+1}}\right)\left(\frac{-1+\sqrt{\frac{r_0^2}{L^2}+1}}{1+\sqrt{\frac{r_0^2}{L^2}+1}}\right)\right] \\
&\approx \frac{\lambda}{4\pi\epsilon_0}\ln\left[\left(\frac{1+\left(1+\frac{1}{2}\frac{r^2}{L^2}+\ldots\right)}{-1+\left(1+\frac{1}{2}\frac{r^2}{L^2}+\ldots\right)}\right)\left(\frac{-1+\left(1+\frac{1}{2}\frac{r_0^2}{L^2}+\ldots\right)}{1+\left(1+\frac{1}{2}\frac{r_0^2}{L^2}+\ldots\right)}\right)\right] \\
&= \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{\left(2+\frac{1}{2}\frac{r^2}{L^2}+\ldots\right)}{\left(\frac{1}{2}\frac{r^2}{L^2}+\ldots\right)}\frac{\left(\frac{1}{2}\frac{r_0^2}{L^2}+\ldots\right)}{\left(2+\frac{1}{2}\frac{r_0^2}{L^2}+\ldots\right)}\right] \\
&= \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{\left(\frac{1}{2}\frac{r_0^2}{L^2}+\ldots\right)}{\left(\frac{1}{2}\frac{r^2}{L^2}+\ldots\right)}\frac{\left(2+\frac{1}{2}\frac{r^2}{L^2}+\ldots\right)}{\left(2+\frac{1}{2}\frac{r_0^2}{L^2}+\ldots\right)}\right] \\
&= \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{\left(r_0^2+\ldots\right)}{\left(r^2+\ldots\right)}\frac{\left(1+\frac{1}{4}\frac{r^2}{L^2}+\ldots\right)}{\left(1+\frac{1}{4}\frac{r_0^2}{L^2}+\ldots\right)}\right] \\
&\approx \frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{r_0^2}{r^2}\right) \\
&= \frac{2\lambda}{4\pi\epsilon_0}\ln\left(\frac{r_0}{r}\right)
\end{aligned} \tag{4}$$

In the second to the last line, we kept only the highest order term in each of the four power series inside the logarithm. Why was it ok to do this? What if we wanted to keep a second term in the power series.

Notice that if $r > r_0$, then the argument of the logarithm is a fraction less than one and the result is negative, i.e. the electrostatic potential is less at r than it is at r_0. Why is this expected? If $r < r_0$, then the opposite is true.

We performed the calculation above with $z = z_0$. Why was it ok to do this?