Chapter 2
Black Holes

2.1 Dark Stars: A Historical Note

It is usual in textbooks to credit John Michell and Pierre-Simon Laplace for the idea of black holes, in the XVIII Century. The idea of a body so massive that even light could not escape was put forward by geologist Rev. John Michell in a letter written to Henry Cavendish in 1783 to the Royal Society:

> If the semi-diameter of a sphere of the same density as the Sun were to exceed that of the Sun in the proportion of 500 to 1, a body falling from an infinite height toward it would have acquired at its surface greater velocity than that of light, and consequently supposing light to be attracted by the same force in proportion to its inertia, with other bodies, all light emitted from such a body would be made to return toward it by its own proper gravity. (Michell 1784).

In 1796, the mathematician Pierre-Simon Laplace promoted the same idea in the first and second editions of his book *Exposition du système du Monde* (it was removed from later editions). Such "dark stars" were largely ignored in the nineteenth century, since light was then thought to be a massless wave and therefore not influenced by gravity. Unlike the modern concept of black hole, the object behind the horizon in black stars is assumed to be stable against collapse. Moreover, no equation of state was adopted neither by Michell nor by Laplace. Hence, their dark stars were Newtonian objects, infinitely rigid, and they have nothing to do with the nature of space and time, which were considered by them as absolute concepts. Nonetheless, Michel and Laplace could calculate correctly the size of such objects from the simple device of equating the potential and escape energy from a body of mass M:

$$\frac{1}{2}mv^2 = \frac{GMm}{r^2}. \tag{2.1}$$

Just setting $v = c$ and assuming that the gravitational and the inertial mass are the same, we get

$$r_{\text{dark star}} = \sqrt{\frac{2GM}{c^2}}. \tag{2.2}$$

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Dark stars are objects conceivable only within the framework of the Newtonian theory of matter and gravitation. In the context of general relativistic theories of gravitation, collapsed objects have quite different properties. Before exploring particular situations that can be represented by different solutions of Einstein's field equations, it is convenient to introduce a general definition of a collapsed gravitational system in a general space-time framework. This is what we do in the next section.

2.2 A General Definition of Black Hole

We shall now provide a general definition of a black hole, independently of the coordinate system adopted in the description of space-time, and even of the exact form of the field equations. First, we shall introduce some preliminary useful definitions (e.g. Hawking and Ellis 1973; Wald 1984).

Definition A causal curve in a space-time $(M, g_{\mu\nu})$ is a curve that is non space-like, that is, piecewise either time-like or null (light-like).

We say that a given space-time $(M, g_{\mu\nu})$ is *time-orientable* if we can define over M a smooth non-vanishing time-like vector field.

Definition If $(M, g_{\mu\nu})$ is a time-orientable space-time, then $\forall p \in M$, the causal future of p, denoted $J^+(p)$, is defined by:

$$J^+(p) \equiv \left\{q \in M \mid \exists \text{ a future-directed causal curve from } p \text{ to } q\right\}. \tag{2.3}$$

Similarly,

Definition If $(M, g_{\mu\nu})$ is a time-orientable space-time, then $\forall p \in M$, the causal past of p, denoted $J^-(p)$, is defined by:

$$J^-(p) \equiv \{q \in M \mid \exists \text{ a past-directed causal curve from } p \text{ to } q\}. \tag{2.4}$$

The causal future and past of any set $S \subset M$ are given by:

$$J^+(S) = \bigcup_{p \in S} J^+(P) \tag{2.5}$$

and,

$$J^-(S) = \bigcup_{p \in S} J^-(P). \tag{2.6}$$

A set S is said *achronal* if no two points of S are time-like related. A Cauchy surface (Sect. 1.8) is an achronal surface such that every non space-like curve in M

crosses it once, and only once. A space-time $(M, g_{\mu\nu})$ is *globally hyperbolic* if it admits a space-like hypersurface $S \subset M$ which is a Cauchy surface for M.

Causal relations are invariant under conformal transformations of the metric. In this way, the space-times $(M, g_{\mu\nu})$ and $(M, \widetilde{g}_{\mu\nu})$, where $\widetilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, with Ω a non-zero C^r function, have the same causal structure.

Particle horizons occur whenever a particular system never gets to be influenced by the whole space-time. If a particle crosses the horizon, it will not exert any further action upon the system with respect to which the horizon is defined.

Definition For a causal curve γ the associated future (past) particle horizon is defined as the boundary of the region from which the causal curves can reach some point on γ.

Finding the particle horizons (if one exists at all) requires a knowledge of the global space-time geometry.

Let us now consider a space-time where all null geodesics that start in a region $\mathcal{J}^-$ end at $\mathcal{J}^+$. Then, such a space-time, $(M, g_{\mu\nu})$, is said to contain a *black hole* if M *is not* contained in $J^-(\mathcal{J}^+)$. In other words, there is a region from where no null geodesic can reach the *asymptotic flat*[1] future space-time, or, equivalently, there is a region of M that is causally disconnected from the global future. The *black hole region*, BH, of such space-time is $BH = [M - J^-(\mathcal{J}^+)]$, and the boundary of BH in M, $H = J^-(\mathcal{J}^+) \bigcap M$, is the *event horizon*.

Notice that a black hole is conceived as a space-time *region*, i.e. what characterizes the black hole is its metric and, consequently, its curvature. What is peculiar of this space-time region is that it is causally disconnected from the rest of the space-time: no events in this region can make any influence on events outside the region. Hence the name of the boundary, event horizon: events inside the black hole are separated from events in the global external future of space-time. The events in the black hole, nonetheless, as all events, are causally determined by past events. A black hole does not represent a breakdown of classical causality. As we shall see, even when closed time-like curves are present, *local* causality still holds along with global consistency constrains. And in case of singularities, they do not belong to space-time, so they are not *predicable* (i.e. we cannot attach any predicate to them, nothing can be said about them) in the theory. More on this in Sect. 3.6.

2.3 Schwarzschild Black Holes

The first exact solution of Einstein's field equations was found by Karl Schwarzschild in 1916. This solution describes the geometry of space-time outside a spherically symmetric matter distribution.

[1] Asymptotic flatness is a property of the geometry of space-time which means that in appropriate coordinates, the limit of the metric at infinity approaches the metric of the flat (Minkowskian) space-time.

2.3.1 Schwarzschild Solution

The most general spherically symmetric metric is

$$ds^2 = \alpha(r,t)dt^2 - \beta(r,t)dr^2 - \gamma(r,t)d\Omega^2 - \delta(r,t)dr\,dt, \tag{2.7}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. We are using spherical polar coordinates. The metric (2.7) is invariant under rotations (isotropic).

The invariance group of general relativity is formed by the group of general transformations of coordinates of the form $x'^\mu = f^\mu(x)$. This yields 4 degrees of freedom, two of which have been used when adopting spherical coordinates (the transformations that do not break the central symmetry are $r' = f_1(r,t)$ and $t' = f_2(r,t)$). With the two available degrees of freedom we can freely choose two metric coefficients, whereas the other two are determined by Einstein's equations. Some possibilities are:

- *Standard gauge.*

$$ds^2 = c^2 A(r,t)dt^2 - B(r,t)dr^2 - r^2 d\Omega^2.$$

- *Synchronous gauge.*

$$ds^2 = c^2 dt^2 - F^2(r,t)dr^2 - R^2(r,t)d\Omega^2.$$

- *Isotropic gauge.*

$$ds^2 = c^2 H^2(r,t)dt^2 - K^2(r,t)\left[dr^2 + r^2(r,t)d\Omega^2\right].$$

- *Co-moving gauge.*

$$ds^2 = c^2 W^2(r,t)dt^2 - U(r,t)dr^2 - V(r,t)d\Omega^2.$$

Adopting the standard gauge and a static configuration (no dependence of the metric coefficients on t), we can get equations for the coefficients A and B of the standard metric:

$$ds^2 = c^2 A(r)dt^2 - B(r)dr^2 - r^2 d\Omega^2. \tag{2.8}$$

Since we are interested in the solution *outside* the spherical mass distribution, we only need to require the Ricci tensor to vanish:

$$R_{\mu\nu} = 0.$$

According to the definition of the curvature tensor and the Ricci tensor, we have:

$$R_{\mu\nu} = \partial_\nu \Gamma^\sigma_{\mu\sigma} - \partial_\sigma \Gamma^\sigma_{\mu\nu} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu} - \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\rho\sigma} = 0. \tag{2.9}$$

If we remember that the affine connection depends on the metric as

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu}\right),$$

we see that we have to solve a set of differential equations for the components of the metric $g_{\mu\nu}$.

The metric coefficients are:

$$
\begin{aligned}
g_{00} &= A(r), \\
g_{11} &= -B(r), \\
g_{22} &= -r^2, \\
g_{33} &= -r^2 \sin^2\theta, \\
g^{00} &= 1/A(r), \\
g^{11} &= -1/B(r), \\
g^{22} &= -1/r^2, \\
g^{33} &= -1/r^2 \sin^2\theta.
\end{aligned}
$$

Then, only nine of the 40 independent connection coefficients are different from zero. They are:

$$
\begin{aligned}
\Gamma^1_{01} &= A'/(2A), \\
\Gamma^1_{22} &= -r/B, \\
\Gamma^2_{33} &= -\sin\theta\cos\theta, \\
\Gamma^1_{00} &= A'/(2B), \\
\Gamma^1_{33} &= -\big(r\sin^2/B\big), \\
\Gamma^3_{13} &= 1/r, \\
\Gamma^1_{11} &= B'/(2B), \\
\Gamma^2_{12} &= 1/r, \\
\Gamma^3_{23} &= \cot\theta.
\end{aligned}
$$

Replacing in the expression for $R_{\mu\nu}$:

$$
\begin{aligned}
R_{00} &= -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB}, \\
R_{11} &= \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB}, \\
R_{22} &= \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right), \\
R_{33} &= R_{22}\sin^2\theta.
\end{aligned}
$$

Einstein's field equations for the region of empty space then become:

$$R_{00} = R_{11} = R_{22} = 0$$

(the fourth equation has no additional information). Multiplying the first equation by B/A and adding the result to the second equation, we get:

$$A'B + AB' = 0,$$

from which $AB =$ constant. We can write then $B = \alpha A^{-1}$. Going to the third equation and replacing B we obtain: $A + rA' = \alpha$, or:

$$\frac{d(rA)}{dr} = \alpha.$$

The solution of this equation is:

$$A(r) = \alpha\left(1 + \frac{k}{r}\right),$$

with k another integration constant. For B we get:

$$B = \left(1 + \frac{k}{r}\right)^{-1}.$$

If we now consider the Newtonian limit:

$$\frac{A(r)}{c^2} = 1 + \frac{2\Phi}{c^2},$$

with $\Phi = -GM/r$ the Newtonian gravitational potential, we conclude that

$$k = -\frac{2GM}{c^2}$$

and

$$\alpha = c^2.$$

Therefore, the Schwarzschild solution for a static mass M can be written in spherical coordinates (t, r, θ, ϕ) as

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \quad (2.10)$$

As mentioned, this solution corresponds to the vacuum region exterior to the spherical object of mass M. Inside the object, space-time will depend on the peculiarities of the physical object.

