The arrangement field theory (AFT) Part 2

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Abstract

In this work we apply the formalism developed in the previous paper ("The arrangement field theory") to describe the content of standard model plus gravity. We discover a triality between *Arrangement Field Theory*, *String Theory* and *Loop Quantum Gravity* which appear as different manifestations of the same theory. Finally we show as three families of fields arise naturally and we discover a new road toward unification of gravity with gauge and matter fields.

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Contents

1 Introduction

The arrangement field paradigm describes universe by means of a graph, ie an ensemble of vertices and edges. However there is a considerable difference between this framework and the usual modeling with spin-foams or spin-networks. The existence of an edge which connects two vertices is in fact probabilistic. In this framework the fundamental quantity is an invertible matrix M with dimension $n \times n$, where n is the number of vertices. In the entry ij of such matrix we have a quaternionic number which gives the probability amplitude for the existence of an edge connecting vertex i to vertex j. In the introductory work [1] we have developed a simple scalar field theory in this probabilistic graph (we call it "non-ordered space"). We have seen that a space-time metric emerges spontaneously when we fix an ensemble of edges. Moreover, the quantization of metric descends naturally from quantization of M in the non-ordered space. In section 2 we summarize these results.

In section 3 we express Ricci scalar as a simple quadratic function of M. We discover how the gravitational field emerges from diagonal components of M, in contrast to gauge fields which come out from non-diagonal components.

In section 4 we define a quartic function of M which develops a Gauss Bonnet term for gravity and the usual kinetic term for gauge fields.

In section 5 we discover a triality between *Arrangement Field Theory*, *String Theory* and *Loop Quantum Gravity* which appear as different manifestations of the same theory.

In section 6 we show that a grassmanian extension of M generates automatically all known fermionic fields, divided exactly in three families. We see how gravitational field exchanges homologous particles in different families. The resulting scheme finds an analogue in supersymmetryc theories, with known fermionic fields which take the role of gauginos for known bosons.

In the subsequent sections we explore some practical implications of arrangement field theory, in connection to inflation, dark matter and quantum entanglement. Moreover we explain how deal with theory perturbatively by means of Feynman diagrams.

We warmly invite the reader to see introductory work [1] before proceeding.

2 Formalism

In paper [1] we have considered an euclidean 4-dimensional space represented by a graph with n vertices. We restrict our attention to euclidean spaces also in this work, giving in the end a suggestion for moving in lorentzian spaces. Moreover we assume the Einstein convention, summing over repeated indices.

In proof of **theorem 8** in [1] we have demonstrated the equivalence between the following actions:

$$S_1 = (M\varphi)^\dagger (M\varphi) \tag{1}$$

$$S_2 = \sum_{i=1}^{n} \sqrt{|h|} h^{\mu\nu}(x^i)(\nabla_\mu \varphi^i)^*(\nabla_\nu \varphi^i). \tag{2}$$

M is any invertible matrix while the field φ is represented by a column array, with an entry for every vertex in the graph:

$$\varphi = \begin{pmatrix} \varphi(x^1) \\ \varphi(x^2) \\ \varphi(x^3) \\ \vdots \\ \varphi(x^N) \end{pmatrix}. \tag{3}$$

The entries of both M and ϕ take values in the division ring of quaternions, usually indicated with $\mathbf{H}$. The first action considers the universe as an abstract ensemble of vertices, numbered from 1 to n, where n is the total number of space vertices. The entry (ij) in the matrix M represents the probability amplitude for the existence of an edge which connects the vertex number i to the vertex number j. We admit non-commutative geometries, which in this framework implies a possible inequivalence $|M^{ij}| \neq |M^{ji}|$. More, the first action is invariant under transformations $(U_1, U_2) \in U(n, \mathbf{H}) \otimes U(n, \mathbf{H})$ which send M in $U_2 M U_1^\dagger$.

In action (2) a covariant derivative for $U(n, \mathbf{H}) \otimes U(n, \mathbf{H})$ appears, represented by a skew hermitian matrix ∇ which expands according to $\nabla_\mu = \tilde{M}_\mu + A_\mu$. Here $\tilde{M}_\mu$ is a linear operator such that $lim_{\Delta \to 0} \tilde{M}_\mu = \partial_\mu$. If we number the space vertices along direction μ, $\tilde{M}_\mu$ becomes

$$\tilde{M}_\mu^{ij} = \delta^{(i+1)j} - \delta^{(i-1)j} \tag{4}$$

$$\sum_j \tilde{M}^{ij} \varphi^j = \sum_j \delta^{(i+1)j} \varphi^j - \delta^{(i-1)j} \varphi^j = \varphi(i+1) - \varphi(i-1).$$

The gauge fields A act as skew hermitian matrices too:

$$A = (A^{ij}) = (A(x^i, x^j))$$

$$(A\phi)^i = A^{ij} \phi^j.$$

In proof of **theorem 5** we have discovered that for every normal matrix $\hat{M}$, which is neither hermitian nor skew hermitian, four couples (U_1, D^μ) exist, with U_1 unitary and D^μ diagonal, such that

$$U_1^\dagger D^\mu \nabla_\mu U_1 = \hat{M} \tag{5}$$

$$\sqrt{|h|} h^{\mu\nu}(x^i) = \frac{1}{2} d_i^{*\mu} d_i^\nu + c.c. \qquad D_\mu^{ij} = d_i^\mu \delta^{ij}. \tag{6}$$

Here h is a non degenerate metric while the first relation determines uniquely the values of gauge fields. The matrices ∇_μ, U_1, D^μ act on field arrays via matricial product and the ensemble of four couples (U_1, D^μ) is called "space arrangement".

Further, in proof of **theorem 6**, we have seen that for every invertible matrix M we can always find an unitary transformation U_M and a normal matrix $\hat{M}$, which is neither hermitian nor skew hermitian, such that $M = U_M \hat{M}$. If we define $U_2 = U_1 U_M^\dagger$, we have

$$M^\dagger M = \hat{M}^\dagger \hat{M} \tag{7}$$

$$U_2^\dagger D^\mu \nabla_\mu U_1 = M. \tag{8}$$

It's sufficient to substitute (8) in (1) to verify its equivalence with (2). We have called $\hat{M}$ the "associated normal matrix" of M.

The action of a transformation (U_1, U_2) on ∇ follows from its action on M. We can always use the invariance under $U(n, \mathbf{H}) \otimes U(n, \mathbf{H})$ to put M in the form $M = D^\mu \nabla_\mu$. Starting from this we have

$$U_2 M U_1^\dagger = U_2 D^\mu \nabla_\mu U_1^\dagger = U_2 D^\mu U_1^\dagger U_1 \nabla_\mu U_1^\dagger.$$

We define $\nabla' = U_1 \nabla_\mu U_1^\dagger$ the transformed of ∇ under (U_1, U_2) and $D'^\mu = U_2 D^\mu U_1^\dagger$ the transformed of D^μ. We assume that A_μ inside ∇_μ transforms correctly as a gauge field, so that

$$\nabla[A]_\mu \phi = \nabla[A] U_1^\dagger \phi' = U_1^\dagger \nabla[A_{U1}]_\mu \phi'$$

$$\phi' = U_1 \phi.$$

We want D'^μ remain diagonal and $h' = h[D'] = h[D]$. In this case there are two relevant possibilities:

1. D is a matrix made by blocks $m \times m$ with m integer divisor of n and every block proportional to identity. In this case the residual symmetry is $U(1, \mathbf{H})^n \times U(m, \mathbf{H})^{n/m}$ with elements (sV, V), s both diagonal and unitary, $V \in U(m, \mathbf{H})^{n/m}$;

2. h is any diagonal matrix. The symmetry reduces to $U(1, \mathbf{H})^n \otimes U(1, \mathbf{H})^n$ which is local $U(1, \mathbf{H}) \otimes U(1, \mathbf{H}) \sim SU(2) \otimes SU(2) \sim SO(4)$.

In this way, if we keep fixed the metric h and keep diagonal D, the action (2) will be invariant at least under $U(1,\mathbf{H})^n \otimes U(1,\mathbf{H})^n$ which doesn't modify h.

We have supposed that a potential for M breaks the $U(n,\mathbf{H}) \otimes U(n,\mathbf{H})$ symmetry in $U(1,\mathbf{H})^n \otimes U(m,\mathbf{H})^{n/m}$ where m is an integer divisor of n. We'll see in fact that the more natural potential has the form $tr\,(\alpha M^\dagger M - \beta M^\dagger M M^\dagger M)$, known as "mexican hat potential". This potential is a very typical potential for a spontaneous symmetry breaking. In this way all the vertices are grouped in n/m ensembles $\mathcal{U}^a$:

$$\mathcal{U}^a = \{x_1^a, x_2^a, x_3^a, \ldots, x_m^a\}$$

$$\varphi = (\varphi(x_i^a)) = \begin{pmatrix} \varphi(x_1^1) & \varphi(x_2^1) & \varphi(x_3^1) & \ldots & \varphi(x_m^1) \\ \varphi(x_1^2) & \varphi(x_2^2) & \varphi(x_3^2) & \ldots & \varphi(x_m^2) \\ \varphi(x_1^3) & \varphi(x_2^3) & \varphi(x_3^3) & \ldots & \varphi(x_m^3) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \varphi(x_1^{n/m}) & \varphi(x_2^{n/m}) & \varphi(x_3^{n/m}) & \ldots & \varphi(x_4^{n/m}) \end{pmatrix} \tag{9}$$

$$A = (A_{ij}^{ab}) = (A(x_i^a, x_j^b)).$$

Now the indices a, b of A act on the columns of φ, while the indices i, j act on the rows. The fields A_{ij}^{ab} with $a = b$ maintain null masses and then they continue to behave as gauge fields for $U(m,\mathbf{H})^{n/m}$. Every $U(m,\mathbf{H})$ term in $U(m,\mathbf{H})^{n/m}$ acts independently inside a single $\mathcal{U}^a$. So, if we consider the ensembles $\mathcal{U}^a$ as the "real" physical points, we can interpret $U(m,\mathbf{H})^{n/m}$ as a local $U(m,\mathbf{H})$.

It's simple to verify:

$$h^{\mu\nu}(x_i^a) = h^{\mu\nu}(x_j^a) \qquad \forall x_i^a, x_j^a \in \mathcal{U}^a$$

$$h^{\mu\nu}(x^a) \stackrel{!}{=} h^{\mu\nu}(\mathcal{U}^a) = h^{\mu\nu}(x_i^a) \qquad \forall x_i^a \in \mathcal{U}^a$$

$$A_{ij}(x^a) \stackrel{!}{=} Tr\,\left[A(x^a)T^{ij}\right], \quad where$$

$$A(x^a) \;=\; \sum_{ij} A(x^a_i, x^a_j)T^{ij}, \quad \text{with } T^{ij} \text{ generator of } U(m, \mathbf{H}) \tag{10}$$

3 Ricci scalar in the arrangement field paradigm

3.1 Two simple rules

In this subsection we demonstrate two simple rules which will permit us to simplify considerably the structure of Ricci scalar, making it suitable for the arrangement field formalism.

