The *product* fg of the transformations f and g of a set is the transformation obtained by applying first g, then f, i.e., $(fg)(x) = f(g(x))$.

Problem 2. Give an example in which fg is not the same as gf.

The transformation f^{-1} *inverse* to f is defined by the condition that if f takes x to y, then f^{-1} takes y to x.

A collection of transformations of a set is called a *transformation group* if it contains the inverse of each of its transformations and the product of any two of its transformations.

Problem 3. Is the set of the three reflections about the vertices of an equilateral triangle a transformation group?

Problem 4. How many elements are there in the group of isometries[12] of an equilateral triangle? In the group of rotations of a tetrahedron?

Answer. 6, 12.

The concept of a transformation group is one of the most fundamental in all of mathematics and at the same time one of the simplest: the human mind naturally thinks in terms of invariants of transformation groups (this is connected with both the visual apparatus and our power of abstraction).

Let A be a transformation group on the set X. Multiplication and inversion define mappings $A \times A \to A$ and $A \to A$ (the pair (f, g) goes to fg, and the element g to g^{-1}). A set A endowed with these two mappings is called an *abstract group* (or briefly, simply a *group*). Thus a *group* is obtained from a *transformation group* by simply *ignoring* the set that is transformed.

Problem 5. Prove that the set $\boldsymbol{R}$ of all real numbers becomes a group when equipped with the operations of ordinary addition and changing the sign.

Algebraists usually define a group as a set with two operations satisfying a collection of axioms such as $f(gh) = (fg)h$. These axioms automatically hold for transformation groups. Actually these axioms mean simply that the group is formed from some transformation group by ignoring the set that is transformed. Such axioms, together with other unmotivated definitions, serve mathematicians mainly by making it difficult for the uninitiated to master their subject, thereby elevating its authority.

Let G be a group and M a set. We say that an *action of the group G on the set M* is defined if to each element g of the group G there corresponds a transformation $T_g : M \to M$ of the set M, to the product of any two elements of the group corresponds the product of the transformations corresponding to these elements, and to any two mutually inverse elements correspond mutually inverse transformations: $T_{fg} = T_f T_g$, $T_{g^{-1}} = (T_g)^{-1}$.

[12] An isometry is a transformation that preserves distances (so that the distance between the images of two points equals the distance between the points themselves).