The metric given by Eq. (2.10) has some interesting properties. Let's assume that the mass M is concentrated at $r = 0$. There seems to be two singularities at which the metric diverges: one at $r = 0$ and the other at

$$r_{\rm Schw} = \frac{2GM}{c^2}. \tag{2.11}$$

The length $r_{\rm Schw}$ is know as the *Schwarzschild radius* of the object of mass M. Usually, at normal densities, $r_{\rm Schw}$ is well inside the outer radius of the physical system, and the solution does not apply in the interior but only to the exterior of the object. For instance, for the Sun $r_{\rm Schw} \sim 3$ km. However, for a point mass, the Schwarzschild radius is in the vacuum region and space-time has the structure given by (2.10). In general, we can write

$$r_{\rm Schw} \sim 3\left(\frac{M}{M_\odot}\right) \text{ km},$$

where $M_\odot = 1.99 \times 10^{33}$ g is the mass of the Sun.

It is easy to see that strange things occur close to $r_{\rm Schw}$. For instance, for the proper time we get:

$$d\tau = \left(1 - \frac{2GM}{rc^2}\right)^{1/2} dt, \tag{2.12}$$

or

$$dt = \left(1 - \frac{2GM}{rc^2}\right)^{-1/2} d\tau. \tag{2.13}$$

When $r \to \infty$ both times agree, so t is interpreted as the proper time measured from an infinite distance. As the system with proper time τ approaches to $r_{\rm Schw}$, dt tends to infinity according to Eq. (2.13). The object never reaches the Schwarzschild surface when seen by an infinitely distant observer. The closer the object is to the Schwarzschild radius, the slower it moves for the external observer.

A direct consequence of the difference introduced by gravity in the local time with respect to the time at infinity is that the radiation that escapes from a given radius $r > r_{\rm Schw}$ will be redshifted when received by a distant and static observer. Since the frequency (and hence the energy) of the photon depends on the time interval, we can write, from Eq. (2.13):

$$\lambda_\infty = \left(1 - \frac{2GM}{rc^2}\right)^{-1/2} \lambda. \tag{2.14}$$

Since the redshift is:

$$z = \frac{\lambda_\infty - \lambda}{\lambda}, \tag{2.15}$$

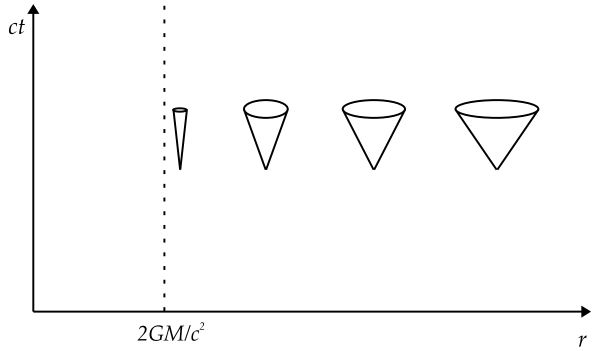

Fig. 2.1 Space-time diagram in Schwarzschild coordinates showing the light cones of events at different distances from the event horizon. Adapted form Carroll (2003)

then

$$1+z=\left(1-\frac{2GM}{rc^2}\right)^{-1/2}, \tag{2.16}$$

and we see that when $r \to r_{\rm Schw}$ the redshift becomes infinite. This means that a photon needs infinite energy to escape from inside the region determined by $r_{\rm Schw}$. Events that occur at $r < r_{\rm Schw}$ are disconnected from the rest of the universe. Hence, we call the surface determined by $r = r_{\rm Schw}$ an *event horizon*. Whatever crosses the event horizon will never return. This is the origin of the expression "black hole", introduced by John A. Wheeler in the mid 1960s. The black hole is the region of space-time inside the event horizon. We can see in Fig. 2.1 what happens with the light cones as an event is closer to the horizon of a Schwarzschild black hole. The shape of the cones can be calculated from the metric (2.10) imposing the null condition $ds^2 = 0$. Then,

$$\frac{dr}{dt}=\pm\left(1-\frac{2GM}{r}\right), \tag{2.17}$$

where we made $c = 1$. Notice that when $r \to \infty$, $dr/dt \to \pm 1$, as in Minkowski space-time. When $r \to 2GM$, $dr/dt \to 0$, and light moves along the surface $r = 2GM$, which is consequently a null surface. For $r < 2GM$, the sign of the derivative is inverted. The inward region of $r = 2GM$ is time-like for any physical system that has crossed the boundary surface.

What happens to an object when it crosses the event horizon? According to Eq. (2.10), there is a singularity at $r = r_{\rm Schw}$. The metric coefficients, however, can be made regular by a change of coordinates. For instance we can consider Eddington-Finkelstein coordinates. Let us define a new radial coordinate r_* such that radial null rays satisfy $d(ct \pm r_*) = 0$. Using Eq. (2.10) it can be shown that:

$$r_* = r + \frac{2GM}{c^2}\log\left|\frac{r-2GM/c^2}{2GM/c^2}\right|.$$

Then, we introduce:

$$v = ct + r_*.$$

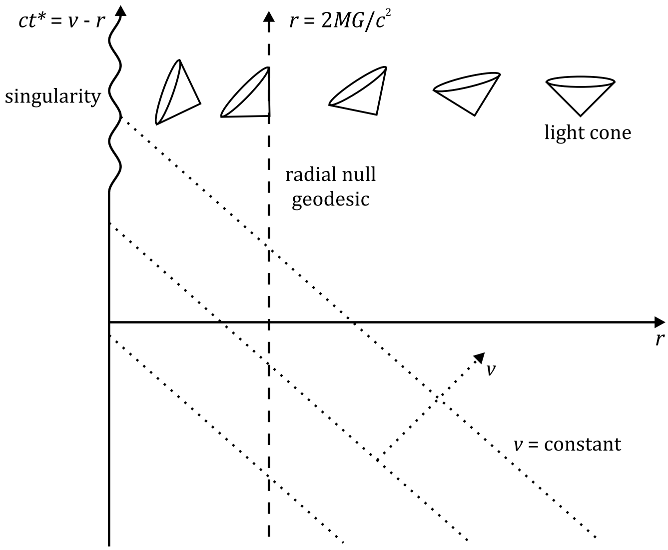

Fig. 2.2 Space-time diagram in Eddington-Finkelstein coordinates showing the light cones close to and inside a black hole. Here, $r = 2GM/c^2 = r_{\rm Schw}$ is the Schwarzschild radius where the event horizon is located. Adapted form Townsend (1997)

The new coordinate v can be used as a time coordinate replacing t in Eq. (2.10). This yields:

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right)\left(c^2 dt^2 - dr_*^2\right) - r^2 d\Omega^2$$

or

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right) dv^2 - 2dr dv - r^2 d\Omega^2, \tag{2.18}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2.$$

Notice that in Eq. (2.18) the metric is non-singular at $r = 2GM/c^2$. The only real singularity is at $r = 0$, since there the Riemann tensor diverges. In order to plot the space-time in a (t, r)-plane, we can introduce a new time coordinate $ct_* = v - r$. From the metric (2.18) or from Fig. 2.2 we see that the line $r = r_{\rm Schw}$, $\theta =$ constant, and $\phi =$ constant is a null ray, and hence, the surface at $r = r_{\rm Schw}$ is a null surface. This null surface is an event horizon because inside $r = r_{\rm Schw}$ all cones have $r = 0$ in their future (see Fig. 2.2). The object in $r = 0$ is the source of the gravitational field and is called the *singularity*. We shall say more about it in Sect. 3.6. For the moment, we only remark that everything that crosses the event horizon will end at the singularity. This is the inescapable fate for everything inside a Schwarzschild black hole. There is no way to avoid it: in the future of every event inside the event

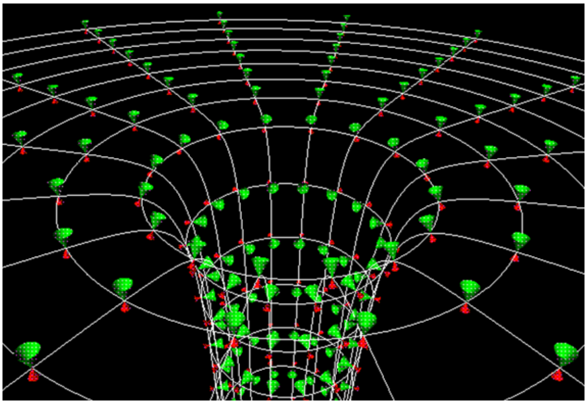

Fig. 2.3 Embedding space-time diagram in Eddington-Finkelstein coordinates showing the light cones of events at different distances from a Schwarzschild black hole. From http://www.oglethorpe.edu/faculty/~m_rulison/ChangingViews/Lecture7.htm

horizon is the singularity. There is no escape, no hope, no freedom, inside the black hole. There is just the singularity, whatever such a thing might be.

We see now that the name "black hole" is not strictly correct for space-time regions isolated by event horizons. There is no hole to other place. Whatever falls into the black hole, goes to the singularity. The central object increases its mass and energy with the accreted bodies and fields, and then the event horizon grows. This would not happen if what falls into the hole were able to pass through, like through a hole in a wall. A black hole is more like a space-time precipice, deep, deadly, and with something unknown at the bottom. A graphic depiction with an embedding diagram of a Schwarzschild black hole is shown in Fig. 2.3. An embedding is an immersion of a given manifold into a manifold of lower dimensionality that preserves the metric properties.

2.3.2 Birkhoff's Theorem

If we consider the isotropic but *not static* line element,

$$ds^2 = c^2 A(r,t)dt^2 - B(r,t)dr^2 - r^2 d\Omega^2, \tag{2.19}$$

and substitute it into Einstein's empty-space field equations $R_{\mu\nu} = 0$ to obtain the functions $A(r,t)$ and $B(r,t)$, the result would be exactly the same:

$$A(r,t) = A(r) = \left(1 - \frac{2GM}{rc^2}\right),$$

and

$$B(r,t) = B(r) = \left(1 - \frac{2GM}{rc^2}\right)^{-1}.$$

This result is general and known as Birkhoff's theorem:

The space-time geometry outside a general spherically symmetric matter distribution is the Schwarzschild geometry.

Birkhoff's theorem implies that strictly radial motions do not perturb the space-time metric. In particular, a pulsating star, if the pulsations are strictly radial, does not produce gravitational waves.

The converse of Birkhoff's theorem is not true, i.e.,

If the region of space-time is described by the metric given by expression (2.10), then the matter distribution that is the source of the metric does not need to be spherically symmetric.