Theorem 1 *For every couple of antisymmetric tensors in Minkowski space-time, $A^{ij} = -A^{ji}$ and $B^{ij} = -B^{ji}$, two fundamental relations are verified as follows:*

$$-2A^{ij}B_{ij} = Re\,(\hat{A}^\gamma \hat{B}^\gamma) \tag{11}$$

$$2i\widehat{[A,B]}^\gamma = \varepsilon^{\alpha\beta\gamma}\hat{A}^\alpha \hat{B}^\beta \tag{12}$$

with

$$\hat{A}^\gamma = i\varepsilon^{\alpha\beta\gamma}A^{\alpha\beta} + 2A^{0\gamma}$$

$$\hat{B}^\gamma = i\varepsilon^{\tau\sigma\gamma}B^{\tau\sigma} + 2B^{0\gamma}.$$

Proof. We have only to explicit calculations. If we take signature $(+---)$, the first relation is easily verified:

$$\begin{aligned}
Re\,(\hat{A}^\gamma \hat{B}^\gamma) &= -\varepsilon^{\alpha\beta\gamma}\varepsilon^{\tau\sigma\gamma}A^{\alpha\beta}B^{\tau\sigma} + 4A^{0\gamma}B^{0\gamma} \\
&= -(\delta^{\alpha\tau}\delta^{\beta\sigma} - \delta^{\alpha\sigma}\delta^{\beta\tau})A^{\alpha\beta}B^{\tau\sigma} + 4A^{0\gamma}B^{0\gamma} \\
&= -A^{\alpha\beta}B^{\alpha\beta} + A^{\alpha\beta}B^{\beta\alpha} + 2A^{0\gamma}B^{0\gamma} + 2A^{\gamma 0}B^{\gamma 0} \\
&= -2A^{\alpha\beta}B^{\alpha\beta} + 2A^{0\gamma}B^{0\gamma} + 2A^{\gamma 0}B^{\gamma 0} \\
&= -2A^{\alpha\beta}B_{\alpha\beta} - 2A^{0\gamma}B_{0\gamma} - 2A^{\gamma 0}B_{\gamma 0} = -2A^{ij}B_{ij}.
\end{aligned} \tag{13}$$

For the second relation we separate real and imaginary components:

$$
\begin{aligned}
Im\widehat{[A,B]}^{\gamma} &= \varepsilon^{\alpha\beta\gamma}(A^{\alpha}{}_{\tau}B^{\tau\beta} - B^{\alpha}{}_{\tau}A^{\tau\beta} + A^{\alpha}{}_{0}B^{0\beta} - B^{\alpha}{}_{0}A^{0\beta}) \\
&= -2\varepsilon^{\alpha\beta\gamma}A^{\alpha\tau}B^{\tau\beta} + 2\varepsilon^{\alpha\beta\gamma}A^{\alpha 0}B^{0\beta} \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
Re\left(\varepsilon^{\alpha\beta\gamma}\hat{A}^{\alpha}\hat{B}^{\beta}\right) &= -\varepsilon^{\alpha\beta\gamma}\varepsilon^{\sigma\tau\alpha}A^{\sigma\tau}\varepsilon^{\delta\omega\beta}B^{\delta\omega} + 4\varepsilon^{\alpha\beta\gamma}A^{0\alpha}B^{0\beta} \\
&= -(\delta^{\beta\sigma}\delta^{\gamma\tau} - \delta^{\beta\tau}\delta^{\gamma\sigma})A^{\sigma\tau}\varepsilon^{\delta\omega\beta}B^{\delta\omega} + 4\varepsilon^{\alpha\beta\gamma}A^{0\alpha}B^{0\beta} \\
&= -2\varepsilon^{\delta\omega\beta}A^{\beta\gamma}B^{\delta\omega} - 4\varepsilon^{\alpha\beta\gamma}A^{\alpha 0}B^{0\beta} \qquad (15)
\end{aligned}
$$

The first term of (14) for $\gamma = 1$ is $-2(A^{21}B^{13} - A^{31}B^{12})$.
The first term of (15) for $\gamma = 1$ is

$$-2(A^{21}B^{31} - A^{21}B^{13} + A^{31}B^{12} - A^{31}B^{21}) = 4(A^{21}B^{13} - A^{31}B^{12}).$$

The first term of (14) for $\gamma = 2$ is $-2(A^{32}B^{21} - A^{12}B^{23})$.
The first term of (15) for $\gamma = 2$ is

$$-2(A^{32}B^{12} - A^{32}B^{21} + A^{12}B^{23} - A^{12}B^{32}) = 4(A^{32}B^{21} - A^{12}B^{23}).$$

The first term of (14) for $\gamma = 3$ is $-2(A^{13}B^{32} - A^{23}B^{31})$.
The first term of (15) for $\gamma = 3$ is

$$-2(A^{13}B^{23} - A^{13}B^{32} + A^{23}B^{31} - A^{23}B^{13}) = 4(A^{13}B^{32} - A^{23}B^{31}).$$

Hence

$$Re\left(\varepsilon^{\alpha\beta\gamma}\hat{A}^{\alpha}\hat{B}^{\beta}\right) = -2Im\widehat{[A,B]}^{\gamma}$$

Moreover

$$
\begin{aligned}
Re\,\widehat{[A,B]}^{\gamma} &= 2(A^{0}{}_{\alpha}B^{\alpha\gamma} - B^{0}{}_{\alpha}A^{\alpha\gamma}) \\
&= 2(-A^{0\alpha}B^{\alpha\gamma} + B^{0\alpha}A^{\alpha\gamma}) \\
&= 2(B^{0\alpha}A^{\alpha\gamma} - A^{0\alpha}B^{\alpha\gamma})
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
Im\left(\varepsilon^{\alpha\beta\gamma}\hat{A}^{\alpha}\hat{B}^{\beta}\right) &= 2\varepsilon^{\alpha\beta\gamma}(\varepsilon^{\tau\sigma\alpha}A^{\tau\sigma}B^{0\beta} + A^{0\alpha}\varepsilon^{\tau\sigma\beta}B^{\tau\sigma}) \\
&= 2(\delta^{\beta\tau}\delta^{\gamma\sigma} - \delta^{\beta\sigma}\delta^{\gamma\tau})A^{\tau\sigma}B^{0\beta} + 2(\delta^{\alpha\sigma}\delta^{\gamma\tau} - \delta^{\alpha\tau}\delta^{\gamma\sigma})A^{0\alpha}B^{\tau\sigma} \\
&= 2(A^{\beta\gamma}B^{0\beta} - A^{\gamma\beta}B^{0\beta}) + 2(A^{0\sigma}B^{\gamma\sigma} - A^{0\tau}B^{\tau\gamma}) \\
&= 4(A^{\beta\gamma}B^{0\beta} + A^{0\sigma}B^{\gamma\sigma}) \\
&= 4(B^{0\beta}A^{\beta\gamma} - A^{0\sigma}B^{\sigma\gamma})
\end{aligned} \tag{17}
$$

So

$$
Im\left(\varepsilon^{\alpha\beta\gamma}\hat{A}^{\alpha}\hat{B}^{\beta}\right) = 2Re\,\widehat{[A,B]}^{\gamma}
$$

and finally

$$
\varepsilon^{\alpha\beta\gamma}\hat{A}^{\alpha}\hat{B}^{\beta} = 2i\widehat{[A,B]}^{\gamma} \tag{18}
$$

■ QED

3.2 Ricci scalar with Barbero connection

The target of this section is to rewrite Ricci scalar using only a $SU(2)$ connection (called Barbero connection) instead of a $SO(1,3)$ or $SO(4)$ connection. Classical gravity doesn't depend on formulation. This is not the case for a quantum theory of gravity: however we don't know what is the correct classical theory to quantize, so that every classical equivalent formulation has the same chances.

We start from Ricci scalar in the Palatini formulation of General Relativity:

$$R(x) = \{\partial_\mu \omega_\nu^{ij}(x) - \partial_\nu \omega_\mu^{ij}(x) + [\omega_\mu, \omega_\nu]^{ij}(x)\} e_i^\mu e_j^\nu. \tag{19}$$

The ω are gauge fields for $SO(1,3)$ and e_i^μ are vielbein fields. We define $E_{ij}^{\mu\nu} = e_{[i}^\mu e_{j]}^\nu$ and apply (11):

$$R(x) = -\frac{1}{4}\{\partial_\mu \hat{\omega}_\nu^\gamma(x) - \partial_\nu \hat{\omega}_\mu^\gamma(x) + \widehat{[\omega_\mu, \omega_\nu]}^\gamma(x)\} \hat{E}^{\gamma\mu\nu} + c.c.. \tag{20}$$

Applying now (12):

$$R(x) = -\frac{1}{4}\{\partial_\mu \hat{\omega}_\nu^\gamma(x) - \partial_\nu \hat{\omega}_\mu^\gamma(x) - \frac{i}{2}\varepsilon^{\alpha\beta\gamma}\hat{\omega}_\mu^\alpha \hat{\omega}_\nu^\beta(x)\} \hat{E}^{\gamma\mu\nu} + c.c.. \tag{21}$$

We obtain a classically equivalent action by adding to Palatini action an Immirzi term which doesn't change the motion equations:

$$\begin{aligned}
R_{\mu\nu}^{ij}(x)E_{ij}^{\mu\nu} &\rightarrow R_{\mu\nu}^{ij}(x)E_{ij}^{\mu\nu} + \frac{1}{\gamma}\varepsilon^{ij}{}_{kl}R_{\mu\nu}^{kl}(x)E_{ij}^{\mu\nu} \\
&= R_{\mu\nu}^{ij}(x)E_{ij}^{\mu\nu} - \frac{1}{2\gamma}(Re\,\hat{\varepsilon}^{ij\alpha}R_{\mu\nu}^\alpha(x))E_{ij}^{\mu\nu} \\
&= -\frac{1}{4}\hat{R}_{\mu\nu}^\gamma(x)\hat{E}^{\gamma\mu\nu} + \frac{1}{8\gamma}(Re\,\widehat{\hat{\varepsilon}^\alpha R_{\mu\nu}^\alpha}(x))^\gamma \hat{E}^{\gamma\mu\nu} + c.c..
\end{aligned} \tag{22}$$

Here γ is a constant called "Immirzi parameter". Moreover

$$\begin{aligned}
\hat{\varepsilon}^{ij\alpha} &= i\varepsilon^{\sigma\tau\alpha}\varepsilon^{ij\sigma\tau} + 2\varepsilon^{ij0\alpha} \\
&= i\varepsilon^{0\sigma\tau\alpha}\varepsilon^{ij\sigma\tau} + 2\varepsilon^{ij0\alpha} \\
&= i\varepsilon^{\alpha0\sigma\tau}\varepsilon^{ij\sigma\tau} + 2\varepsilon^{ij0\alpha} \\
&= 2i(\delta^{\alpha i}\delta^{0j} - \delta^{\alpha j}\delta^{0i}) + 2\varepsilon^{0\alpha ij}
\end{aligned} \tag{23}$$

$$Re\,\hat{\varepsilon}^{ij\alpha}R^\alpha = -2(\delta^{\alpha i}\delta^{0j} - \delta^{\alpha j}\delta^{0i})Im(R^\alpha) + 2\varepsilon^{0\alpha ij}Re(R^\alpha) \tag{24}$$

$$
\begin{aligned}
\left(\widehat{Re\,\hat{\varepsilon}^{\alpha} R^{\alpha}}\right)^{\gamma} &= -4i\varepsilon^{\sigma\tau\gamma}(\delta^{\alpha\sigma}\delta^{0\tau}-\delta^{\alpha\tau}\delta^{0\sigma})Im(R^{\alpha})+2i\varepsilon^{\sigma\tau\gamma}\varepsilon^{0\alpha\sigma\tau}Re(R^{\alpha})- \\
&\quad -4(\delta^{\alpha 0}\delta^{0\gamma}-\delta^{\alpha\gamma}\delta^{00})Im(R^{\alpha})+4\varepsilon^{0\alpha 0\gamma}Re(R^{\alpha}) \\
&= 4iRe\,R^{\gamma}+4Im\,R^{\gamma}=4iR^{*\gamma} \qquad (25)
\end{aligned}
$$

