PROOF. By virtue of Theorem 3.1, we need only show that cycling is impossible when the smallest-subscript rule is used. We shall do this by deriving a contradiction from the assumption that the smallest-subscript rule leads from some dictionary D_0 to itself in a sequence of degenerate iterations. For definiteness, let us say that this sequence of iterations produces dictionaries D_1, $D_2, \ldots, D_k$ such that $D_k = D_0$. A variable will be called *fickle* if it is nonbasic in some of these dictionaries and basic in others. Among all the fickle variables, let x_t have the largest subscript. In the sequence $D_0, D_1, \ldots, D_k$, there is a dictionary D with x_t leaving (basic in D but nonbasic in the next dictionary), and some other fickle variable x_s entering (nonbasic in D but basic in the

next dictionary). Further along in the sequence $D_0, D_1, \ldots, D_k, D_1, D_2, \ldots, D_k$, there must be a dictionary D^* with x_t entering. Let us record D as

$$\begin{array}{l} x_i = b_i - \sum_{j \notin B} a_{ij} x_j \qquad (i \in B) \\ \hline z = v + \sum_{j \notin B} c_j x_j \, . \end{array}$$

Since all the iterations leading from D to D^* are degenerate, the objective function z must have the same value v in both dictionaries. Thus, the last row of D^* may be recorded as

$$z = v + \sum_{j=1}^{n+m} c_j^* x_j$$

with $c_j^* = 0$ whenever x_j is basic in D^*. Since this equation has been obtained from D by algebraic manipulations, it must be satisfied by every solution of D. In particular, it must be satisfied by $x_s = y$, $x_j = 0$ ($j \notin B$ but $j \neq s$), $x_i = b_i - a_{is} y$ ($i \in B$) and $z = v + c_s y$ for every choice of y.

Thus we have

$$v + c_s y = v + c_s^* y + \sum_{i \in B} c_i^*(b_i - a_{is} y)$$

and, after simplification,

$$\left(c_s - c_s^* + \sum_{i \in B} c_i^* a_{is}\right) y = \sum_{i \in B} c_i^* b_i$$

'or every choice of y. Since the right-hand side of the last equation is a constant independent of y, ve conclude that

$$c_s - c_s^* + \sum_{i \in B} c_i^* a_{is} = 0. \tag{3.10}$$

The rest is easy. Since x_s is entering in D, we have $c_s > 0$. Since x_s is not entering in D^* and yet $< t$, we have $c_s^* \leq 0$. Hence (3.10) implies that

$$c_r^* a_{rs} < 0 \qquad \text{for some} \quad r \in B. \tag{3.11}$$

ince $r \in B$, the variable x_r is basic in D; since $c_r^* \neq 0$, the same variable is nonbasic in D^*. Hence, $_r$ is fickle and we have $r \leq t$. Actually, x_r is different from x_t: since x_t is leaving in D, we have $_{ts} > 0$ and so $c_t^* a_{ts} > 0$. Now $r < t$ and yet x_r is not entering in D^*. Thus, we cannot have $c_r^* > 0$. rom (3.11), we conclude that

$$a_{rs} > 0.$$

ince all the iterations leading from D to D^* are degenerate, the two dictionaries describe the ame solution. In particular, the value of x_r is zero in both dictionaries (x_r is nonbasic in D^*) and $b_r = 0$. Hence x_r was a candidate for leaving the basis of D—yet we picked x_t, even though $r < t$. his contradiction completes the proof. ■

One further point: termination of the simplex method can be guaranteed even without abiding y the smallest-subscript rule in every single iteration. We might resort to the smallest-subscript ıle, for instance, only when the last fifty or so iterations were degenerate, and abandon it after ıe next nondegenerate iteration in favor of any other way of choosing the entering and leaving ıriables. Although cycling might conceivably take place in this case, each block of consecutive egenerate iterations would be followed by a nondegenerate iteration, and so each dictionary ould be constructed only a finite number of times.