2.3.3 Orbits

Orbits around a Schwarzschild black hole can be easily calculated using the metric and the relevant symmetries (see. e.g. Raine and Thomas 2005; Frolov and Zelnikov 2011). Let us call k^{μ} a vector in the direction of a given symmetry (i.e. k^{μ} is a Killing vector). A static situation is symmetric in the time direction, hence we can write $k^{\mu} = (1, 0, 0, 0)$. The 4-velocity of a particle with trajectory $x^{\mu} = x^{\mu}(\tau)$ is $u^{\mu} = dx^{\mu}/d\tau$. Then, since $u^0 = E/c$, where E is the energy, we have:

$$g_{\mu\nu}k^{\mu}u^{\nu} = g_{00}k^0u^0 = g_{00}u^0 = \eta_{00}\frac{E}{c} = \frac{E}{c} = \text{constant}. \tag{2.20}$$

If the particle moves along a geodesic in a Schwarzschild space-time, we obtain from Eq. (2.20):

$$c\left(1 - \frac{2GM}{c^2r}\right)\frac{dt}{d\tau} = \frac{E}{c}. \tag{2.21}$$

Similarly, for the symmetry in the azimuthal angle ϕ we have $k^{\mu} = (0, 0, 0, 1)$, in such a way that:

$$g_{\mu\nu}k^{\mu}u^{\nu} = g_{33}k^3u^3 = g_{33}u^3 = -L = \text{ constant}. \tag{2.22}$$

In the Schwarzschild metric we find, then,

$$r^2\frac{d\phi}{d\tau} = L = \text{constant}. \tag{2.23}$$

If we now divide the Schwarzschild interval (2.10) by $c^2d\tau^2$ we get

$$1 = \left(1 - \frac{2GM}{c^2r}\right)\left(\frac{dt}{d\tau}\right)^2 - c^{-2}\left(1 - \frac{2GM}{c^2r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - c^{-2}r^2\left(\frac{d\phi}{d\tau}\right)^2, \tag{2.24}$$

and using the conservation equations (2.21) and (2.23) we obtain:

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{c^2} - \left(c^2 + \frac{L^2}{r^2}\right)\left(1 - \frac{2GM}{c^2r}\right). \tag{2.25}$$

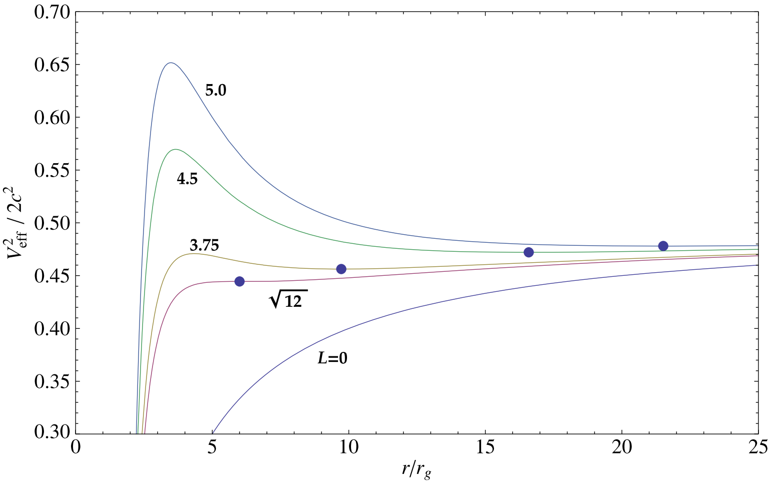

Fig. 2.4 General relativistic effective potential plotted for several values of angular momentum

Then, expressing the energy in units of mc^2 and introducing an effective potential V_{eff},

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{c^2} - V_{\text{eff}}^2. \tag{2.26}$$

For circular orbits of a massive particle we have the conditions

$$\frac{dr}{d\tau} = 0 \quad \text{and} \quad \frac{d^2r}{d\tau^2} = 0.$$

The orbits are possible only at the turning points of the effective potential:

$$V_{\text{eff}} = \sqrt{\left(c^2 + \frac{L^2}{r^2}\right)\left(1 - \frac{2r_{\text{g}}}{r}\right)}, \tag{2.27}$$

where L is the angular momentum in units of mc and $r_{\text{g}} = GM/c^2$ is the gravitational radius. Then,

$$r = \frac{L^2}{2cr_{\text{g}}} \pm \frac{1}{2}\sqrt{\frac{L^4}{c^2 r_{\text{g}}^2} - 12L^2}. \tag{2.28}$$

The effective potential is shown in Fig. 2.4 for different values of the angular momentum.

For $L^2 > 12c^2 r_{\text{g}}^2$ there are two solutions. The negative sign corresponds to a maximum of the potential and is unstable. The positive sign corresponds to a minimum, which is, consequently, stable. At $L^2 = 12c^2 r_{\text{g}}^2$ there is a single stable orbit. It is the innermost marginally stable orbit, and it occurs at $r = 6r_{\text{g}} = 3r_{\text{Schw}}$. The specific angular momentum of a particle in a circular orbit at r is:

$$L = c\left(\frac{r_g r}{1 - 3r_g/r}\right)^{1/2}.$$

Its energy (units of mc^2) is:

$$E = \left(1 - \frac{2r_g}{r}\right)\left(1 - \frac{3r_g}{r}\right)^{-1/2}.$$

The proper and observer's periods are:

$$\tau = \frac{2\pi}{c}\left(\frac{r^3}{r_g}\right)^{1/2}\left(1 - \frac{3r_g}{r}\right)^{1/2}$$

and

$$T = \frac{2\pi}{c}\left(\frac{r^3}{r_g}\right)^{1/2}.$$

Notice that when $r \to 3r_g$ both L and E tend to infinity, so only massless particles can orbit at such a radius.

The local velocity at r of an object falling from rest to the black hole is (e.g. Raine and Thomas 2005):

$$v_{\rm loc} = \frac{\text{proper distance}}{\text{proper time}} = \frac{dr}{(1 - 2GM/c^2 r)dt}.$$

Hence, using the expression for dr/dt from the metric (2.10)

$$\frac{dr}{dt} = -c\left(\frac{2GM}{c^2 r}\right)^{1/2}\left(1 - \frac{2GM}{c^2 r}\right), \tag{2.29}$$

we have,

$$v_{\rm loc} = \left(\frac{2r_g}{r}\right)^{1/2} \quad \text{(in units of } c\text{)}. \tag{2.30}$$

Then, the differential acceleration the object will experience along an element dr is:[2]

$$dg = \frac{2r_g}{r^3}c^2 dr. \tag{2.31}$$

The tidal acceleration on a body of finite size Δr is simply $(2r_g/r^3)c^2\Delta r$. This acceleration and the corresponding force becomes infinite at the singularity. As the object falls into the black hole, tidal forces act to tear it apart. This painful process is

[2]Notice that $dv_{\rm loc}/d\tau = (dv_{\rm loc}/dr)(dr/d\tau) = (dv_{\rm loc}/dr)v_{\rm loc} = r_g c^2/r^2$.

known as "spaghettification". The process can last significant long before the object crosses the event horizon, depending on the mass of the black hole.

The energy of a particle in the innermost stable orbit can be obtained from the above equation for the energy setting $r = 6r_g$. This yields (units of mc^2):

$$E = \left(1 - \frac{2r_g}{6r_g}\right)\left(1 - \frac{3r_g}{6r_g}\right)^{-1/2} = \frac{2}{3}\sqrt{2}.$$

Since a particle at rest at infinity has $E = 1$, then the energy that the particle should release to fall into the black hole is $1 - (2/3)\sqrt{2} = 0.057$. This means 5.7 % of its rest mass energy, significantly higher than the energy release that can be achieved through nuclear fusion.

An interesting question is what is the gravitational acceleration at the event horizon as seen by an observer from infinity. The acceleration relative to a hovering frame system of a freely falling object at rest at r is (Raine and Thomas 2005):

$$g_r = -c^2\left(\frac{GM/c^2}{r^2}\right)\left(1 - \frac{2GM/c^2}{r}\right)^{-1/2}.$$

So, the energy spent to move the object a distance dl will be $dE_r = mg_r dl$. The energy expended respect to a frame at infinity is $dE_\infty = mg_\infty dl$. Because of the conservation of energy, both quantities should be related by a redshift factor:

$$\frac{E_r}{E_\infty} = \frac{g_r}{g_\infty} = \left(1 - \frac{2GM/c^2}{r}\right)^{-1/2}.$$

Hence, using the expression for g_r we get:

$$g_\infty = c^2\frac{GM/c^2}{r^2}. \tag{2.32}$$

Notice that for an observer at r, $g_r \to \infty$ when $r \to r_{\rm Schw}$. From infinity, however, the required force to hold the object hovering at the horizon is

$$mg_\infty = c^2\frac{GmM/c^2}{r_{\rm Schw}^2} = \frac{mc^4}{4GM}.$$

This is the *surface gravity* of the black hole.

2.3.4 Radial Motion of Photons

In the case of photons we have that $ds^2 = 0$. The radial motion, then, satisfies:

$$\left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1}dr^2 = 0. \tag{2.33}$$

From here,

$$\frac{dr}{dt} = \pm c\left(1 - \frac{2GM}{rc^2}\right). \tag{2.34}$$

Integrating, we have:

$$ct = r + \frac{2GM}{c^2}\ln\left|\frac{rc^2}{2GM} - 1\right| + \text{constant outgoing photons,} \tag{2.35}$$

$$ct = -r - \frac{2GM}{c^2}\ln\left|\frac{rc^2}{2GM} - 1\right| + \text{constant incoming photons.} \tag{2.36}$$

Notice that in a (ct, r)-diagram the photons have world-lines with slopes ± 1 as $r \to \infty$, indicating that space-time is asymptotically flat. As the events that generate the photons approach to $r = r_{\rm Schw}$, the slopes tend to $\pm\infty$. This means that the light cones become thinner and thinner for events close to the event horizon. At $r = r_{\rm Schw}$ the photons cannot escape and they move along the horizon (see Fig. 2.1). An observer in the infinity will never detect them.