Hence:

$$
R^{ij}_{\mu\nu}(x)E^{\mu\nu}_{ij}+\frac{1}{\gamma}\varepsilon^{ij}{}_{kl}R^{kl}_{\mu\nu}(x)E^{\mu\nu}_{ij} = \left(-\frac{1}{4}R^{\gamma}_{\mu\nu}+\frac{i}{2\gamma}R^{*\gamma}_{\mu\nu}\right)\hat{E}^{\gamma\mu\nu} \qquad (26)
$$

We fix $\gamma=2i$:

$$
R(x)=-\frac{1}{4}(R^{\gamma}_{\mu\nu}-R^{*\gamma}_{\mu\nu})\hat{E}^{\gamma\mu\nu}+c.c.=(Im\,R^{\gamma}_{\mu\nu})(Im\,\hat{E}^{\gamma\mu\nu}). \qquad (27)
$$

If we write $\hat{\omega}^{\gamma}=B^{\gamma}+iC^{\gamma}$ with B,C real tensors, the Ricci scalar becomes

$$
R(x) = \{\partial_{\mu}C^{\gamma}_{\nu}-\partial_{\nu}C^{\gamma}_{\mu}+\frac{1}{2}\varepsilon^{\alpha\beta\gamma}C^{\alpha}_{\mu}C^{\beta}_{\nu}-\frac{1}{2}\varepsilon^{\alpha\beta\gamma}B^{\alpha}_{\mu}B^{\beta}_{\nu}\}(Im\,\hat{E}^{\gamma\mu\nu}). \qquad (28)
$$

We vary the action with respect to B and achieve the classical value of B in the vacuum:

$$
\varepsilon^{\alpha\beta\gamma}B^{\alpha}_{\mu}Im\,\hat{E}^{\gamma\mu\nu}=\varepsilon^{\alpha\beta\gamma}B^{\alpha}_{\mu}\varepsilon^{\sigma\tau\gamma}E^{\sigma\tau\mu\nu}=0
$$

$$
(\delta^{\alpha\sigma}\delta^{\beta\tau}-\delta^{\alpha\tau}\delta^{\beta\sigma})B^{\alpha}_{\mu}E^{\sigma\tau\mu\nu}=B^{\alpha}_{\mu}E^{\alpha\beta\mu\nu}-B^{\alpha}_{\mu}E^{\beta\alpha\mu\nu}=0
$$

$$
2B^{\alpha}_{\mu}E^{\alpha\beta\mu\nu}=0\longrightarrow B^{\alpha}_{\mu}=0.
$$

Substituting the solution $B=0$, the action reduces to

$$
R(x)=\{\partial_{\mu}C^{\gamma}_{\nu}-\partial_{\nu}C^{\gamma}_{\mu}+\frac{1}{2}\varepsilon^{\alpha\beta\gamma}C^{\alpha}_{\mu}C^{\beta}_{\nu}\}(Im\,\hat{E}^{\gamma\mu\nu}). \qquad (29)
$$

Classically, one can solve $\delta S/\delta\omega = 0$ in the vacuum and substitute the solution inside action to obtain General Relativity with null torsion. In presence of matters, the two actions

$$S_1 = S_1[\omega, e] \qquad S_2 = S_2[e] = S_1\left[\omega \left| \frac{\delta S_1}{\delta\omega} = 0, e\right.\right]$$

give a different classical physics, although it is in both cases compatible with modern measurements. What we have obtained in (29) is an action which sits halfway between S_1 and S_2 and then it's also compatible with measurements. We define

$$A_\mu = \frac{1}{4} C^\gamma_\mu (i\sigma)^\gamma \qquad A^\dagger_\mu = -A_\mu$$
$$f^{\mu\nu} = (Im\,\hat{E})^{\gamma\mu\nu}\sigma^\gamma$$

and use the relations

$$tr\,(\sigma^\alpha\sigma^\beta) = 2\delta^{\alpha\beta}$$
$$tr\,(\sigma^\alpha\sigma^\beta\sigma^\gamma) = 2i\varepsilon^{\alpha\beta\gamma}.$$

It's straightforward

$$\begin{aligned} R(x) &= -2i\,tr\,\{\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - 2A_\mu A_\nu(x)\} f^{\mu\nu} \\ &= -2i\,tr\,\{\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - [A_\mu, A_\nu](x)\} f^{\mu\nu} \\ &= 2i\,tr\,\left([\nabla_\mu, \nabla_\nu] f^{\mu\nu}(x)\right) \end{aligned} \tag{30}$$

with

$$\nabla_\mu = \partial_\mu - A_\mu$$

$$\begin{aligned} f^{\mu\nu} &= (Im\,\hat{E})^{\gamma\mu\nu}\sigma^\gamma = \varepsilon^{\alpha\beta\gamma}\sigma^\gamma E^{\alpha\beta\mu\nu} \\ &= -i\sigma^\alpha\sigma^\beta E^{\alpha\beta\mu\nu} = -i(e^{\alpha\mu}\sigma^\alpha)(e^{\beta\nu}\sigma^\beta). \end{aligned} \tag{31}$$

We can set $d^\mu = \sqrt{e}e^{i\mu}(i\sigma)^i$ with $\sigma^0 = \mathbf{1}$, where $e = |det\,(e^{i\mu})|^{-1}$. In this way

$$2i\sqrt{|h|}f^{\mu\nu} = \left(d^{\dagger\mu}d^\nu - d^{\dagger\nu}d^\mu\right).$$

This is because $\sigma^\alpha\sigma^\beta = -\sigma^\alpha\sigma^\beta$ but $\sigma^\alpha\mathbf{1} = \mathbf{1}\sigma^\alpha$. In the end:

$$\sqrt{-h}R(x) = tr\,\left([\nabla_\mu, \nabla_\nu](d^{\dagger\mu}d^\nu(x) - d^{\dagger\nu}d^\mu(x))\right) \tag{32}$$

To move from lorentzian to euclidean spaces we have to substitute dx^0 with $-idx^0$. This is equal to exchange σ^0 with $-i\sigma^0$ where it appears, ie inside d^μ. We choose to maintain the same definition

$$d^\mu = \sqrt{e}e^{i\mu}(i\sigma^i)$$

by redefining $\sigma^0 = -i\mathbf{1}$.

We use now the usual isomorphism between quaternions and Pauli matrices, substituting $i(\sigma^0, \sigma^1, \sigma^2, \sigma^3)$ with $(1, i, j, k)$ and $tr\,(**)$ with $2(**)$. The gauge fields A_μ become purely imaginary quaternions and the Ricci scalar simplifies in

$$\begin{aligned}\sqrt{h}R(x) &= 2[\nabla_\mu, \nabla_\nu](d^{*\mu}d^\nu(x) - c.c.)\\ &= 2[\nabla_\mu, \nabla_\nu]d^{*\mu}d^\nu(x) + c.c..\end{aligned} \tag{33}$$

3.3 Ricci scalar in the new paradigm

Consider this definition

$$S_{HE} = 8tr\,(M^\dagger M) = 8tr\,(\hat{M}^\dagger\hat{M})$$

where $\hat{M}$ is the associated normal matrix of M as defined in [1]. $MM^\dagger$ is self-adjoint, so its track is real. We insert in S_{HE} the usual expansion $\hat{M} = UD^\mu\nabla_\mu U^\dagger$, obtaining

$$
\begin{aligned}
S_{HE} &= 8tr\,[(UD^{\mu}\nabla_{\mu}U^{\dagger})^{\dagger}(UD^{\nu}\nabla_{\nu}U^{\dagger})] \\
&= 8tr\,[U\nabla_{\mu}^{\dagger}D^{\dagger\mu}U^{\dagger}UD^{\nu}\nabla_{\nu}U^{\dagger}] \\
&= 8tr\,[\nabla_{\nu}\nabla_{\mu}^{\dagger}D^{\dagger\mu}D^{\nu}].
\end{aligned} \tag{34}
$$

This term is gauge invariant only if $tr\,(\nabla_{\mu}^{\dagger}\nabla_{\nu}\sqrt{h}h^{\mu\nu}) = 0$. Hence we'll impose this condition on physical states of quantum theory and we'll discard all terms proportional to it.

$$
\begin{aligned}
S_{HE} &= 8tr\,[\nabla_{\nu}\nabla_{\mu}^{\dagger}\sqrt{h}(h^{\mu\nu}+E^{\mu\nu})] \\
&= 8tr\,[\nabla_{\nu}\nabla_{\mu}^{\dagger}\sqrt{h}E^{\mu\nu}] \\
&= -4tr\,\{[\nabla_{\mu}^{\dagger},\nabla_{\nu}]\sqrt{h}E^{\mu\nu}\}.
\end{aligned} \tag{35}
$$

Expanding the covariant derivatives we obtain

$$
\begin{aligned}
S_{HE} &= 2\sum_{a,b,c}\{\partial_{\mu}^{\dagger}A_{\nu}(x^a,x^b) - \partial_{\nu}A_{\mu}^{\dagger}(x^a,x^b) - \\
&\qquad -[A_{\mu}^{\dagger},A_{\nu}](x^a,x^b)\}d^{*\mu}(x^b)\delta^{bc}d^{\nu}(x^c)\delta^{ca} + c.c. \\
&= 2\sum_{a}\{\partial_{\mu}^{\dagger}A_{\nu}(x^a) - \partial_{\nu}A_{\mu}^{\dagger}(x^a) - \\
&\qquad -[A_{\mu}^{\dagger},A_{\nu}](x^a,x^a)\}d^{*\mu}(x^a)d^{\nu}(x^a) + c.c. \\
&= 2\sum_{a,b\neq a}\{\partial_{\mu}^{\dagger}A_{\nu}(x^a) - \partial_{\nu}A_{\mu}^{\dagger}(x^a) - [A_{\mu}^{\dagger}(x^a),A_{\nu}(x^a)] - \\
&\qquad -[A_{\mu}^{\dagger}(x^a,x^b),A_{\nu}(x^b,x^a)]\}\cdot d^{*\mu}(x^a)d^{\nu}(x^a) + c.c.
\end{aligned} \tag{36}
$$

Consider now a symmetry breaking with residual group $U(m,\mathbf{H})^{n/m}$ which regroups vertices in ensembles $\mathcal{U}^a = \{x_1^a, x_2^a, \ldots, x_m^a\}$. We assume that fields $A(x_i^a, x_j^b)$ with $a \neq b$ acquire big masses and thus we can neglect them. The symbol $\sum_a$ becomes $\sum_{a,i}$, while $\sum_{a,b\neq a}$ becomes $\sum_{a,i,b,j|(a,i)\neq(b,j)}$. After neglecting heavy fields, the last one is simply $\sum_{a,i,j\neq i}$.

$$\begin{aligned} S_{HE} &= 2\sum_a \{\partial^\dagger_\mu tr\, A_\nu(x^a) - \partial_\nu tr\, A^\dagger_\mu(x^a) - [tr\, A^\dagger_\mu(x^a), tr\, A_\nu(x^a)] - \\ &\quad - \sum_{i,j\neq i} [A^{\dagger ij}_\mu(x^a) A^{ji}_\nu(x^a) - A^{ij}_\nu(x^a) A^{\dagger ji}_\mu](x^a)\} \cdot d^{*\mu}(x^a) d^\nu(x^a) + c.c. \end{aligned} \tag{37}$$

