VARIATION OF EH LAGRANGIAN - RESOLUTION

I think I have now arrived at a solution to proposition (ii)

$\delta L_{EH} = \delta \mathfrak{g}^{\mu\nu} R_{\mu\nu} + \mathfrak{g}^{\mu\nu} \delta R_{\mu\nu}$

so, using

$\delta g^{\mu\nu} = -\delta g_{\alpha\beta} g^{\mu\alpha} g^{\nu\beta}$

$\delta L_{EH} = -\delta \mathfrak{g}_{\alpha\beta} R^{\alpha\beta} + \mathfrak{g}^{\mu\nu} \delta R_{\mu\nu}$

Therefore

$\delta L_{EH} + \delta \mathfrak{g}_{\alpha\beta} R^{\alpha\beta} = \mathfrak{g}^{\mu\nu} \delta R_{\mu\nu}$

If we now integrate both sides over an arbitrary volume element, we have already proved that

$\int_\Omega \mathfrak{g}^{\mu\nu} \delta R_{\mu\nu} d\Omega = 0$

and as Ω is arbitrary, then

$\delta L_{EH} + \delta \mathfrak{g}_{\alpha\beta} R^{\alpha\beta} = 0$

so

$\frac{\delta L_{EH}}{\delta \mathfrak{g}_{\alpha\beta}} = -R^{\alpha\beta}$