2.3.5 Circular Motion of Photons

In this case, fixing $\theta =$ constant due to the symmetry, we have that photons will move in a circle of $r =$ constant and $ds^2 = 0$. Then, from (2.10), we have:

$$\left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 - r^2d\phi^2 = 0. \tag{2.37}$$

This means that

$$\dot{\phi} = \frac{c}{r}\sqrt{\left(1 - \frac{2GM}{rc^2}\right)} = \text{constant.}$$

The circular velocity is:

$$v_{\rm circ} = \frac{r\dot{\phi}}{\sqrt{g_{00}}} = \frac{\Omega r}{(1 - 2GM/c^2r)^{1/2}}. \tag{2.38}$$

Setting $v_{\rm circ} = c$ for photons and using $\Omega = (GM/r^3)^{1/2}$, we get that the only possible radius for a circular photon orbit is:

$$r_{\rm ph} = \frac{3GM}{c^2}. \tag{2.39}$$

For a compact object of 1 $M_\odot$, $r_{\rm ph} \approx 4.5$ km, in comparison with the Schwarzschild radius of 3 km. Photons moving at this distance form the “photosphere” of the black hole. The orbit, however, is unstable, as it can be seen from the effective potential:

$$V_{\text{eff}} = \frac{L_{\text{ph}}^2}{r^2}\left(1 - \frac{2r_{\text{g}}}{r}\right). \tag{2.40}$$

Notice that the four-acceleration for circular motion is $a_\mu = u_\mu u_\nu;^\nu$. The radial component in the Schwarzschild metric is:

$$a_r = \frac{GM/r^2 - \Omega^2 r}{1 - 2GM/c^2 r - \Omega^2 r^2/c^2}. \tag{2.41}$$

The circular motion along a geodesic line corresponds to the case $a_r = 0$ (free motion). This gives from Eq. (2.41) the usual expression for the Keplerian angular velocity

$$\Omega_{\text{K}} = \left(\frac{GM}{r^3}\right)^{1/2},$$

already used in deriving r_{ph}. The angular velocity, however, can have any value determined by the metric and can be quite different from the corresponding Keplerian value. In general:

$$v = \frac{r\Omega_{\text{K}}}{(1 - 2GM/c^2 r)^{1/2}} = \left(\frac{GM}{r}\right)^{1/2}\left(1 - \frac{2GM}{c^2 r}\right)^{-1/2}. \tag{2.42}$$

From this latter equation and the fact that $v \le c$ it can be concluded that pure Keplerian motion is only possible for $r \ge 1.5 r_{\text{Schw}}$. At $r \le 1.5 r_{\text{Schw}}$ any massive particle will find its mass increased by special relativistic effects in such a way that the gravitational attraction will outweigh any centrifugal force.

2.3.6 Gravitational Capture

A particle coming from infinity is captured if its trajectory ends in the black hole. The angular momentum of a non-relativistic particle with velocity v_∞ at infinity is $L = m v_\infty b$, where b is an impact parameter. The condition $L/mcr_{\text{Schw}} = 2$ defines $b_{\text{cr,non-rel}} = 2r_{\text{Schw}}(c/v_\infty)$. Then, the capture cross section is:

$$\sigma_{\text{non-rel}} = \pi b_{\text{cr}}^2 = 4\pi \frac{c^2 r_{\text{Schw}}^2}{v_\infty^2}. \tag{2.43}$$

For an ultra-relativistic particle, $b_{\text{cr}} = 3\sqrt{3} r_{\text{Schw}}/2$, and then

$$\sigma_{\text{rel}} = \pi b_{\text{cr}}^2 = \frac{27}{4}\pi r_{\text{Schw}}^2. \tag{2.44}$$

2.3.7 Other Coordinate Systems

Other coordinates can be introduced to study additional properties of black holes. We refer the reader to the books of Frolov and Novikov (1998), Raine and Thomas (2005), and Frolov and Zelnikov (2011) for further details. Here we shall only introduce the Kruskal-Szekeres coordinates. These coordinates have the advantage that they cover the entire space-time manifold of the maximally extended Schwarzschild solution and are well-behaved everywhere outside the physical singularity. They allow to remove the non-physical singularity at $r = r_{\rm Schw}$ and provide new insights on the interior solution, on which we shall return later.

Let us consider the following coordinate transformation:

$$\begin{aligned} u &= \left(\frac{r}{r_{\rm Schw}} - 1\right)^{1/2} e^{\frac{r}{2r_{\rm Schw}}} \cosh\left(\frac{ct}{2r_{\rm Schw}}\right), \\ v &= \left(\frac{r}{r_{\rm Schw}} - 1\right)^{1/2} e^{\frac{r}{2r_{\rm Schw}}} \sinh\left(\frac{ct}{2r_{\rm Schw}}\right), \\ &\text{if } r > r_{\rm Schw}, \end{aligned} \tag{2.45}$$

and

$$\begin{aligned} u &= \left(1 - \frac{r}{r_{\rm Schw}}\right)^{1/2} e^{\frac{r}{2r_{\rm Schw}}} \sinh\left(\frac{ct}{2r_{\rm Schw}}\right), \\ v &= \left(1 - \frac{r}{r_{\rm Schw}}\right)^{1/2} e^{\frac{r}{2r_{\rm Schw}}} \cosh\left(\frac{ct}{2r_{\rm Schw}}\right), \\ &\text{if } r < r_{\rm Schw}. \end{aligned} \tag{2.46}$$

The line element in the Kruskal-Szekeres coordinates is completely regular, except at $r = 0$:

$$ds^2 = \frac{4r_{\rm Schw}^3}{r} e^{\frac{r}{r_{\rm Schw}}} \left(dv^2 - du^2\right) - r^2 d\Omega^2. \tag{2.47}$$

The curves at $r =$ constant are hyperbolic and satisfy:

$$u^2 - v^2 = \left(\frac{r}{r_{\rm Schw}} - 1\right)^{1/2} e^{\frac{r}{r_{\rm Schw}}}, \tag{2.48}$$

whereas the curves at $t =$ constant are straight lines that pass through the origin:

$$\begin{aligned} \frac{u}{v} &= \tanh \frac{ct}{2r_{\rm Schw}}, \quad r < r_{\rm Schw}, \\ \frac{u}{v} &= \coth \frac{ct}{2r_{\rm Schw}}, \quad r > r_{\rm Schw}. \end{aligned} \tag{2.49}$$

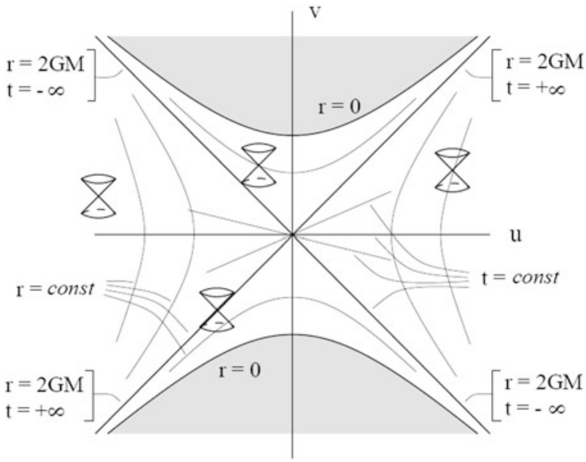

Fig. 2.5 The Schwarzschild metric in Kruskal-Szekeres coordinates ($c = 1$)

In Fig. 2.5 we show the Schwarzschild space-time in Kruskal-Szekeres coordinates. Each hyperbola represents a set of events of constant radius in Schwarzschild coordinates. A radial worldline of a photon in this diagram ($ds = 0$) is represented by a straight line forming an angle of $\pm 45°$with the u axis. A time-like trajectory has always a slope larger than that of 45°; and a space-like one, a smaller slope. A particle falling into the black hole crosses the line at 45° and reaches the future singularity at $r = 0$. For an external observer this occurs in an infinite time. The Kruskal-Szekeres coordinates have the useful feature that outgoing null geodesics are given by $u =$ constant, whereas ingoing null geodesics are given by $v =$ constant. Furthermore, the (future and past) event horizon(s) are given by the equation $uv = 0$, and the curvature singularity is given by the equation $uv = 1$.

A closely related diagram is the so-called Penrose or Penrose-Carter diagram. This is a two-dimensional diagram that captures the causal relations between different points in space-time. It is an extension of a Minkowski diagram (light cone) where the vertical dimension represents time, and the horizontal dimension represents space, and slanted lines at an angle of 45° correspond to light rays. The biggest difference with a Minkowski diagram is that, locally, the metric on a Penrose diagram is conformally equivalent[3] to the actual metric in space-time. The conformal factor is chosen such that the entire infinite space-time is transformed into a Penrose diagram of finite size. For spherically symmetric space-times, every point in the diagram corresponds to a 2-sphere. In Fig. 2.6 we show a Penrose diagram of a Minkowskian space-time.

This type of diagrams can be applied to Schwarzschild black holes. The result is shown in Fig. 2.7. The trajectory represents a particle that goes from some point

[3]We remind that two geometries are conformally equivalent if there exists a conformal transformation (an angle-preserving transformation) that maps one geometry to the other. More generally, two (pseudo) Riemannian metrics on a manifold M are conformally equivalent if one is obtained from the other through multiplication by a function on M.

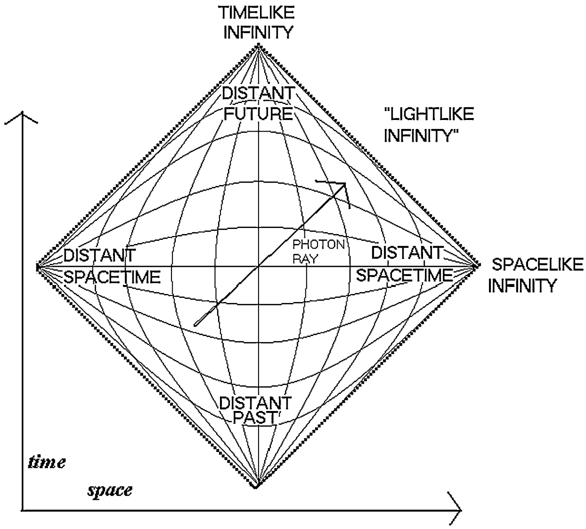

Fig. 2.6 Penrose diagram of a Minkowskian space-time

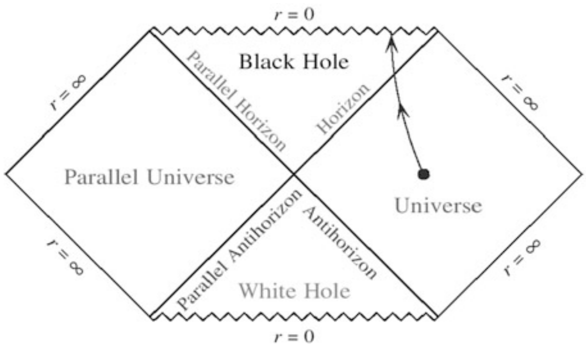

Fig. 2.7 Penrose diagram of a Schwarzschild black hole

in our universe into the black hole, ending in the singularity. Notice that there is a mirror extension, also present in the Kruskal-Szekeres diagram, representing a white hole and a parallel, but inaccessible universe. A white hole presents a naked singularity. These type of extensions of solutions of Einstein's field equations will be discussed later.

Now, we turn to axially symmetric (rotating) solutions of the field equations.

2.4 Kerr Black Holes

A Schwarzschild black hole does not rotate. The solution of the field equations (1.36) for a rotating body of mass M and angular momentum per unit mass a was found by Roy Kerr (1963):

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi - g_{\phi\phi}d\phi^2 - \Sigma\Delta^{-1}dr^2 - \Sigma d\theta^2 \tag{2.50}$$

$$g_{tt} = \left(c^2 - 2GMr\Sigma^{-1}\right) \tag{2.51}$$

$$g_{t\phi} = 2GMac^{-2}\Sigma^{-1}r\sin^2\theta \tag{2.52}$$

$$g_{\phi\phi} = \left[\left(r^2 + a^2c^{-2}\right)^2 - a^2c^{-2}\Delta\sin^2\theta\right]\Sigma^{-1}\sin^2\theta \tag{2.53}$$

$$\Sigma \equiv r^2 + a^2c^{-2}\cos^2\theta \tag{2.54}$$

$$\Delta \equiv r^2 - 2GMc^{-2}r + a^2c^{-2}. \tag{2.55}$$

This is the Kerr metric in Boyer-Lindquist coordinates (t, r, θ, ϕ), which reduces to Schwarzschild metric for $a = 0$. In Boyer-Lindquist coordinates the metric is approximately Lorentzian at infinity (i.e. we have a Minkowski space-time in the usual coordinates of Special Relativity).