For what follows we write $S_{HE} = 2\sum_a R^{ik}_{\mu\nu}\delta^{ik}(d^{*\mu}d^\nu - c.c.)$ with

$$\begin{aligned} R^{ik}_{\mu\nu} &= \partial^\dagger_\mu tr\, A_\nu(x^a) - \partial_\nu tr\, A^\dagger_\mu(x^a) - [tr\, A^\dagger_\mu(x^a), tr\, A_\nu(x^a)] - \\ &\quad - \sum_{i,j\neq i,k\neq j} [A^{\dagger ij}_\mu(x^a) A^{jk}_\nu(x^a) - A^{ij}_\nu(x^a) A^{\dagger jk}_\mu](x^a). \end{aligned} \tag{38}$$

$R^{ik}_{\mu\nu}$ is a generalization of curvature tensor. We have indicated with $tr\, A$ the track on ij, ie $\delta^{ij}A^{ij}(x^a) = \delta^{ij}A(x^a_i, x^a_j)$. Note that $[A^{\dagger ii}, A^{jj}] = 0$ when $i \neq j$ and then $\sum_i [A^{\dagger ii}_\mu, A^{ii}_\nu] = \sum_{ij}[A^{\dagger ii}_\mu, A^{jj}_\nu] = [tr\, A^\dagger_\mu, tr\, A_\nu]$. Consider now any skew hermitian matrix W_μ with elements $W^{ij}_\mu = A^{ij}_\mu$ for $i \neq j$ and $W^{ij}_\mu = 0$ for $i = j$. It belongs to the subalgebra of $u(m, \mathbf{H})$ made by all null track generators. This means that commutators between null track generators are null track generators too. In this way

$$\sum_{i,i\neq j} [A^\dagger_\mu(x^i, x^j), A_\nu(x^j, x^i)] = tr[W^\dagger_\mu, W_\nu] = 0.$$

Hence we can delete the mixed term in S_{EH}.

$$\begin{aligned} S_{HE} &= 2\sum_a \{\partial^\dagger_\mu tr\, A_\nu(x^a) - \partial_\nu tr\, A^\dagger_\mu(x^a) - [tr\, A^\dagger_\mu(x^a), tr\, A_\nu(x^a)]\} \cdot \\ &\quad \cdot (d^{*\mu}(x^a) d^\nu(x^a) - c.c.) \end{aligned}$$

In the arrangement field paradigm, the operator $\dagger$ transposes also rows with columns in matrices which represent ∂ and A. As we have seen, the fields A

which intervene in R are only the diagonal ones, so the transposition of rows with columns is trivial. Conversely, if we consider the matrix which represents ∂ (we have called it $\tilde{M}$), we note that $\partial^\dagger = -\partial$. Hence

$$\nabla_\nu^\dagger = (\partial_\nu + A_\nu)^\dagger = \partial_\nu^\dagger + A_\nu^\dagger = -\partial_\nu - A_\nu = -\nabla_\nu.$$

Applying this to S_{HE},

$$\begin{aligned} S_{HE} &= -2\sum_a \{\partial_\mu tr\, A_\nu(x^a) - \partial_\nu tr\, A_\mu(x^a) - [tr\, A_\mu(x^a), tr\, A_\nu(x^a)]\} \cdot \\ &\qquad\qquad\qquad\qquad\qquad\qquad \cdot(d^{*\mu}(x^a)d^\nu(x^a) - c.c.) \\ &= 2[\overset{G}{\nabla}_\mu, \overset{G}{\nabla}_\nu](d^{*\mu}(x^a)d^\nu(x^a) - c.c.) \\ &= \sum_a \sqrt{h}R(x^a) \to \int d^4x\,\sqrt{h}R(x) \end{aligned} \tag{39}$$

Here $\overset{G}{\nabla}$ is the gravitational covariant derivative $\overset{G}{\nabla} = \partial - tr\, A$. It's very remarkable that gauge fields in R are only the diagonal ones. First, this is the unique possibility to obtain $\overset{G}{\nabla}{}^\dagger{}_\nu = -\overset{G}{\nabla}_\nu$. Moreover, while gauge fields in R are tracks of matrices $(A_{ij})(x^a)$, we'll see as the other gauge fields in Standard Model correspond to non diagonal components.

4 The kinetic term

If we expand the field M as follows, with

$$M = \partial + \delta M, \tag{40}$$

the Ricci scalar becomes proportional to

$$\partial\delta M^\dagger - \partial\delta M + \delta M\delta M^\dagger. \tag{41}$$

Thus we have a quadratic term of "mass" and two superficial terms. These last become null for appropriate boundary conditions on the field.

We lack a kinetic term for M. How can we build it? One option is as follows:

$$\begin{aligned} S_{GB} &= 2tr\,(M^\dagger M M^\dagger M) \\ &= 2tr\,\left[U\nabla^\dagger_\mu D^{\dagger\mu} U^\dagger U D^\nu \nabla_\nu U^\dagger U \nabla^\dagger_\alpha D^{\alpha\dagger} U^\dagger U D^\beta \nabla_\beta U^\dagger\right] \\ &= 2tr\,\left[\nabla^\dagger_\mu D^{\dagger\mu} D^\nu \nabla_\nu \nabla^\dagger_\alpha D^{\dagger\alpha} D^\beta \nabla_\beta\right] \end{aligned} \tag{42}$$

We assume a residual symmetry under $U(m, \mathbf{H})^{n/m}$. This means that D^μ are matrices made of blocks $m \times m$ where every block is a quaternionic multiple of identity. We use newly the correspondence between $(1, i, j, k)$ and $i(\sigma^0, \sigma^1, \sigma^2, \sigma^3)$, writing any quaternion d as

$$d = d_a(i\sigma^a) + d_0(i\sigma^0)$$

We use letters a, b, c, d for indices with run on Pauli matrices, α, β, μ, ν for spatial coordinates indices and ijk for gauge indices (ie indices with run inside a single $\mathcal{U}^a$). Moreover we have to consider two gauge conditions

$$tr\,(\nabla^\dagger_\mu \nabla_\nu \sqrt{h} h^{\mu\nu}) = 0$$

$$tr\,(d^\beta \{\nabla_\beta, \nabla^\dagger_\mu\} d^{*\mu} d^\nu \nabla_\nu \nabla^\dagger_\alpha d^{*\alpha}) = 0.$$

The first one is what we have used for S_{HE}; the second one permits us to ignore terms proportional to $\{\nabla_\beta, \nabla^\dagger_\mu\}$ inside S_{GB}. Pay attention to not confuse the index a in the first group with the index a which runs over the vertices like in x^a_i.

We will see that physical fields are complex field which arise in three families, one for every possible choice of imaginary unit. This is true both for fermionic and bosonic fields. Thus the indices with letters a, b, c, d run over the three families. We take

$$Q^{ij}_{a\beta\mu} = \varepsilon^{abc}\nabla^{ik}_{\beta b}\nabla^{*jk}_{\mu c}$$
$$W^{\alpha\beta}_a = \frac{\sqrt{h}}{2}\varepsilon^{abc}E^{\alpha\beta}_{bc} + \sqrt{h}E^{\alpha\beta}_{a0} = \frac{1}{2}\varepsilon^{abc}D^{\dagger\alpha}_b D^\beta_c + \frac{1}{2}D^{\dagger\alpha}_a D^\beta_0 - \frac{1}{2}D^{\dagger\beta}_0 D^\alpha_a$$
$$W^{\alpha\beta}_0 = \sqrt{h}h^{\alpha\beta} = \frac{1}{2}D^{\dagger\alpha}_a D^\beta_a + \frac{1}{2}D^{\dagger\alpha}_a D^\beta_a + \frac{1}{2}D^{\dagger\alpha}_0 D^\beta_0 + \frac{1}{2}D^{\dagger\beta}_0 D^\alpha_0$$
$$S_{GB} = \sum_a L_{GB}(x^a)$$

Then

$$\begin{aligned}
L_{GB} &= \frac{1}{2}\left[Q^{ij}_{\beta\mu}W^{\mu\nu}Q^{ji}_{\nu\alpha}W^{\alpha\beta}\right] \\
&= \frac{1}{2}tr\left(\sigma^a\sigma^b\sigma^c\sigma^d\right)\left[Q^{ij}_{a\beta\mu}W^{\mu\nu}_b Q^{ji}_{c\nu\alpha}W^{\alpha\beta}_d\right] - \left[Q^{ij}_{a\beta\mu}W^{\mu\nu}_0 Q^{ji}_{a\nu\alpha}W^{\alpha\beta}_0\right] \\
&= \left(\delta^{ab}\delta^{cd} - \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\right)\left[Q^{ij}_{a\beta\mu}W^{\mu\nu}_b Q^{ji}_{c\nu\alpha}W^{\alpha\beta}_d\right] - \left[Q^{ij}_{a\beta\mu}W^{\mu\nu}_0 Q^{ji}_{a\nu\alpha}W^{\alpha\beta}_0\right]
\end{aligned}$$

We analyze each term one by one

$$\begin{aligned}
L^1_{GB} &= \left[Q^{ij}_{a\beta\mu}W^{\mu\nu}_a Q^{ji}_{c\nu\alpha}W^{\alpha\beta}_c\right] \\
&= \frac{1}{2}\left[Q^{ij}_{ab\beta\mu}W^{\mu\nu}_{ab}Q^{ji}_{cd\nu\alpha}W^{\alpha\beta}_{cd}\right] = 2hR^{ij\nu}_\beta R^{ji\beta}_\nu
\end{aligned} \tag{43}$$

$$\begin{aligned}
L^2_{GB} &= -\left[Q^{ij}_{a\beta\mu}W^{\mu\nu}_b Q^{ji}_{a\nu\alpha}W^{\alpha\beta}_b + Q^{ij}_{a\beta\mu}W^{\mu\nu}_0 Q^{ji}_{a\nu\alpha}W^{\alpha\beta}_0\right] \\
&= -\frac{h}{2}R^{ij}_{ac\beta\mu}R^{jiac}_{\nu\alpha}(h^{\mu\alpha}h^{\nu\beta} - h^{\mu\beta}h^{\nu\alpha} + E^{\beta\mu}_{bd}E^{\nu\alpha}_{bd}) - \frac{\sqrt{h}}{4!}\varepsilon^{\beta\mu\nu\alpha}R^{ij}_{ac\beta\mu}R^{acji}_{\nu\alpha} \\
&= -hR^{ij}_{ac\beta\mu}R^{jiac}_{\nu\alpha}h^{\mu\alpha}h^{\nu\beta} - hR^{ij}R_{ji} - hR^{ij}_{ac\beta\mu}R^{*jiac\beta\mu}
\end{aligned} \tag{44}$$

$$L^3_{GB} = \left[Q^{ij}_{a\beta\mu}W^{\mu\nu}_b Q^{ji}_{b\nu\alpha}W^{\alpha\beta}_a\right] = 2hR^{ij\alpha}_\mu R^{ji\mu}_\alpha \tag{45}$$

$R^{ij}_{\beta\mu}$ was defined in (38), while $\sqrt[4]{h}R^{ij}_\mu = R^{ij}_{\beta\mu}d^\beta$ and $\sqrt{h}R^{ij} = R^{ij}_{\beta\mu}d^\beta d^{*\mu}$. You understand in a moment that for $i \neq j$ we have $R^{ij}_{ac\beta\mu}R^{jiac}_{\nu\alpha}h^{\mu\alpha}h^{\nu\beta} = \sum_b F^b_{\mu\nu}F^{b\mu\nu}$

with $\varepsilon(bac) = 1$. The index b runs over three fields families and $F_{b\mu\nu}$ is a strength field tensor. In this way the terms $R^{ij\nu}_{\beta} R^{ji\beta}_{\nu}$ and $R^{ij} R_{ji}$ are terms which mix families.

