Time evolution of $|0\rangle|\alpha\rangle$ where $H = (b^\dagger a + a^\dagger b)\hbar$

The initial state is

$$|\psi(0)\rangle = |0\rangle|\alpha\rangle \longrightarrow (1)$$

The time evolved state is

$$|\psi(t)\rangle = e^{-\frac{iHt}{\hbar}} |0\rangle_a |\alpha\rangle_b \longrightarrow (2)$$

$$= e^{-\frac{iHt}{\hbar}} e^{(\alpha a^+ - \alpha^* a)} |0\rangle_a |0\rangle_b \longrightarrow (3)$$

Here $U = e^{-iHt/\hbar}$

Inserting UU^+ in (3)

$$|\psi(t)\rangle = e^{-\frac{iHt}{\hbar}} e^{(\alpha a^+ - \alpha^* a)} e^{\frac{iHt}{\hbar}} e^{-\frac{iHt}{\hbar}} |0\rangle_a |0\rangle_b \longrightarrow (4)$$

$$= e^{-\frac{iHt}{\hbar}} e^{(\alpha a^+ - \alpha^* a)} e^{+\frac{iHt}{\hbar}} |0\rangle_a |0\rangle_b \longrightarrow (5)$$

This looks like $e^{x\hat{X}} \hat{Y} e^{-x\hat{X}}$

$$e^{x\hat{X}} \hat{Y} e^{-x\hat{X}} = Y + \frac{x}{1!}[\hat{X},\hat{Y}] + \frac{x^2}{2!}[\hat{X},[\hat{X},\hat{Y}]] + \longrightarrow (6)$$

Here $X = b^+ a + a^+ b$; $Y = e^{(\alpha a^+ - \alpha^* a)}$; $x = -it$

Applying this in eqn (6)

$$|\psi(t)\rangle = \Big\{ e^{(\alpha a^+ - \alpha^* a)} + \frac{(-it)}{1!}\left[b^+ a + a^+ b, e^{(\alpha a^+ - \alpha^* a)}\right] + \frac{(-it)^2}{2!}\left[b^+ a + a^+ b, \left[b^+ a + a^+ b, e^{(\alpha a^+ - \alpha^* a)}\right]\right] + \cdots \Big\} |0,0\rangle \longrightarrow (7)$$

First simplifying the term $\left[b^{+}a + a^{+}b, e^{(\alpha a^{+} - \alpha^{*} a)}\right]$ which is in eqn (7)

This is in the form

$$\left[b^{+}a + a^{+}b, e^{(\alpha a^{+} - \alpha^{*} a)}\right] = \int_0^1 e^{(\alpha a^{+} - \alpha^{*} a)} \left[b^{+}a + a^{+}b, \alpha a^{+} - \alpha^{*} a\right] ds$$

$$= \int_0^1 e^{(\alpha a^{+} - \alpha^{*} a)} \left[b^{+}a + a^{+}b, \alpha a^{+} - \alpha^{*} a\right] ds \longrightarrow (8)$$

$$= e^{(\alpha a^{+} - \alpha^{*} a)} \left[b^{+}a + a^{+}b, \alpha a^{+} - \alpha^{*} a\right]$$

$$= e^{(\alpha a^{+} - \alpha^{*} a)} \left(\alpha b^{+} + \alpha^{*} b\right) \longrightarrow (9)$$

Therefore

$$\left[b^{+}a + a^{+}b, e^{(\alpha a^{+} - \alpha^{*} a)}\right] = e^{(\alpha a^{+} - \alpha^{*} a)} \left(\alpha b^{+} + \alpha^{*} b\right) \longrightarrow (10)$$

Next simplifying the term $\left[b^{+}a + a^{+}b, \left[b^{+}a + a^{+}b, e^{(\alpha a^{+} - \alpha^{*} a)}\right]\right]$

This is in the form

$$\left.\begin{array}{l} \left[\hat{P}, \left[P, e^{Q}\right]\right] \text{ where} \\ \hat{P} = b^{+}a + a^{+}b \; ; \; Q = \alpha a^{+} - \alpha^{*} a \end{array}\right\} \longrightarrow (11)$$

$$[P,[P,e^Q]] = [P,[P,Q]e^Q] \longrightarrow (12)$$

$$= [P,[P,Q]]e^Q + [P,Q][P,e^Q] \longrightarrow (13)$$

$$= [P,[P,Q]]e^Q + [P,Q][P,Q]e^Q \longrightarrow (14)$$

commutating of P with $[P,Q]$ would give Q

(i.e) $[P,[P,Q]] = Q \longrightarrow (15)$

Sub (15) in (14)

$$[P,[P,e^Q]] = \left(Q + [P,Q]^2\right)e^Q \longrightarrow (16)$$

$$= \left(\alpha a^+ - \alpha^* a + (\alpha b^+ + \alpha^* b)^2\right) e^{\alpha a^+ - \alpha^* a} \longrightarrow (17)$$

Substituting (10) and (17) in eqn (7)

$$|\psi(t)\rangle = \Big\{ e^{(\alpha a^+ - \alpha^* a)} + \frac{(-it)}{1!}\left(\alpha b^+ + \alpha^* b\right) e^{(\alpha a^+ - \alpha^* a)}$$

$$+ \frac{(-it)^2}{2!}\left((\alpha a^+ - \alpha^* a) + (\alpha b^+ + \alpha^* b)^2\right) e^{\alpha a^+ - \alpha^* a}$$

$$+ \cdots\cdots \Big\} |0,0\rangle \longrightarrow (18)$$

How to simplify further?