The element $g_{t\phi}$ no longer vanishes. Even at infinity this element remains (hence we wrote *approximately* Lorentzian above). The Kerr parameter ac^{-1} has dimensions of length. The larger the ratio of this scale to GMc^{-2} (the *spin parameter* $a_* \equiv ac/GM$), the more aspherical the metric. Schwarzschild's black hole is the special case of Kerr's for $a = 0$. Notice that, with the adopted conventions, the angular momentum J is related to the parameter a by:

$$J = Ma. \tag{2.56}$$

Just as the Schwarzschild solution is the unique static vacuum solution of Eqs. (1.36) (a result called Israel's theorem), the Kerr metric is the unique stationary axisymmetric vacuum solution (Carter-Robinson theorem).

The horizon, the surface which cannot be crossed outward, is determined by the condition $g_{rr} \to \infty$ $(\Delta = 0)$. It lies at $r = r_{\rm h}^{\rm out}$ where

$$r_{\rm h}^{\rm out} \equiv GMc^{-2} + \left[\left(GMc^{-2}\right)^2 - a^2c^{-2}\right]^{1/2}. \tag{2.57}$$

Indeed, the track $r = r_{\rm h}^{\rm out}$, $\theta =$ constant with $d\phi/d\tau = a(r_{\rm h}^2 + a^2)^{-1}dt/d\tau$ has $ds = 0$ (it represents a photon circling azimuthally *on* the horizon, as opposed to hovering at it). Hence the surface $r = r_{\rm h}^{\rm out}$ is tangent to the local light cone. Because of the square root in Eq. (2.57), the horizon is well defined only for $a_* = ac/GM \leq 1$. An *extreme* (i.e. maximally rotating) Kerr black hole has a spin parameter $a_* = 1$. Notice that for $(GMc^{-2})^2 - a^2c^{-2} > 0$ we have actually two horizons. The second, the *inner* horizon, is located at:

$$r_{\rm h}^{\rm inn} \equiv GMc^{-2} - \left[\left(GMc^{-2}\right)^2 - a^2c^{-2}\right]^{1/2}. \tag{2.58}$$

This horizon is not seen by an external observer, but it hides the singularity to any observer that has already crossed r_h and is separated from the rest of the universe. For $a = 0$, $r_{\rm h}^{\rm inn} = 0$ and $r_{\rm h}^{\rm out} = r_{\rm Schw}$. The case $(GMc^{-2})^2 - a^2c^{-2} < 0$ corresponds to no horizons and it is thought to be unphysical.

A study of the orbits around a Kerr black hole is beyond the limits of the present text (the reader is referred to Frolov and Novikov 1998; Pérez et al. 2013), but we shall mention several interesting features. One is that if a particle initially falls radially with no angular momentum from infinity to the black hole, it gains angular motion during the infall. The angular velocity as seen from a distant observer is:

$$\Omega(r,\theta) = \frac{d\phi}{dt} = \frac{(2GM/c^2)ar}{(r^2 + a^2c^{-2})^2 - a^2c^{-2}\Delta \sin^2\theta}. \tag{2.59}$$

The particle will acquire angular velocity in the direction of the spin of the black hole. As the black hole is approached, the particle will find an increasing tendency to get carried away in the same sense in which the black hole is rotating. To keep the particle stationary with respect to the distant stars, it will be necessary to apply a force against this tendency. The closer the particle will be to the black hole, the stronger the force. At a point $r_{\rm e}$ it becomes impossible to counteract the rotational sweeping force. The particle is in a kind of space-time maelstrom. The surface determined by $r_{\rm e}$ is the *static limit*: from there in, you cannot avoid rotating. Space-time is rotating here in such a way that you cannot do anything in order to not co-rotate with it. You can still escape from the black hole, since the outer event horizon has not been crossed, but rotation is inescapable. The region between the static limit and the event horizon is called the *ergosphere*. The ergosphere is not spherical but its shape changes with the latitude θ. It can be determined through the condition $g_{tt} = 0$. Consider a stationary particle, $r =$ constant, $\theta =$ constant, and $\phi =$ constant. Then:

$$c^2 = g_{tt}\left(\frac{dt}{d\tau}\right)^2. \tag{2.60}$$

When $g_{tt} \leq 0$ this condition cannot be fulfilled, and hence a massive particle cannot be stationary inside the surface defined by $g_{tt} = 0$. For photons, since $ds = c\,d\tau = 0$, the condition is satisfied at the surface. Solving $g_{tt} = 0$ we obtain the shape of the ergosphere:

$$r_{\rm e} = \frac{GM}{c^2} + \frac{1}{c^2}\left(G^2M^2 - a^2c^2\cos^2\theta\right)^{1/2}. \tag{2.61}$$

The static limit lies outside the horizon except at the poles where both surfaces coincide. The phenomenon of "frame dragging" is common to all axially symmetric metrics with $g_{t\phi} \neq 0$.

Roger Penrose (1969) suggested that a projectile thrown from outside into the ergosphere begins to rotate acquiring more rotational energy than it originally had. Then the projectile can break up into two pieces, one of which will fall into the black hole, whereas the other can go out of the ergosphere. The piece coming out will then have more energy than the original projectile. In this way, we can extract energy from a rotating black hole. In Fig. 2.8 we illustrate the situation and show the static limit, the ergosphere and the outer/inner horizons of a Kerr black hole.

The innermost marginally stable circular orbit $r_{\rm ms}$ around an extreme rotating black hole ($ac^{-1} = GM/c^2$) is given by Raine and Thomas (2005):

$$\left(\frac{r_{\rm ms}}{GM/c^2}\right)^2 - 6\left(\frac{r_{\rm ms}}{GM/c^2}\right) \pm 8\left(\frac{r_{\rm ms}}{GM/c^2}\right)^{1/2} - 3 = 0. \tag{2.62}$$

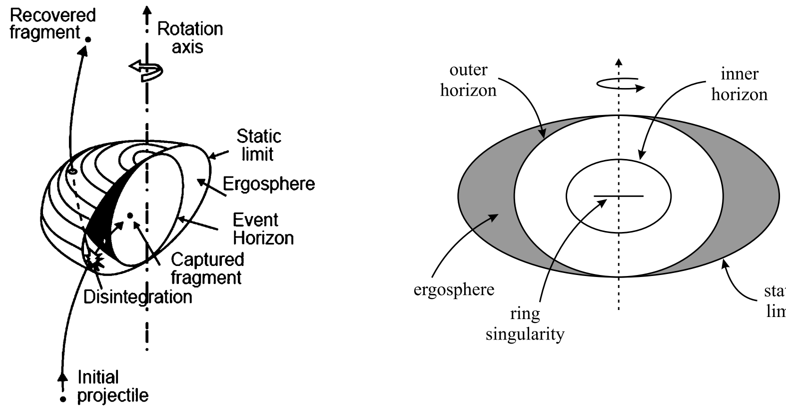

Fig. 2.8 *Left*: a rotating black hole and the Penrose process. From Luminet (1998). *Right*: sketch of the interior of a Kerr black hole

For the "+" sign this is satisfied by $r_{\rm ms} = GM/c^2$, whereas for the "−‘" sign the solution is $r_{\rm ms} = 9GM/c^2$. The first case corresponds to a co-rotating particle and the second one to a counter-rotating particle. The energy of the co-rotating particle in the innermost orbit is $1/\sqrt{3}$ (units of mc^2). The binding energy of a particle in an orbit is the difference between the orbital energy and its energy at infinity. This means a binding energy of 42 % of the rest energy at infinity! For the counter-rotating particle, the binding energy is 3.8 %, smaller than for a Schwarzschild black hole.

An essential singularity occurs when $g_{tt} \to \infty$; this happens if $\Sigma = 0$. This condition implies:

$$r^2 + a^2 c^{-2} \cos^2\theta = 0. \tag{2.63}$$

Such a condition is fulfilled only by $r = 0$ and $\theta = \frac{\pi}{2}$. This translates in Cartesian coordinates to:[4]

$$x^2 + y^2 = a^2 c^{-2} \quad \text{and} \quad z = 0. \tag{2.64}$$

The singularity is a ring of radius ac^{-1} on the equatorial plane. If $a = 0$, then Schwarzschild's point-like singularity is recovered. If $a \neq 0$ the singularity is not necessarily in the future of all events at $r < r_{\rm h}^{\rm inn}$: the singularity can be avoided by some geodesics.

[4]The relation with Boyer-Lindquist coordinates is $z = r\cos\theta$, $x = \sqrt{r^2 + a^2c^{-2}}\sin\theta\cos\phi$, $y = \sqrt{r^2 + a^2c^{-2}}\sin\theta\sin\phi$.

2.4.1 Pseudo-Newtonian Potentials for Black Holes

The full effective general relativistic potential for particle orbits around a Kerr black hole is quite complex. Instead, pseudo-Newtonian potentials can be used. The first of such potentials, derived by Bohdam Paczyński and used by first time by Paczyński and Wiita (1980), for a non-rotating black hole with mass M, is:

$$\Phi = -\frac{GM}{r - 2r_{\mathrm{g}}}, \tag{2.65}$$

where as before $r_{\mathrm{g}} = GM/c^2$ is the gravitational radius. With this potential one can use Newtonian theory and obtain the same behavior of the Keplerian circular orbits of free particles as in the exact theory: orbits with $r < 9r_{\mathrm{g}}$ are unstable, and orbits with $r < 6r_{\mathrm{g}}$ are unbound. However, velocities of material particles obtained with the potential (2.65) are not accurate, since special relativistic effects are not included (Abramowicz et al. 1996). The velocity $v_{\mathrm{p-N}}$ calculated with the pseudo-Newtonian potential should be replaced by the corrected velocity $v_{\mathrm{p-N}}^{\mathrm{corr}}$ such that

$$v_{\mathrm{p-N}} = v_{\mathrm{p-N}}^{\mathrm{corr}} \gamma_{\mathrm{p-N}}^{\mathrm{corr}}, \quad \gamma_{\mathrm{p-N}}^{\mathrm{corr}} = \frac{1}{\sqrt{1 - \left(\frac{v_{\mathrm{p-N}}^{\mathrm{corr}}}{c}\right)^2}}. \tag{2.66}$$

This re-scaling works amazingly well (see Abramowicz et al. 1996) compared with the actual velocities. The agreement with General Relativity is better than 5 %.