The trouble with S_{GB} is that it generates a factor h instead of $\sqrt{h}$. However, we can solve the problem imposing the gauge condition $h = 1$. Note that for $i = j$ we have

$$L_{GB} = -R_{ac\beta\mu} R^{ac\beta\mu} - R^2 - R_{ac\beta\mu} R^{*ac\beta\mu} + 4 R^{\alpha}_{\mu} R^{\mu}_{\alpha}$$

which is a topological term and it doesn't change the Einstein equations. The Gauss-Bonnet theorem on a 4 dimensional manifold says that

$$S_{GB} = \int L_{GB} = 8\pi^2 \chi(\Lambda^4).$$

This remains true with $S_{GB} = tr\,(M^{\dagger} M M^{\dagger} M)$ only if gauge conditions are satisfied, so that we can substitute them with $S_{GB} = 8\pi^2 \chi(\Lambda^4)$. In this way we obtain a Faddev-Popov term like $FP = tr\,(\bar{c} M^{\dagger} M M^{\dagger} M b)$ with $\bar{c}, b$ ghost fields.

Expanding M we obtain terms from S_{GB} of the following form:

$$\begin{aligned} (\partial \delta M)\,(\partial \delta M) &\longrightarrow \text{ Kinetic} \\ (\delta M)^2\,(\partial \delta M) &\longrightarrow \text{ Mixed} \\ (\delta M)^4 &\longrightarrow \text{ Potential} \end{aligned} \tag{46}$$

The potential term combines with mass term to generate a non-trivial potential of form

$$(\delta M)^4 - (\delta M)^2 \tag{47}$$

It has non trivial minimums which should correspond to classical solutions for gauge fields and Einstein equations solutions for the metric.

5 Connections with Strings and Loop Gravity

We have seen in [1], at **Remark 13**, that some similarities exist between diagonal components of M (loops) and closed strings in string theory. Now we have discovered that such diagonal components describe a gravitational field. Is then a case that the lower energy state for closed string is the graviton? We think no. Moreover, we have seen that gauge fields correspond to non-diagonal components of M, ie open edge in the graph. This finds also a connection with open strings, whose lower energy states are gauge fields. We have shown that a symmetry $U(m, \mathbf{H})$ arises when vertices are grouped in ensembles $\mathcal{U}^a$ containing m vertices. This seems to represent a superimposition of m universes or branes. Gauge fields for such symmetry correspond to open edge which connect vertices in the same $\mathcal{U}^a$. Is then a case that the same symmetry arises in open strings with endpoints in m superimposed branes? We still think no. Until now we have supposed that open edges between vertices in the same $\mathcal{U}^a$ have length zero, so that we haven't to introduce extra dimensions. However, by $T-duality$ such edges correspond to open strings with $U(m, \mathbf{H})$ Chan-Paton which moves in an infinite extended extra dimension. This happens because an absente extra dimension is a compactified dimension with $R = 0$ and $T-duality$ sends R in $1/R$. Regarding edges between vertices in different $\mathcal{U}^a$, we see that they have a mass proportional to separation between endpoints. This is true both in our model and string theory.

The following two theorems emphasize a triality between *Arrangement Field Theory*, *String Theory* and *Loop Quantum Gravity*. We can see as they are different manifestations of the same theory.

Theorem 2 *Every element M^{ij} in the arrangement matrix can be written as a state in the Hilbert space of* Loop Quantum Gravity*, ie an holonomy for a $SU(2)$ gauge field*[1]*. In this way, every field (gauge or gravitational) becomes a manifes-*

[1] In Loop Gravity the gauge field appears usually in the form iA with A real. We incorporate

tation of only gravitational field.

Proof. An element M^{ij} can always be written in the following form:

$$M^{ij} = |M^{ij}| exp \left(\int_{x_i}^{x_j} A_\mu dx^\mu \right) \tag{48}$$

with $\mu = 1, 2, 3$ and

$$|M^{ij}| = exp \left(\int_{x_i}^{x_j} A_0 dx^0 \right).$$

Here A_μ is a purely imaginary $SU(2)$ connection and A_0 is a real field. Obviously, we take A_μ quaternionic by using the usual correspondence with Pauli matrices. The integration is intended over the edge which goes from vertex i to vertex j, parametrized by any $\tau \in [0, 1]$. If you look (48), you see on the left a discrete space (the graph) with discrete derivatives and fields which are defined only on the vertices. On the right you find instead a Hausdorff space with continuous paths, continuous derivatives and fields which are defined everywhere. Applying eventually a transformation in $U(n, \mathbf{H}) \otimes U(n, \mathbf{H})$, we have

$$M^{ij} = D^{ik\mu} \nabla_\mu^{kj} = D^{ii\mu} \nabla_\mu^{ij} = d^\mu(x_i) \nabla_\mu^{ij}. \tag{49}$$

In *Loop Quantum Gravity* we consider any space-time foliation defined by some temporary parameter and then we quantize the theory on a tridimensional slice. The simpler choice is a foliation along x_0: in this case the metric on the slice is simply the spatial block 3×3 inside the four dimensional metric when it's taken in temporary gauge. In such framework we have $d^0 = \mathbf{1}$ and $[d^\mu(x), A_\nu(x')] = \delta_\nu^\mu \delta^3(x - x')$ with $\mu, \nu = 1, 2, 3$. We deduce the relation $d^\mu(x) = \delta/\delta A_\mu(x)$ and apply it to (49) when vertices i and j sit on the same slice. We obtain

$$d^\mu(x_i) \nabla_\mu^{ij} = \frac{\delta}{\delta A_\mu(x_i)} \nabla_\mu^{ij} = |M^{ij}| exp \left(\int_{x_i}^{x_j} A_\mu dx^\mu \right) \tag{50}$$

the i inside A so that $A^i \sigma^i$ corresponds to a quaternionic number.

with $\mu = 1, 2, 3$. Note that $x_0(x_i) = x_0(x_j)$ when i and j sit on the same slice. Hence

$$|M^{ij}| = exp\left(\int_{x_i}^{x_j} A_0 dx^0\right) = exp\left(\oint A_0 dx^0\right).$$

Consider now the following relation:

$$exp\left(\int_{x_i}^{x_j} A_\mu dx^\mu\right) = \frac{\delta}{\delta A_\nu}\int_\Omega d^2 s\, n_\nu exp\left(\int_{x_i}^{x_j} A_\mu dx^\mu\right) \tag{51}$$

with

$$n_\nu = \frac{1}{2}\varepsilon_{\nu\mu\alpha}\frac{\partial x^\mu}{\partial s^a}\frac{\partial x^\alpha}{\partial s^b}\varepsilon^{ab}.$$

Here s^a with $a = 1, 2$ are coordinates on a two dimensional surface Ω. Ω is intended to shrink around point x_i until it contains only that point. We substitute (51) in (50), obtaining

$$\frac{\delta}{\delta A^i_\nu}\nabla^{ij}_\nu = \frac{\delta}{\delta A_\nu}\int_\Omega d^2 s\, n_\nu |M^{ij}| exp\left(\int_{x_i}^{x_j} A_\mu dx^\mu\right)$$

and then

$$\begin{aligned}\nabla^{ij}_\nu &= \int_\Omega d^2 s\, n_\nu |M^{ij}| exp\left(\int_{x_i}^{x_j} A_\mu dx^\mu\right) + K_\nu \\ &= \int_\Omega d^2 s\, n_\nu exp\left(\int_{x_i}^{x_j} A_\mu dx^\mu\right) + K_\nu.\end{aligned}$$

In the second line we have taken $\mu = 0, 1, 2, 3$. For diagonal components this becomes

$$A^{ii}_\nu = \int_\Omega d^2 s\, n_\nu exp\left(\oint A_\mu dx^\mu\right) + K_\nu. \tag{52}$$

We have used $\partial^{ii} = 0$ because the matrix which represents the discrete derivative is null along diagonal. The circle in $\oint$ is infinitely small and then

$$\begin{aligned} A^{ii}_{\nu} &= \int_{\Omega} d^2 s\, n_{\nu} \left(1 + \oint A_{\mu} dx^{\mu}\right) + K_{\nu} \\ &= (\Sigma_{\Omega} + A_0) n_{\nu}(x_i) + A_{\nu}(x_i) + K_{\nu}. \end{aligned} \tag{53}$$

The surface Ω shrinks on x_i and then $\Sigma \to 0$. If we set $K_{\nu} = A_0 n_{\nu}(x_i)$, we obtain

$$A^{ii}_{\nu} = A_{\nu}(x_i).$$

This verifies the consistence of our definition and proves the theorem. ■

Remark 3 (third quantization) *Note that canonical quantization of gauge fields implies*

$$\left[\partial_0 A^{ij}_{\alpha}(x_a), A^{ij}_{\nu}(x_b)\right] = \left[\partial_0 A_{\mu} \frac{\delta \nabla^{ij}_{\alpha}}{\delta A_{\mu}}(x_a), \nabla^{ij}_{\nu}(x_b)\right] = \delta_{\alpha\nu} \delta^3(x_a - x_b).$$

Here we have used the fact that $\partial^{ij} = 0$ not only for $i = j$ but also for i and j in the same ensemble $\mathcal{U}^a$. This implies $\nabla^{ij} = A^{ij}$. Moreover ∇^{ij} is a state in the Hilbert space of Loop Quantum Gravity and hence we have a sort of third quantization which applies on gravitational states and creates gauge fields:

$$\left[\dot{A}_{\mu} \frac{\delta \Psi[A]}{\delta A_{\mu}}, \Psi[A']\right] = \delta(A - A').$$

Theorem 4 *The actions $tr\,(M^{\dagger} M)$ and $tr\,(M^{\dagger} M M^{\dagger} M)$ are sums of exponentiated string actions.*

Proof. We obtain from theorem 2:

$$\begin{aligned} M^{ij} M^{*jk} M^{kl} M^{*li} &= exp\left(\int_{\partial\Box} A_{\mu} dx^{\mu}\right) \\ &= exp\left(\int_{\Box} F_{\mu\nu} dx^{\mu} \wedge dx^{\nu}\right) \\ &= exp\left(\int_{\Box} \varepsilon^{ab} F_{\mu\nu} X^{\mu}_{,a} X^{\nu}_{,b}\, d^2 s\right) \end{aligned} \tag{54}$$

This is the exponential of an action for open strings whose worldsheet is a square made by edges (ij), (jk), (kl), (li). The strings move in a curved background with antisymmetric metric $F_{\mu\nu} = (d \wedge A)_{\mu\nu}$. In a similar manner

$$M^{ij} M^{*jk} M^{ki} \;=\; exp\left(\int_{\triangle} \varepsilon^{ab} F_{\mu\nu} X^{\mu}_{,a} X^{\nu}_{,b}\, d^2 s\right) \tag{55}$$

This is the exponential of an action for open strings whose worldsheet is a triangle.

$$M^{ij} M^{*ji} \;=\; exp\left(\int_{O} \varepsilon^{ab} F_{\mu\nu} X^{\mu}_{,a} X^{\nu}_{,b}\, d^2 s\right) \tag{56}$$

This is the exponential of an action for open strings whose worldsheet is a circle.

$$M^{ii} \;=\; exp\left(\int_{O} \varepsilon^{ab} F_{\mu\nu} X^{\mu}_{,a} X^{\nu}_{,b}\, d^2 s\right) \tag{57}$$

The same of above.

$$M^{ii} M^{jj} \;=\; exp\left(\int_{Cil} \varepsilon^{ab} F_{\mu\nu} X^{\mu}_{,a} X^{\nu}_{,b}\, d^2 s\right) \tag{58}$$

This is the exponential of an action for closed strings whose worldsheet is a cilinder. This concludes the proof. ■

6 Standard model interactions

We suppose that a residual symmetry for $U(1, \mathbf{H})^n \otimes U(6, \mathbf{H})^{n/6}$ survives. $U(1, \mathbf{H})$ correspond to a $SU(2)$ in the $SU(2) \otimes SU(2) \sim SO(4)$ which is the gravitational gauge group. The second $SU(2)$ is comprised in $U(6, \mathbf{H})^{n/6}$ If we consider the ensembles $\mathcal{U}^a = (x^a_1, x^a_2, x^a_3, x^a_4, x^a_5, x^a_6)$ as the real physical points, $U(6, \mathbf{H})^{n/6}$ can be considered as a local $U(6, \mathbf{H})$.