For the Kerr black hole, a pseudo-Newtonian potential was found by Semerák and Karas (1999). It can be found in the expression (19) of their paper. However, the use of this potential is almost as complicated as dealing with the full effective potential of the Kerr metric in General Relativity.

2.5 Reissner-Nordström Black Holes

The Reissner-Nordström metric is a spherically symmetric solution of Eqs. (1.36). However, it is not a vacuum solution, since the source has an electric charge Q, and hence there is an electromagnetic field. The energy-momentum tensor of this field is:

$$T_{\mu\nu} = -\mu_0^{-1}\left(F_{\mu\rho}F_{\nu}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right), \tag{2.67}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor and A_μ is the electromagnetic 4-potential. Outside the charged object the 4-current j^μ is zero, so Maxwell's equations are:

$$F^{\mu\nu}_{;\mu} = 0, \tag{2.68}$$

$$F_{\mu\nu;\sigma} + F_{\sigma\mu;\nu} + F_{\nu\sigma;\mu} = 0. \tag{2.69}$$

Einstein's and Maxwell's equations are coupled since $F^{\mu\nu}$ enters into the gravitational field equations through the energy-momentum tensor and the metric $g_{\mu\nu}$ enters into the electromagnetic equations through the covariant derivative. Because of the symmetry constraints we can write:

$$[A^{\mu}] = \left(\frac{\varphi(r)}{c^2}, a(r), 0, 0\right), \tag{2.70}$$

where $\varphi(r)$ is the electrostatic potential, and $a(r)$ is the radial component of the 3-vector potential as $r \to \infty$.

The solution for the metric is given by

$$ds^2 = \Delta c^2 dt^2 - \Delta^{-1} dr^2 - r^2 d\Omega^2, \tag{2.71}$$

where

$$\Delta = 1 - \frac{2GM/c^2}{r} + \frac{q^2}{r^2}. \tag{2.72}$$

In this expression, M is once again interpreted as the mass of the hole and

$$q = \frac{GQ^2}{4\pi\epsilon_0 c^4} \tag{2.73}$$

is related to the total electric charge Q.

The metric has a coordinate singularity at $\Delta = 0$, in such a way that:

$$r_{\pm} = r_{\rm g} \pm \left(r_{\rm g}^2 - q^2\right)^{1/2}. \tag{2.74}$$

Here, $r_{\rm g} = GM/c^2$ is the gravitational radius. For $r_{\rm g} = q$, we have an *extreme* Reissner-Nordström black hole with a unique horizon at $r = r_{\rm g}$. Notice that a Reissner-Nordström black hole can be more compact than a Schwarzschild black hole of the same mass. For the case $r_{\rm g}^2 > q^2$, both $r_{\pm}$ are real and there are two horizons as in the Kerr solution. Finally, in the case $r_{\rm g}^2 < q^2$ both $r_{\pm}$ are imaginary there is no coordinate singularities, no horizon hides the intrinsic singularity at $r = 0$. It is thought, however, that naked singularities do not exist in Nature (see Sect. 3.6 below).

2.6 Kerr-Newman Black Holes

The Kerr-Newman metric of a charged spinning black hole is the most general black hole solution. It was found by Ezra "Ted" Newman in 1965 (Newman et al. 1965). This metric can be obtained from the Kerr metric (2.50) in Boyer-Lindquist coordinates with the replacement:

$$\frac{2GM}{c^2} r \longrightarrow \frac{2GM}{c^2} r - q^2,$$

where q is related to the charge Q by Eq. (2.73).

The full expression reads:

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dt d\phi - g_{\phi\phi}d\phi^2 - \Sigma\Delta^{-1}dr^2 - \Sigma d\theta^2 \tag{2.75}$$

$$g_{tt} = c^2\left[1 - \left(2GMrc^{-2} - q^2\right)\Sigma^{-1}\right] \tag{2.76}$$

$$g_{t\phi} = a\sin^2\theta\,\Sigma^{-1}\left(2GMrc^{-2} - q^2\right) \tag{2.77}$$

$$g_{\phi\phi} = \left[\left(r^2 + a^2c^{-2}\right)^2 - a^2c^{-2}\Delta\sin^2\theta\right]\Sigma^{-1}\sin^2\theta \tag{2.78}$$

$$\Sigma \equiv r^2 + a^2c^{-2}\cos^2\theta \tag{2.79}$$

$$\Delta \equiv r^2 - 2GMc^{-2}r + a^2c^{-2} + q^2 \equiv \left(r - r_{\rm h}^{\rm out}\right)\left(r - r_{\rm h}^{\rm inn}\right), \tag{2.80}$$

where all symbols have the same meaning as in the Kerr metric and the outer horizon is located at

$$r_{\rm h}^{\rm out} = GMc^{-2} + \left[\left(GMc^{-2}\right)^2 - a^2c^{-2} - q^2\right]^{1/2}. \tag{2.81}$$

The inner horizon is located at:

$$r_{\rm h}^{\rm inn} = GMc^{-2} - \left[\left(GMc^{-2}\right)^2 - a^2c^{-2} - q^2\right]^{1/2}. \tag{2.82}$$

The Kerr-Newman solution is a non-vacuum solution either. It shares with the Kerr and Reissner-Nordström solutions the existence of two horizons, and as the Kerr solution it presents an ergosphere. At a latitude θ, the radial coordinate for the ergosphere is:

$$r_{\rm e} = GMc^{-2} + \left[\left(GMc^{-2}\right)^2 - a^2c^{-2}\cos^2\theta - q^2\right]^{1/2}. \tag{2.83}$$

Like the Kerr metric for an uncharged rotating mass, the Kerr-Newman interior solution exists mathematically but is probably not representative of the actual metric of a physically realistic rotating black hole due to stability problems (see next chapter). The surface area of the horizon is:

$$A = 4\pi\left(r_{\rm h}^{\rm out2} + a^2c^{-2}\right). \tag{2.84}$$

The Kerr-Newman metric represents the simplest stationary, axisymmetric, asymptotically flat solution of Einstein's equations in the presence of an electromagnetic field in four dimensions. It is sometimes referred to as an "electrovaccuum" solution of Einstein's equations. Any Kerr-Newman source has its rotation axis aligned with its magnetic axis (Punsly 1998a). Thus, a Kerr-Newman source is different from commonly observed astronomical bodies, for which there might be a substantial angle between the rotation axis and the magnetic moment.

Since the electric field cannot remain static in the ergosphere, a magnetic field is generated as seen by an observer outside the static limit. This is illustrated in Fig. 2.9. You can find the expressions for the components of the fields in Punsly (2001).

Pekeris and Frankowski (1987) have calculated the interior electromagnetic field of the Kerr-Newman source, i.e., the ring singularity. The electric and magnetic

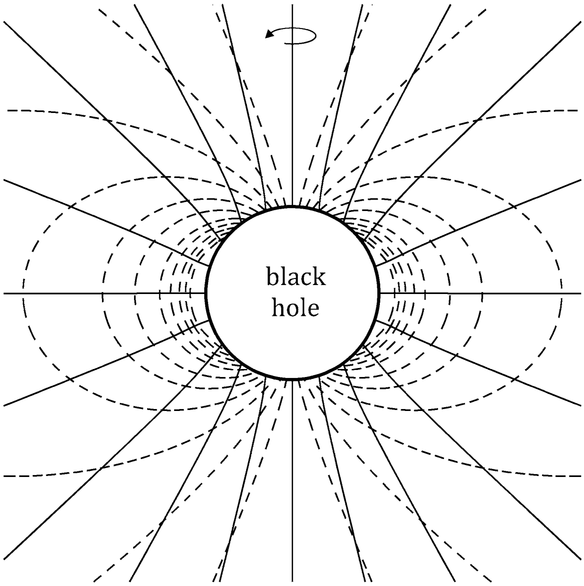

Fig. 2.9 The electric (*solid*) and magnetic (*dashed*) field lines of a Kerr-Newman black hole. The rotation axis of the hole is indicated

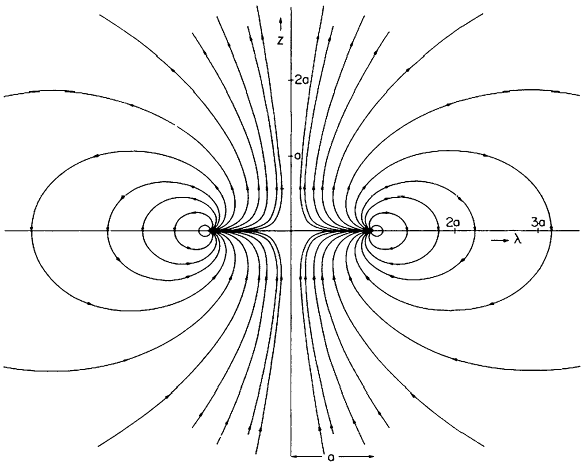

Fig. 2.10 Magnetic field of a Kerr-Newman source. See text for units. Reprinted figure with permission from Pekeris and Frankowski (1987). Copyright (1987) by the American Physical Society

fields are shown in Figs. 2.10 and 2.11, in a (λ, z)-plane, with $\lambda = (x^2 + y^2)^{1/2}$. The general features of the magnetic field are that at distances much larger than ac^{-1} it resembles closely a dipole field, with a dipolar magnetic moment $\mu_\mathrm{d} = Qac^{-1}$. On the disc of radius ac^{-1} the z-component of the field vanishes, in contrast with the interior of Minkowskian ring-current models. The electric field for a positive charge distribution is attractive for positive charges toward the interior disc. At the ring there is a charge singularity and at large distances the field corresponds to that of a point-like charge Q.