The fields $A(x_i^a, x_j^b)$ with $a = b$ (we call them $A(x^a)$) can be written as a combination of 6×6 skew adjoint quaternionic matrices. These matrices form the $U(6, \mathbf{H})$ algebra which has 78 generators ω with $\omega^\dagger = -\omega$.

$$\omega = \begin{pmatrix} \vec{y} & b+\vec{b} & c+\vec{c} & d+\vec{d} & e+\vec{e} & m+\vec{m} \\ -b+\vec{b} & \vec{a}_1 & f+\vec{f} & g+\vec{g} & h+\vec{h} & p+\vec{p} \\ -c+\vec{c} & -f+\vec{f} & \vec{a}_2 & s+\vec{s} & q+\vec{q} & r+\vec{r} \\ -d+\vec{d} & -g+\vec{g} & -s+\vec{s} & \vec{a}_3 & k+\vec{k} & t+\vec{t} \\ -e+\vec{e} & -h+\vec{h} & -q+\vec{q} & -k+\vec{k} & \vec{a}_4 & v+\vec{v} \\ -m+\vec{m} & -p+\vec{p} & -r+\vec{r} & -t+\vec{t} & -v+\vec{v} & \vec{a}_5 \end{pmatrix}$$

Consider now the subalgebra of the following form with complex (not quaternionic) components except for y which remains quaternionic:

$$\omega = \begin{pmatrix} \vec{y} & 0 & 0 & 0 & 0 & 0 \\ 0 & \vec{a}_1 & f+\vec{f} & g+\vec{g} & h+\vec{h} & p+\vec{p} \\ 0 & -f+\vec{f} & \vec{a}_2 & s+\vec{s} & q+\vec{q} & r+\vec{r} \\ 0 & -g+\vec{g} & -s+\vec{s} & \vec{a}_3 & k+\vec{k} & t+\vec{t} \\ 0 & -h+\vec{h} & -q+\vec{q} & -k+\vec{k} & \vec{a}_4 & v+\vec{v} \\ 0 & -p+\vec{p} & -r+\vec{r} & -t+\vec{t} & -v+\vec{v} & \vec{a}_5 \end{pmatrix}$$

Moreover we put the additional condition $\vec{a} = \sum_l \vec{a}_l = 0$. The field $y = tr\,\omega$ is the only one which contributes to Ricci scalar. Conversely, all other fields belong to a $SU(5)$ subgroup, which defines the Georgi - Glashow grand unification theory. The symmetry breaking in Georgi - Glashow model is induced by Higgs bosons in representations which contain triplets of color. These color triplet Higgs can mediate a proton decay that is suppressed by only two powers of GUT scale. However, our mechanism of symmetry breaking doesn't use such Higgs bosons, but descends from the expectation values of quadratic terms AA, which derive from non trivial minimums of a potential $AA - AAAA$. So we circumvent the problem.

Restrict now the attention to the $SU(2)_{GRAVITY} \otimes SU(2) \otimes U(1) \otimes SU(3)$ generators, that are the generators of standard model plus gravity.

$$\omega = \begin{pmatrix} \vec{y} & 0 & 0 & 0 & 0 & 0 \\ 0 & \vec{a}_1 & f+\vec{f} & 0 & 0 & 0 \\ 0 & -f+\vec{f} & \vec{a}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \vec{a}_3 & k+\vec{k} & t+\vec{t} \\ 0 & 0 & 0 & -k+\vec{k} & \vec{a}_4 & v+\vec{v} \\ 0 & 0 & 0 & -t+\vec{t} & -v+\vec{v} & \vec{a}_5 \end{pmatrix}$$

It's easy to see that all the standard model fields transform under this subgroup as the adjoint representation of $U(6,\mathbf{H})$. In this way themselves are elements of $U(6,\mathbf{H})$ algebra, explicitly:

$$\psi = \begin{pmatrix} 0 & e & -\nu & d^c_R & d^c_G & d^c_B \\ -e^* & 0 & e^c & -u_R & -u_G & -u_B \\ \nu^* & -e^{c*} & 0 & -d_R & -d_G & -d_B \\ -d^{c*}_R & u^*_R & d^*_R & 0 & u^c_B & -u^c_G \\ -d^{c*}_G & u^*_G & d^*_G & -u^{c*}_B & 0 & u^c_R \\ -d^{c*}_B & u^*_B & d^*_B & u^{c*}_G & -u^{c*}_R & 0 \end{pmatrix}$$

We have used the standard formalism for Georgi - Glashow model, where the basic fields are all left. In place of right fields it uses their charge conjugates ^{c}, that are left fields. The subscripts R, G, B indicates the color charge for the strong interacting particles (R=red, G=green, B=blue). The sub matrix (ω^{ij}) with $i, j = 2, 3, 4, 5, 6$ is the representation 10 of $SU(5)$ (the adjoint representation), while the array (ω^{1j}) with $j = 2, 3, 4, 5, 6$ is the representation $\bar{5}$ of the same group (the fundamental representation).

In this formalism, given $\omega \in su(3)\otimes su(2)\otimes u(1)$, the transformation $\delta\psi = [\omega, \psi]$ corresponds to the usual transformation $\delta\psi = \omega\psi$ in the standard model formalism. We see that the only fields which transform correctly under $SU(2)$ gravity are e, ν and d^c. For now we do not care.

We note rather that, when we restrict the elements of ω from the quaternions to the complex numbers, we have 3 possibility to do it. A complex number is not only in the form $a + ib$, with $a, b \in R$, but also $a + jb$ and $a + kb$. The same is true for a fixed linear combination $a + (ci + dj + fk)b$, where $c, d, f \in R$ and $c^2 + d^2 + f^2 = 1$.

The choice of j in place of i determines another set of (ω, ψ) isomorphic to the first one. So we have a second family of fermionic fields. In the same way we obtain a third set choosing k. Here's how emerge the three families of fermionic fields found in the modern experiments.

The three families are related by the group $SU(2)$ which rotates an unitary vector in R^3 with coordinates (c, d, f). Its generators are

$$\omega = \frac{\vec{y}}{6} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} .$$

Their diagonal form suggests an identification between this group and gravitational $SU(2)$. If the two groups coincided, all fields would transform correctly under gravitational $SU(2)$.

Note that three families have to exist also for bosonic particles (photon, $W^{\pm}$, Z, gluons) although they are probably indistinguishable.

Other interesting thing is that we have no warranty for the persistence of $U(6, \mathbf{H})$ in the entire universe. However we have surely at least the symmetry $U(1, \mathbf{H}) \otimes U(1, \mathbf{H})$, which implies the secure existence of gravity.

We can introduce grassmann coordinate with derivatives ∂_g and $\partial_g^\dagger$, and covariant derivatives $\theta\nabla_g = \theta\partial_g + \theta\psi$ and $\theta^\dagger\nabla_g^\dagger = \theta^\dagger\partial_g^\dagger + \theta^\dagger\psi^\dagger$. In the arrangement field formalism these descend from a grassmanian matrix M_g or $M_g^\dagger$.

We can consider a unique generalized matrix $\hat{M} = M_g + M$ that, up to a generalized $U(n, \mathbf{H})$, becomes

$$
\begin{aligned}
\hat{M} &= \theta\nabla_g + d^\mu\nabla_\mu = \theta\partial_g + \theta\psi + d^\mu\nabla_\mu \\
\hat{M}^\dagger &= \nabla_g^\dagger\theta^\dagger + \nabla_\mu^\dagger d^{*\mu} = \partial_g^\dagger\theta^\dagger + \psi^\dagger\theta^\dagger + \nabla_\mu^\dagger d^{*\mu}.
\end{aligned} \tag{59}
$$

θ belongs to the ensemble of grassmanian quaternions (we call it **HG**). This means

$$
\begin{aligned}
&\theta = \theta_0 + i'\theta_1 + j'\theta_2 + k'\theta_3 \qquad \text{with } \theta_0, \theta_1, \theta_2, \theta_3 \text{ grassmanian} \\
&i'^\dagger = -i' \qquad j'^\dagger = -j' \qquad k'^\dagger = -k' \\
&\theta^\dagger = \theta_0 - i'\theta_1 - j'\theta_2 - k'\theta_3 \\
&\partial_g\theta = \partial_g^\dagger\theta^\dagger = 1 \qquad \partial_g\theta^\dagger = \partial_g^\dagger\theta = 0 \\
&i'j' = j'i' = k' \qquad j'k' = k'j' = i' \qquad k'i' = i'k' = j' \\
&ii' = i'i = jj' = j'j = kk' = k'k = -1 \\
&ij' = j'i = k' \qquad jk' = k'j = i' \qquad ki' = i'k = j' \\
&i'^2 = j'^2 = k'^2 = \theta^2 = 0 \\
&\theta\tilde{\theta} = -\tilde{\theta}\theta \qquad \forall\theta, \tilde{\theta} \in \mathbf{HG} \\
&\theta\theta^\dagger = -\theta^\dagger\theta = -2\theta_0(i'\theta_1 + j'\theta_2 + k'\theta_3)
\end{aligned} \tag{60}
$$

Remark 5 *Note that every quaternion commutes with a grassmann quaternion, ie* $a\theta = \theta a$, $a \in \mathbf{H}, \theta \in \mathbf{HG}$.

θ is the equivalent of d in the grassmanian part of $\hat{M}$. Obviously it is a grassmann function of grassmann coordinates $\theta_0, \theta_1, \theta_2, \theta_3$. Expanding θ in polynomial series, we have to stop at the power 1, because the square of a grassmann variable is zero. Moreover the constant corresponding to the power 0 has to be zero, because θ is grassmanian. So, it is proportional to the coordinates. We set the proportionality constants respectively to $1, i', j', k'$.

If we take two grassmanian arrays, f and g, we interpret the product $fg^\dagger$ as the product $g^\dagger f$, yet inverting the factors order in the products of components.

Expanding $tr\,(\hat{M}^\dagger \hat{M})$ we obtain

$$tr\,(\hat{M}^\dagger \hat{M}) \;=\; tr\,\left(d^\nu d^{*\mu} \nabla^\dagger_\mu \nabla_\nu\right) = \sum_a \sqrt{h} R(x^a). \tag{61}$$

To calculate $tr\,(\hat{M}^\dagger \hat{M} \hat{M}^\dagger \hat{M})$ we write first M^2 and $M^{\dagger 2}$.

$$\begin{aligned} M^2 &= \theta\partial_g + \theta\psi + \theta d^\mu\{\nabla_\mu, \psi\} + d^\mu \nabla_\mu d^\nu \nabla_\nu \\ M^{\dagger 2} &= \partial^\dagger_g \theta^\dagger + \psi^\dagger \theta^\dagger + \{\psi^\dagger, \nabla^\dagger_\alpha\} d^{*\alpha} \theta^\dagger + \nabla^\dagger_\alpha d^{*\alpha} \nabla^\dagger_\beta d^{*\beta} \end{aligned} \tag{62}$$

Here ψ is a matrix with entries in **HG**. If M has the form (59), then it is normal and satisfies $tr\,(M^\dagger M M^\dagger M) = tr\,(M^2 M^{\dagger 2})$. We calculate its value starting from the following product

$$tr\,(\theta d^\mu\{\nabla_\mu, \psi\}\{\psi^\dagger, \nabla^\dagger_\alpha\} d^{*\alpha}\theta^\dagger) \;=\; tr\,(\theta\theta^\dagger d^\mu\{\nabla_\mu, \psi\}\{\psi^\dagger, \nabla^\dagger_\alpha\} d^{*\alpha}). \tag{63}$$

Remember that operator tr acts as a sum over vertices. Now every vertex is labeled by a couple (θ, x_i) and then

$$tr\,(\theta\theta^\dagger(***)) = \left(\int d\theta^\dagger d\theta\, \theta\theta^\dagger\right) tr\,(***) = tr\,(***)$$

Hence

$$\begin{aligned} tr\,(\theta d^\mu\{\nabla_\mu, \psi\}\{\psi^\dagger, \nabla^\dagger_\alpha\} d^{*\alpha}\theta^\dagger) &= tr\,(d^\mu\{\nabla_\mu, \psi\}\{\psi^\dagger, \nabla^\dagger_\alpha\} d^{*\alpha}) \\ &= tr\,(d^{*\alpha} d^\mu [\nabla_\mu, \nabla^\dagger_\alpha]\psi\psi^\dagger) \\ &= \sum_a \sqrt{h} R(x^a)\psi^\dagger\psi \end{aligned} \tag{64}$$

In this way

$$
\begin{aligned}
tr\,(\hat{M}\hat{M}^\dagger\hat{M}\hat{M}^\dagger) &= tr\,\left(\psi^\dagger d^\mu\{\nabla_\mu,\psi\}+\{\psi^\dagger,\nabla^\dagger_\alpha\}d^{*\alpha}\psi\right)+\\
&+\sum_a\sqrt{h}R(x^a)\psi^\dagger\psi+S_{GB} \qquad (65)
\end{aligned}
$$

We have seen that every family distinguishes itself by the choice of complex unity. In this way we can write $\psi=\psi_0+i\psi_1+j\psi_2+k\psi_3$, where ψ_0 takes contributes from all the three families (we decompose it in $\psi_0=\phi_1+\phi_2+\phi_3$, where ϕ_i is the real component of the corresponding family). Using the correspondence $(1,i,j,k)\leftrightarrow i(\sigma^0,\sigma^1,\sigma^1,\sigma^2)$ with $\sigma_0=-i1$, the first term becomes

$$
i\,tr\,\left(\psi_l^\dagger(i\sigma^l)^\dagger\left(\sigma^s d_s^\mu\{\overset{G}{\nabla}_\mu,\psi_i(i\sigma^i)\}+\{A_\mu,\psi\}\right)\right) \qquad (66)
$$

$$
\Downarrow\!\!===\!\!\Downarrow
$$

$$
i\,tr\,\left(\psi_l^\dagger(i\sigma^l)^\dagger\left(\sigma^s d_s^\mu\{\overset{G}{\nabla}_\mu,\psi_i(i\sigma^i)\}+\sum_{q,q'=1}^{3}\{A^q_\mu,(\phi_{q'}+i\psi_{q'})\}\right)\right) \qquad (67)
$$

where in the covariant derivative we have included only the gravitational (track) contribution, while A_μ is intended to have null track. Moreover $i_1=i$, $i_2=j$ and $i_3=k$.

In the second line we have divided the 72 generators A_μ in three families of 24 generators. When they act on spinorial fields which belong to their own family, they behave exactly as the 24 generators of $SU(5)$. Conversely, when a generator A^q acts on a q'-field (with $q\neq q'$), it mimics the application of some generator $A^{q'}$ followed by a rotation in $SU(2)_{GRAVITY}$ which sends the family q' in the remaining family q''.

Concentrate now yourself on the derivative term

$$
i\,tr\,\left(\left(\begin{array}{cc}\psi_0^\dagger-i\psi_1^\dagger & -\psi_2^\dagger-i\psi_3^\dagger\\ \psi_2^\dagger-i\psi_3^\dagger & \psi_0^\dagger+i\psi_1^\dagger\end{array}\right)\sigma^i d_i^\mu\overset{G}{\nabla}_\mu\left(\begin{array}{cc}\psi_0+i\psi_1 & \psi_2+i\psi_3\\ -\psi_2+i\psi_3 & \psi_0-i\psi_1\end{array}\right)\right). \qquad (68)
$$

If we define the two components spinor

$$\hat{\psi} = \begin{pmatrix} \psi_0 + i\psi_1 \\ \psi_2 + i\psi_3 \end{pmatrix} = \hat{\psi}_1 + \hat{\psi}_2 + \hat{\psi}_3$$

$$\hat{\psi}_1 = \begin{pmatrix} \phi_1 + i\psi_1 \\ 0 \end{pmatrix}; \qquad \hat{\psi}_2 = \begin{pmatrix} \phi_2 \\ \psi_2 \end{pmatrix}; \qquad \hat{\psi}_3 = \begin{pmatrix} \phi_3 \\ i\psi_3 \end{pmatrix}$$

it becomes

$$i\,\hat{\psi}^\dagger\,\sigma^i d_i^\mu \{\overset{G}{\nabla}_\mu, \hat{\psi}\} + c.c. \tag{69}$$

This is the Weyl action, although with a new interpretation of spinorial components. Adding the other terms

$$\begin{aligned} tr\,(\hat{M}^\dagger \hat{M} \hat{M}^\dagger \hat{M}) \;\; &= \\ &= \int \left(i\,\hat{\psi}^\dagger\,\sigma^i d_i^\mu \{\overset{G}{\nabla}_\mu, \hat{\psi}\} + \sum_{q,q'} \hat{\psi}\{A_\mu^q, \hat{\psi}_{q'}\} + \sqrt{h} R(x) \sum_q \hat{\psi}_q^\dagger \hat{\psi}_q \right) dx + c.c. \end{aligned}$$

This expression includes only the LEFT contribution and we have to sum the RIGHT one to obtain the Dirac action. In this way we include all the contents of standard model as elements in the generalized $U(6, \mathbf{H})$ algebra. Terms which mix families can be used to calculate values in CKM and PMNS matrices. Masses for fermionic fields arise, as usual, from non null expectation values of $A_\mu(x_i^a, x_j^b)$ with $a \neq b$ in ∇_μ.

We obtain a contribute to Hilbert-Einstein action also from $\int d^4x \sqrt{h} R \bar{\psi} \psi$, due to a non null expectation value of $\sum_q \bar{\psi}_q \psi_q$. It contains in fact the chiral condensate, whose non null vacuum value breaks the chiral flavour symmetry of QCD Lagrangian.

Note that known fermionic fields fill up a matrix ψ with null track. However, only if $tr\,\psi \neq 0$ our action has an extra invariance under

$$\begin{aligned} A_\mu &\rightarrow d_\mu^{-1}\theta\psi \\ \psi &\rightarrow \overleftarrow{\partial}_g d^\mu A_\mu. \end{aligned} \tag{70}$$

Here $\overleftarrow{\partial}_g$ is a ∂_g which acts backwards. This means we have the same number of fermions and bosons, so that the vacuum energies erase each other.

Invariance (70) predicts the existence of a new colored fermionic sextuplet which sits on diagonal in ψ. We call it "gravitino", stressing that this particle must have spin 1/2 and so it is different from gravitino with spin 3/2 predicted by most supersymmetric theories.

7 The vector superfield

The invariance (70) suggests a connection with super-symmetric theories. We can think to arrange right and left fields in a two components array

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \qquad \theta = \begin{pmatrix} \theta_L \\ \theta_R \end{pmatrix}.$$

Note that now we have $\theta\theta \neq 0$ while $\theta^m = 0$ for $m > 2$.

We extend the generators of supersymmetric algebra by substituting $i\sigma^\mu$ with d^μ, noting that $d^\mu = i\sigma^\mu$ in a flat space.

$$Q = \partial_g + d^\mu \nabla_\mu \theta^\dagger$$

$$Q^\dagger = \partial_g^\dagger + \theta \nabla_\nu^\dagger d^{*\nu}$$

In the same manner we generalize the definition of vector superfield. In Wess-Zumino gauge it assumes the form

$$V = \theta\theta^\dagger \hat{V}$$

$$\hat{V} = d^\mu A_\mu + \theta\psi - \theta^\dagger\psi^\dagger - \frac{1}{2}\theta\theta^\dagger D.$$

The field-strength superfield is then

$$W = \psi^\dagger + \theta D + \frac{1}{2}\theta d^\nu d^{*\mu}[\nabla_\mu, \nabla_\nu] + \theta\theta d^\mu \nabla^\mu \psi$$

If we ignore the commutator and the non dynamical field D, we have

$$W = \psi^\dagger + \theta M^\dagger M + \theta\theta M\psi$$

It's easy to see that the usual term W^2 of supersymmetric theories generates the same terms we have found in $tr\,(\hat{M}^\dagger \hat{M}) - tr\,(\hat{M}^\dagger \hat{M}\hat{M}^\dagger \hat{M})$. This can mean that our theory includes supersymmetry, with the known fermionic fields which take the role of gauginos. In this way the right up quarks are gauginos for gluons, while right electrons are gauginos for W bosons.

8 Inflation

Our final action is

$$S = tr\left(\frac{MM^\dagger}{2\pi G} - \frac{1}{2}MM^\dagger MM^\dagger\right)$$

This is also an action for an $U(n, \mathbf{H})$ gauge theory with coupling constant $1/G$ in a mono-vertex space-time. In these theories the scaling of coupling constant can be calculated exactly in the limit of large n. In several cases the coupling constant changes its sign for big values of scale: this has considerable consequences for the first times after Big Bang, when a measurement of G has sense only at very high energies (very small distances). What said suggests that such measurement can return a negative value of G, which implies a repulsive force of gravity. In turn, repulsive gravity implies an accelerate expansion for the universe.

Because the entries of M are probability amplitudes, we would be it was dimensionless. However, when we pass from M to ∇, we need a scale Δ to define the matrix ∂. This justify the inclusion of Δ^{-1} inside M. If we extract this factor, the Hilbert Einstein action becomes

$$\frac{\Delta^4}{2\pi G\Delta^2}tr\,(MM^\dagger) = \frac{\Delta^2}{2\pi G}tr\,(MM^\dagger)$$

where we have also added the correct volume form Δ^4. This seems a more natural formulation when M represents probability amplitudes. In this way we can take Δ very small but not zero. The most natural choice is $\Delta^2 \sim G$.

In this case, what does it mean that G is negative? Negative G implies negative $\Delta^2 = ds^2$. In lorentzian spaces $\Delta^2 = dt^2 - ds^2 < 0$. For purely temporal intervals we'll have $dt^2 < 0$, so the time becomes imaginary. An imaginary time is indistinguishable from space. This hypothesis of a "spatial" time had already been espoused by Hawking as a solution for eliminate the singularity in the Big Bang [5].