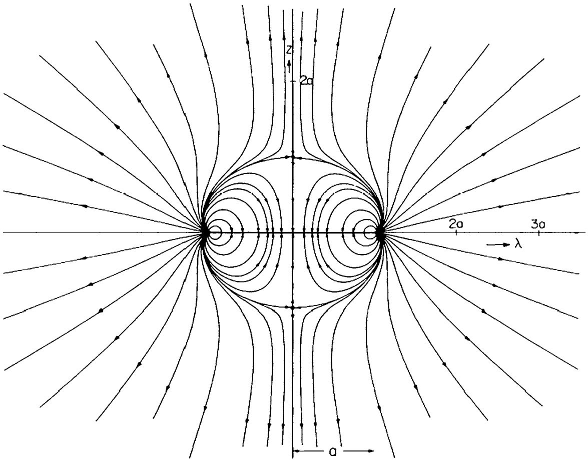

Fig. 2.11 Electric field of a Kerr-Newman source. See text for units. Reprinted figure with permission from Pekeris and Frankowski (1987). Copyright (1987) by the American Physical Society

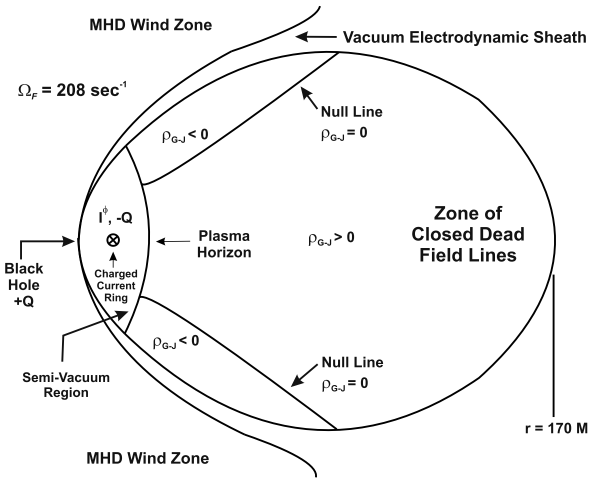

Fig. 2.12 Charged and rotating black hole magnetosphere. The black hole has charge $+Q$ whereas the current ring circulating around it has opposite charge. The figure shows (units $G = c = 1$) the region of closed lines determined by the light cylinder, the open lines that drive a magnetohydrodynamical wind, and the vacuum region in between. Adapted from Punsly (1998a). Reproduced by permission of the AAS

Charged black holes might be a natural result from charge separation during the gravitational collapse of a star. It is thought that an astrophysical charged object would discharge quickly by accretion of charges of opposite sign. There remains the possibility, however, that the charge separation could lead to a configuration where the black hole has a charge and a superconducting ring around it would have the same but opposite charge, in such a way the whole system seen from infinity is neutral. In such a case a Kerr-Newman black hole might survive for some time, depending on the environment. For further details, the reader is referred to the highly technical book by Brian Punsly (2001) and related articles (Punsly 1998a, 1998b, and Punsly et al. 2000). In Figs. 2.12 and 2.13 the magnetic field around a Kerr-Newman black hole surrounded by a charged current ring is shown. The opposite charged black hole and ring are the minimum energy configuration for the system black hole plus magnetosphere. Since the system is neutral from the infinity, it dis-

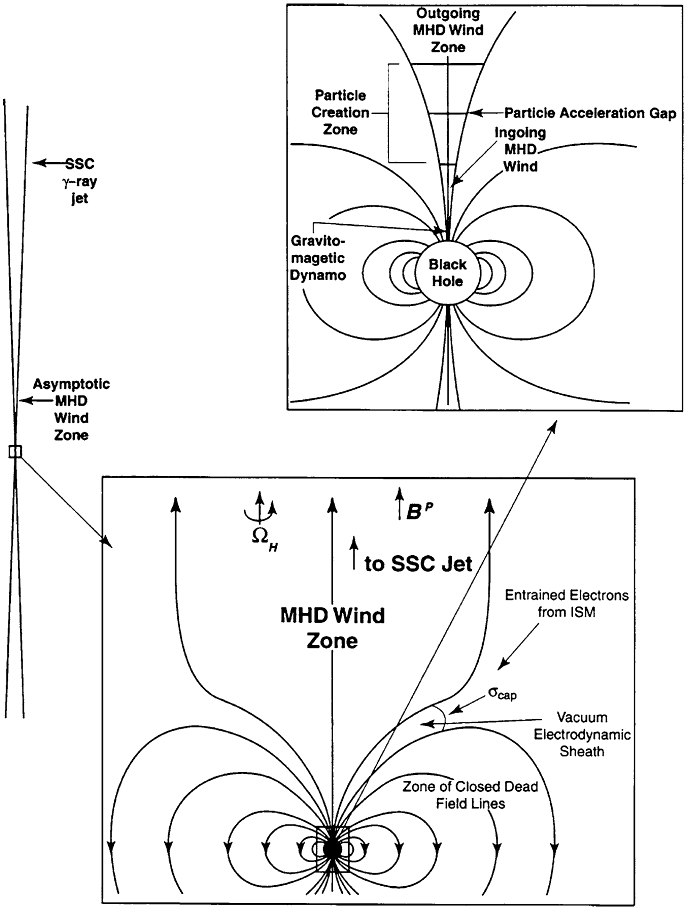

Fig. 2.13 Three different scales of the Kerr-Newman black hole model developed by Brian Punsly. From Punsly (1998a). Reproduced by permission of the AAS

charges slowly and can survive for a few thousand years. During this period, the source can be active through the capture of free electrons from the environment and the production of gamma rays by inverse Compton up-scattering of synchrotron photons produced by electrons accelerated in the polar gap of the hole. In Fig. 2.14 we show the corresponding spectral energy distribution obtained by Punsly et al. (2000) for such a configuration of Kerr-Newman black hole magnetosphere.

2.6.1 Einstein-Maxwell Equations

In order to determine the gravitational and electromagnetic fields over a region of a space-time we have to solve the Einstein-Maxwell equations:

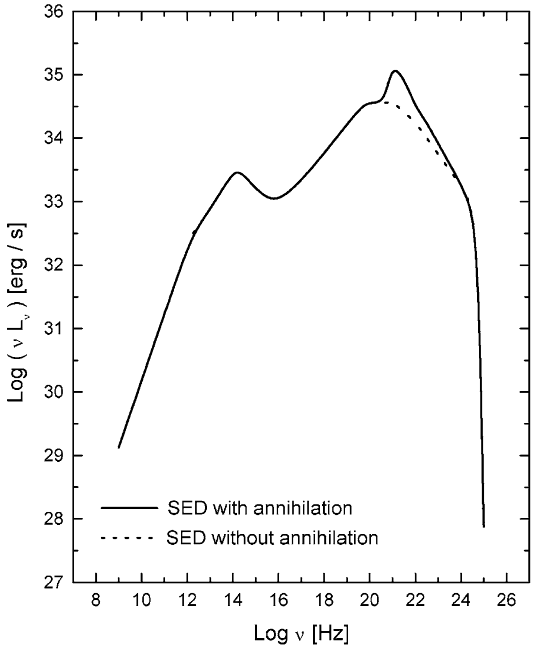

Fig. 2.14 The spectral energy distribution resulting from a Kerr-Newman black hole slowly accreting from the interstellar medium. From Punsly et al. (2000), reproduced with permission ©ESO

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4}(T_{\mu\nu} + E_{\mu\nu}), \tag{2.85}$$

$$\frac{4\pi}{c}E^{\mu}_{\nu} = -F^{\mu\rho}F_{\rho\nu} + \frac{1}{4}\delta^{\mu}_{\nu}F^{\sigma\lambda}F_{\sigma\lambda}, \tag{2.86}$$

$$F_{\mu\nu} = A_{\mu;\nu} - A_{\nu;\mu}, \tag{2.87}$$

$$F^{\nu;\nu}_{\mu} = \frac{4\pi}{c}J_{\mu}. \tag{2.88}$$

Here $T_{\mu\nu}$ and $E_{\mu\nu}$ are the energy-momentum tensors of matter and electromagnetic fields, $F_{\mu\nu}$ and J_{μ} are the electromagnetic field and current density, A_{μ} is the 4-dimensional potential, and Λ is the cosmological constant.

The solution of this system of equations is non-trivial since they are coupled. The electromagnetic field is a source of the gravitational field and this field enters into the electromagnetic equations through the covariant derivatives indicated by the semi-colons. For an exact and relevant solution of the problem see Manko and Sibgatullin (1992).

2.7 Other Black Holes

2.7.1 Born-Infeld Black Holes

Born and Infeld (1934) to avoid the singularities associated with charged point particles in Maxwell theory. Almost immediately, Hoffmann (1935) coupled General

Relativity with Born-Infeld electrodynamics to obtain a spherically symmetric solution representing the gravitational field of a charged object. This solution, forgotten during decades, can represent a charged black hole in nonlinear electrodynamics. In Born-Infeld electrodynamics the trajectories of photons in curved space-times are not null geodesics of the background metric. Instead, they follow null geodesics of an effective geometry determined by the nonlinearities of the electromagnetic field.

The action of Einstein gravity coupled to Born-Infeld electrodynamics has the form (in this section we adopt, for simplicity, $c = G = 4\pi\epsilon_0 = (4\pi)^{-1}\mu_0 = 1$):

$$S = \int dx^4 \sqrt{-g}\left(\frac{R}{16\pi} + L_{\mathrm{BI}}\right), \tag{2.89}$$

with

$$L_{\mathrm{BI}} = \frac{1}{4\pi b^2}\left(1 - \sqrt{1 + \frac{1}{2}F_{\sigma\nu}F^{\sigma\nu}b^2 - \frac{1}{4}\tilde{F}_{\sigma\nu}F^{\sigma\nu}b^4}\right), \tag{2.90}$$

where g is the determinant of the metric tensor, R is the scalar of curvature, $F_{\sigma\nu} = \partial_\sigma A_\nu - \partial_\nu A_\sigma$ is the electromagnetic tensor, $\tilde{F}_{\sigma\nu} = \frac{1}{2}\sqrt{-g}\varepsilon_{\alpha\beta\sigma\nu}F^{\alpha\beta}$ is the dual of $F_{\sigma\nu}$ (with $\varepsilon_{\alpha\beta\sigma\nu}$ the Levi-Civita symbol), and b is a parameter that indicates how much Born-Infeld and Maxwell electrodynamics differ. For $b \to 0$ the Einstein-Maxwell action is recovered. The maximal possible value of the electric field in this theory is b, and the self-energy of point charges is finite. The field equations can be obtained by varying the action with respect to the metric $g_{\sigma\nu}$ and the electromagnetic potential A_ν.

We can write L_{BI} in terms of the electric and magnetic fields:

$$L_{\mathrm{BI}} = \frac{b^2}{4\pi}\left[1 - \sqrt{1 - \frac{B^2 - E^2}{b^2} - \frac{(\mathbf{E}\cdot\mathbf{B})^2}{b^4}}\right]. \tag{2.91}$$

The Lagrangian depends non-linearly of the electromagnetic invariants:

$$F = \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} = \frac{1}{2}\left(B^2 - E^2\right), \tag{2.92}$$

$$\tilde{G} = \frac{1}{4}F_{\alpha\beta}\tilde{F}^{\alpha\beta} = -\mathbf{B}\cdot\mathbf{E}. \tag{2.93}$$

Introducing the Hamiltonian formalism,

$$P^{\alpha\beta} = 2\frac{\partial L}{\partial F_{\alpha\beta}} = \frac{\partial L}{\partial F}F^{\alpha\beta} + \frac{\partial L}{\partial \tilde{G}}\tilde{F}^{\alpha\beta}, \tag{2.94}$$

$$H = \frac{1}{2}P^{\alpha\beta}F_{\alpha\beta} - L\left(F, \tilde{G}^2\right), \tag{2.95}$$

and adopting the notation

$$P = \frac{1}{4}P_{\alpha\beta}P^{\alpha\beta}, \tag{2.96}$$

$$\tilde{Q} = \frac{1}{4} P_{\alpha\beta} \tilde{P}^{\alpha\beta}, \tag{2.97}$$

we can express $F^{\alpha\beta}$ as a function of $P^{\alpha\beta}$, P, and $\tilde{Q}$:

$$F^{\alpha\beta} = 2\frac{\partial H}{\partial P_{\alpha\beta}} = \frac{\partial H}{\partial P} P^{\alpha\beta} + \frac{\partial H}{\partial \tilde{Q}} \tilde{P}^{\alpha\beta}. \tag{2.98}$$