9 Classical solutions

We rewrite our action in the form

$$S = \frac{1}{2}tr\,(MM^\dagger) - \frac{1}{4g}tr\,(MM^\dagger MM^\dagger)$$

where we have defined $g = \frac{\Delta^2}{2\pi G}$. We diagonalize M with a transformation in $U(n, \mathbf{H})$ and define $M^{ii} \equiv \varphi(x_i)$, $\varphi(x) = a(x) + \vec{b}(x)$. The lagrangian becomes:

$$L = \frac{1}{2}\left[a(x_i)^2 + |\vec{b}(x_i)|^2\right] - \frac{1}{4g}\left[a(x_i)^4 + |\vec{b}(x_i)|^2 + 2a(x_i)^2|\vec{b}(x_i)|^2\right]$$

The motion equations are

$$ga(x) - a(x)^3 - a(x)|\vec{b}(x)|^2 = 0$$

$$g\vec{b}(x) - \vec{b}(x)|\vec{b}(x)|^2 - a(x)^2\vec{b}(x) = 0$$

There are two solutions:

$$(1) \qquad a(x) = \vec{b}(x) = 0$$

$$(2) \qquad a(x)^2 + |\vec{b}(x)|^2 = MM^\dagger = g$$

The first one corresponds to the vacuum (all non-gravitational fields equal to zero) plus a solution of Einstein equations in the vacuum:

$$\psi = A_\mu = 0 \qquad R(x) = 0$$

The solution $MM^\dagger = g$ corresponds to a vacuum expectation value for $MM^\dagger$ equal to g. M contains a factor A, so that an expectation value for $MM^\dagger$ corresponds to an expectation value for AA. This means that

$$AAAA =< AA > AA + \text{quantum perturbations}$$

$< AA >$ gives a mass for A. More precisely, for $A \in U(n, \mathbf{H})/U(m, \mathbf{H})^{n/m}$,

$$m_A^2 \sim \frac{< MM^\dagger >}{\Delta^2} = \frac{g}{\Delta^2} = \frac{1}{2\pi G}$$

So the fields $A \in U(n, \mathbf{H})/U(m, \mathbf{H})^{n/m}$ have a mass in the order of Planck mass m_P. Moreover, in the primordial universe, when $k_B T \approx m_p$, all the fields behave like null mass fields. In that time the symmetry was then $U(n, \mathbf{H})$ and no arrangement exists. Our conclusion is that Quantum Gravity cannot be treated as a quantum field theory in an ordinary space. In what follows we explain how overcome this trouble.

10 Feynman diagrams

To quantizing M we make the expansion $M \to \partial + \delta M$, where ∂ acts as a discrete derivation according to the numeration of space time vertices.

$$\partial^{ij} f(x_j) = \frac{f(x_{j+1}) - f(x_{j-1})}{2}$$

This trick can work only in a discrete space-time, for which a bijection with $\mathbf{N}$ always exists. From now we indicate the perturbation δM simply with M, obtaining

$$MM^\dagger = -\partial\partial + \partial M^\dagger - M\partial + MM^\dagger$$

When applied to something else, the first two terms create only superficial terms. So, ignoring these ones, we have

$$MM^\dagger MM^\dagger = -M\partial^2 M^\dagger - M\partial(MM^\dagger) + MM^\dagger \partial M^\dagger + MM^\dagger MM^\dagger$$

Inverting the derivative in the second term and using the cyclicity of the track

$$MM^\dagger MM^\dagger = -M\partial^2 M^\dagger + MM^\dagger \partial M + MM^\dagger \partial M^\dagger + MM^\dagger MM^\dagger$$

The kinetic part of the lagrangian becomes

$$L_K = \frac{1}{4g} \left(2gMM^\dagger + M\partial^2 M^\dagger \right)$$

The potential part is

$$L_V = \frac{1}{4g} \left(MM^\dagger \partial M + MM^\dagger \partial M^\dagger + MM^\dagger MM^\dagger \right).$$

We obtain a propagator for M:

$$P_M = \frac{4g}{(2g - p^2)}$$

We calculate the superficial degree of divergence, by treating this theory as a one-dimensional quantum field theory. This can be done because tr is simply a sum over vertices and a bijection always exists from Λ^4 to $\mathbf{N}$. Obviously this is only an approximate way to doing calculations, substituting the sum in $\mathbf{N}$ with a one-dimensional integral. Conversely, if our intent is to recover a theory on continuous space, we should use first the bijection from $\mathbf{N}$ to $\mathbf{N}^4$ and take after the limit $\Delta \to 0$.

For this reason, p isn't a real momentum, although we'll call it with this name, but it is only a parameter for the Fourier expansion. In the same manner $2g$ isn't a real mass.

Every internal line I in the Feynmann graphs carries a p^{-2} for the propagator and an integration dp. Summing we have a p^{-1}. Every vertex with three legs (v) or four legs (V), carries a Dirac function $\sim p^{-1}$, but the vertex v carries an extra factor p. Moreover we have to consider another Delta function for the overall conservation of momentum. Summing all

$$div = -I - V - v + v - 1 = -I - V - 1$$

Indicating with E the external legs, we have

$$\begin{aligned} 2I + E &= 3v + 4V \\ div &= \frac{1}{2}E - 3V - \frac{3}{2}v - 1 \end{aligned}$$

Feynmann diagrams with loops have $I \geq 2$ and then $E_{MAX} = 4V + 3v - 4$. Divergence is negative (the diagram converges) if $E < 6V + 3v + 2$.

Note however that always $E \leq E_{MAX} = 4V + 3v - 4 < 6V + 3v + 2$ and then all diagrams converge.

11 Quantum Entanglement and Dark Matter

The elements of M which do not reside in or near the diagonal, describe connections between points that are not necessarily adjacent to each other, in the common sense. These connections construct discontinuous paths as in figure 1 and can be considered as quantum perturbations of the ordered space-time.

Such components permit us to describe the quantum entanglement effect, as it could be shown in detail in a complete coverage that goes beyond the purpose of the present paper.

It is remarkable that in this framework the discontinuity of paths is only apparent, and it is a consequence of imposing an arrangement to the space-time points. These discontinuous paths can be considered as continuous paths which cross wormholes. The trait of path inside a wormhole is described by a component of M far away from diagonal. The information seems to travel faster than light, but in reality it only takes a byway.

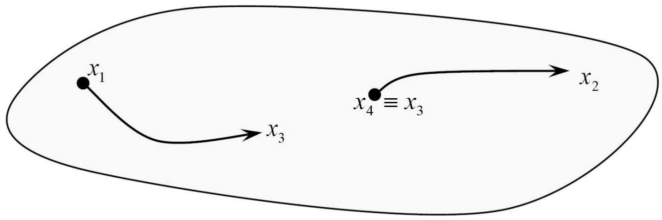

Figure 1: Discontinuous paths. The connection between x_3 and x_4 is done by a component of M far away from diagonal.

Imagine now a gravitational source with mass M_S which emits some gravitons with energy $\sim E_{PLANCK}$, directed to an orbiting body with mass M_B at distance r. In this case (respect such gravitons) the fields $M(x^a, x^b)$ with $a \neq b$ would behave as they had null mass. This implies the probable existence of connections (practicable by such gravitons) between every couple of vertices in the path from the source to the orbiting body. This means that if $r = \Delta j$, $j \in \mathbf{N}$, the graviton could reach the orbiting body by traveling a shorter path $\Delta j'$, $j > j' \in \mathbf{N}$. The question is: what is the average gravitational force perceived by the orbiting body?

The probability for a graviton to reach a distance r passing through m vertices is

$$P_m = (1-a)^{m-1}a \qquad with \ \sum_{m=1}^{\infty} P_m = 1$$

where $a = 1/j$. These are the probabilities for extracting one determined object in a box with j objects at the m-th attempt. In this way the effective length traveled by the graviton will be Δm.

We use these probabilities to compute the average gravitational force in a semiclassical approximation.

$$\begin{aligned} < F > &= G\frac{M_B M_S}{\Delta^2}\frac{a}{1-a}\int_1^\infty \frac{(1-a)^m}{m^2}dm \\ &= G\frac{M_B M_S}{\Delta^2}\frac{a}{1-a}[log(1-a)]\int_{log(1-a)}^{-\infty}\frac{e^x}{x^2}dx \end{aligned} \tag{71}$$

The last integral gives

$$\int_{log(1-a)}^{-\infty}\frac{e^x}{x^2}dx = -Ei(log(1-a)) + \frac{1-a}{log(1-a)}$$

We expand $\langle F\rangle$ near $a = 0$ (which implies $j >> 1$), obtaining

$$\frac{a}{(1-a)}[log(1-a)]\int_{log(1-a)}^{-\infty}\frac{e^x}{x^2}dx \approx a + a^2(log(a)+\gamma) + O(a^3).$$

Here γ is the Eulero-Mascheroni constant. The dominant contribution is then

$$\begin{aligned} < F > &\approx G\frac{M_B M_S}{\Delta^2}\cdot a\cdot(1 + a\,log(a) + a\gamma) \\ &\approx \frac{G}{\Delta}\frac{M_B M_S}{r}\left(1 - \frac{\Delta}{r}\left(log\left(\frac{r}{\Delta}\right) - \gamma\right)\right) \end{aligned} \tag{72}$$

If the massive object orbits at a fix distance r, its centrifugal force has to be equal to the gravitational force. This gives

$$< F > \approx \frac{G}{\Delta}\frac{M_B M_S}{r}\left(1 - \frac{\Delta}{r}\left(log\left(\frac{r}{\Delta}\right) - \gamma\right)\right) = \frac{M_B v^2}{r}$$

$$v^2 = \frac{G M_B M_S}{\Delta}\left(1 - \frac{\Delta}{r}\left(log\left(\frac{r}{\Delta}\right) - \gamma\right)\right)$$

We see that, varying the radius, the velocity remains more or less constant (It increases slightly with r). Can this explain the rotation curves of galaxies without introducing dark matter?

Surely not all gravitons have energy $> E_{PLANCK}$; at the same time we have to consider that G scales for small distances (hence for small m in (71)). It's possible that these factors reduces the extremely high value of r/Δ.

12 Lorentzian theory

After a Wick rotation the generators $i\sigma^j$ in $SU(2)_{BOOSTS}$ become $i(i\sigma^j) = -\sigma^j$. To realize the same thing in $\mathbf{H}$ we have to introduce another unity which takes the role of the i situated ahead σ. It sufficient to define I such that

$$I^2 = -1 \qquad I^\dagger = -I$$

$$[I, i] = [I, j] = [I, k] = 0$$

In total we have seven imaginary unities I, i, j, k, Ii, Ij, Ik, so that a generic number is

$$n = a + bI + ci + dj + ek + fIi + gIj + hIk = q + Ip$$

$$a, b, c, d, e, f, g, h \in \mathbf{R} \qquad q, p \in \mathbf{H}$$

We call the numbers with this form "extended quaternions" and indicate their ring with $\mathbf{H}_E$. It's easy to see that the six imaginary unities i, j, k, Ii, Ij, Ik of an extended quaternion satisfies the Lorentz algebra.

A vector in a Lorentzian space-time can be considered as an extended quaternion with the form

$$V = a + fIi + gIj + hIk$$

$$|V|^2 = V^\dagger V = a^2 - f^2 - g^2 - h^2$$

Note that extended quaternions are very different from split-octonions. Extended quaternions maintain in fact associativity. However they are not a division ring, because (as in lorentzian space-times) exist elements different from zero with null norm.

13 Conclusion

In this work we have applied the framework developed in [1] to describe the contents of our universe, ie forces and matter.

Doing this, we have discovered an unexpected road toward unification, which shows similarities with Loop Gravity, String Theory and Georgi - Glashow model. For the first time a natural symmetry justifies the existence of three particles families, not one more, not one less. Moreover a new version of supersymmetry seems to couple gauge fields with all known fermions, without necessity of imagining new particles never seen by experiments.

Clearly this fact closes the door to dark matter. To compensate this big absence, AFT proposes an explanation to galaxy rotation curves which doesn't make use of dark matter.

We don't say that this theory is exact. However there are several good signals which must be taken into account. We hope that a future teamwork can verify this theory in detail, deepening all its implications.

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