The Hamiltonian equations in the P and $\tilde{Q}$ formalism can be written as:

$$\left(\frac{\partial H}{\partial P} \tilde{P}^{\alpha\beta} + \frac{\partial H}{\partial \tilde{Q}} P^{\alpha\beta} \right)_{,\beta} = 0. \tag{2.99}$$

The coupled Einstein-Born-Infeld equations are:

$$4\pi T_{\mu\nu} = \frac{\partial H}{\partial P} P_{\mu\alpha} P^{\alpha}_{\nu} - g_{\mu\nu} \left(2P \frac{\partial H}{\partial P} + \tilde{Q} \frac{\partial H}{\partial \tilde{Q}} - H \right), \tag{2.100}$$

$$R = 8 \left(P \frac{\partial H}{\partial P} + \tilde{Q} \frac{\partial H}{\partial \tilde{Q}} - H \right). \tag{2.101}$$

The field equations have spherically symmetric black hole solutions given by

$$ds^2 = \psi(r) dt^2 - \psi(r)^{-1} dr^2 - r^2 d\Omega^2, \tag{2.102}$$

with

$$\psi(r) = 1 - \frac{2M}{r} + \frac{2}{b^2 r} \int_r^{\infty} \left(\sqrt{x^4 + b^2 Q^2} - x^2 \right) dx, \tag{2.103}$$

$$D(r) = \frac{Q_E}{r^2}, \tag{2.104}$$

$$B(r) = Q_M \sin\theta, \tag{2.105}$$

where M is the mass, $Q^2 = Q_E^2 + Q_M^2$ is the sum of the squares of the electric Q_E and magnetic Q_M charges, $B(r)$ and $D(r)$ are the magnetic and the electric inductions in the local orthonormal frame. In the limit $b \to 0$, the Reissner-Nordström metric is obtained. The metric (2.102) is also asymptotically Reissner-Nordström for large values of r. With the units adopted above, M, Q and b have dimensions of length. The metric function $\psi(r)$ can be expressed in the form

$$\psi(r) = 1 - \frac{2M}{r} + \frac{2}{3b^2} \left\{ r^2 - \sqrt{r^4 + b^2 Q^2} + \frac{\sqrt{|bQ|^3}}{r} F\left[\arccos\left(\frac{r^2 - |bQ|}{r^2 + |bQ|} \right), \frac{\sqrt{2}}{2} \right] \right\}, \tag{2.106}$$

where $F(\gamma, k)$ is the elliptic integral of the first kind.[5] As in Schwarzschild and Reissner-Nordström cases, the metric (2.102) has a singularity at $r = 0$.

The zeros of $\psi(r)$ determine the position of the horizons, which have to be obtained numerically. For a given value of b, when the charge is small, $0 \leq |Q|/M \leq \nu_1$, the function $\psi(r)$ has one zero and there is a regular event horizon. For intermediate values of charge, $\nu_1 < |Q|/M < \nu_2$, $\psi(r)$ has two zeros, so there are, as in the Reissner-Nordström geometry, an inner horizon and an outer regular event horizon. When $|Q|/M = \nu_2$, there is one degenerate horizon. Finally, if the values of charge are large, $|Q|/M > \nu_2$, the function $\psi(r)$ has no zeros and a naked singularity is obtained. The values of $|Q|/M$ where the number of horizons change, $\nu_1 = (9|b|/M)^{1/3}[F(\pi, \sqrt{2}/2)]^{-2/3}$ and ν_2, which should be calculated numerically from the condition $\psi(r_h) = \psi'(r_h) = 0$, are increasing functions of $|b|/M$. In the Reissner-Nordström limit ($b \to 0$) it is easy to see that $\nu_1 = 0$ and $\nu_2 = 1$.

The paths of photons in nonlinear electrodynamics are not null geodesics of the background geometry. Instead, they follow null geodesics of an effective metric generated by the self-interaction of the electromagnetic field, which depends on the particular nonlinear theory considered. In Einstein gravity coupled to Born-Infeld electrodynamics the effective geometry for photons is given by Bretón (2002):

$$ds^2_{\text{eff}} = \omega(r)^{1/2}\psi(r)dt^2 - \omega(r)^{1/2}\psi(r)^{-1}dr^2 - \omega(r)^{-1/2}r^2 d\Omega^2, \tag{2.107}$$

where

$$\omega(r) = 1 + \frac{Q^2 b^2}{r^4}. \tag{2.108}$$

Then, to calculate the deflection angle for photons passing near the black holes, it is necessary to use the effective metric (2.107) instead of the background metric (2.102). The horizon structure of the effective metric is the same as that of metric (2.102), but the trajectories of photons are different.

2.7.2 Regular Black Holes

Solutions of Einstein's field equations representing black holes where the metric is always regular (i.e. free of intrinsic singularities where $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$ diverges) can be found for some choices of the equation of state. For instance, Mbonye and Kazanas (2005) have suggested the following equation:

$$p_r(\rho) = \left[\alpha - (\alpha + 1)\left(\frac{\rho}{\rho_{\max}}\right)^m\right]\left(\frac{\rho}{\rho_{\max}}\right)^{1/n}\rho. \tag{2.109}$$

[5] $F(\gamma, k) = \int_0^\gamma (1 - k^2 \sin^2\phi)^{-1/2} d\phi = \int_0^{\sin\gamma} [(1 - z^2)(1 - k^2 z^2)]^{-1/2} dz.$

The maximum limiting density $\rho_{\max}$ is concentrated in a region of radius

$$r_0 = \sqrt{\frac{1}{G\rho_{\max}}}. \tag{2.110}$$

At low densities $p_r \propto \rho^{1+1/n}$ and the equation reduces to that of a polytrope gas. At high densities close to $\rho_{\max}$ the equation becomes $p_r = -\rho$ and the system behaves as a gravitational field dominated by a cosmological term in the field equations. The exact values of m, n, and α determine the sound speed in the system. Imposing that the maximum sound speed $c_s = (dp/d\rho)^{1/2}$ be finite everywhere, it is possible to constrain the free parameters. Adopting $m = 2$ and $n = 1$ Eq. (2.109) becomes:

$$p_r(\rho) = \left[\alpha - (\alpha+1)\left(\frac{\rho}{\rho_{\max}}\right)^2\right]\left(\frac{\rho}{\rho_{\max}}\right)\rho. \tag{2.111}$$

The model introduced by Mbonye and Kazanas represents a regular static black hole, with a matter source that smoothly goes from a de Sitter behavior near the origin to Schwarzschild's spacetime outside the object. A space-time metric well-adapted to examine the properties of this system is (Mbonye et al. 2011):

$$ds^2 = -B(r)dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{2.112}$$

where

$$\begin{aligned} B(r) = \exp \int_{r_0}^{r} & \frac{2}{r'^2}\left[m\left(r'\right) + 4\pi r'^3 p_{sfr'}\right] \\ & \times \left[\frac{1}{(1 - \frac{2m(r')}{r'})}\right] dr', \end{aligned} \tag{2.113}$$

and

$$m(r) = 4\pi \int_0^r \rho\left(r'\right) r'^2 dr'. \tag{2.114}$$

Outside the body $\rho \to 0$, and Eq. (2.112) becomes Schwarzschild solution for $R_{\mu\nu} = 0$. When $r \to 0$, $\rho = \rho_{\max}$ and the metric becomes of de Sitter type:

$$ds^2 = \left(1 - \frac{r^2}{r_0^2}\right)c^2 dt^2 - \left(1 - \frac{r^2}{r_0^2}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{2.115}$$

with

$$r_0 = \sqrt{\frac{3}{8\pi G\rho_{\max}}}. \tag{2.116}$$

There is no singularity at $r = 0$ and the black hole is regular. For $0 \le r < 1$ it has constant positive density $\rho_{\max}$ and negative pressure $p_r = -\rho_{\max}$ and space-time becomes asymptotically de Sitter in the innermost region. It might be speculated that the transition in the equation of state occurs because at very high densities the matter field couples with a scalar field that provides the negative pressure.

Other assumptions for the equation of state can lead to different (but still regular) behavior, like a bouncing close to $r = 0$ and the development of an expanding closed universe inside the black hole (Frolov et al. 1990). Unstable behavior, both dynamic and thermodynamic, seems to be a characteristic of these type of black hole solutions (Pérez et al. 2011).

Regular black holes might also be found in $f(R)$ gravity for some suitable function of the curvature scalar.

2.7.3 $f(R)$ Black Holes

2.7.3.1 Rewriting the Field Equations of $f(R)$ Gravity

In order to study the possible black hole solutions obtained from any $f(R)$ theory, we rewrite the action (1.122) in the form

$$S = S_g + S_m, \tag{2.117}$$

where S_g is the gravitational action, with the scalar function now written as $R + f(R)$

$$S_g = \frac{1}{16\pi G} \int d^4x \sqrt{|g|}\big(R + f(R)\big). \tag{2.118}$$

From the matter term S_m, we define the energy momentum tensor as

$$T^{\mu\nu} = -\frac{2}{\sqrt{|g|}} \frac{\delta S_m}{\delta g_{\mu\nu}}. \tag{2.119}$$

By performing variations of (2.117) with respect to the metric tensor, we obtain the field equations in metric formalism in a more convenient way:

$$\begin{aligned} R_{\mu\nu}\big(1 + f'(R)\big) - \frac{1}{2} g_{\mu\nu}\big(R + f(R)\big) \\ + (\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box) f'(R) + 8\pi G T_{\mu\nu} = 0, \end{aligned} \tag{2.120}$$

with $\Box = \nabla_\beta \nabla^\beta$ as before and $f'(R) = df(R)/dR$. Clearly, for $f(R) = 0$ standard General Relativity is recovered. Taking the trace of this equation yields:

$$R\big(1 + f'(R)\big) - 2\big(R + f(R)\big) - 3\Box f'(R) + 8\pi G T = 0, \tag{2.121}$$

where $T = T^\mu_\mu$. Unlike the case of General Relativity, vacuum solutions ($T = 0$) do not necessarily imply a null curvature $R = 0$. From Eq. (2.120) we obtain the condition for vacuum constant scalar curvature $R = R_0$ solutions:

$$R_{\mu\nu}\big(1 + f'(R_0)\big) - \frac{1}{2}g_{\mu\nu}\big(R_0 + f(R_0)\big) = 0. \tag{2.122}$$

and the Ricci tensor becomes proportional to the metric,

$$R_{\mu\nu} = \frac{R_0 + f(R_0)}{2(1 + f'(R_0))}g_{\mu\nu}, \tag{2.123}$$

with $1 + f'(R_0) \neq 0$. Taking the trace on the previous equation:

$$R_0\big(1 + f'(R_0)\big) - 2\big(R_0 + f(R_0)\big) = 0, \tag{2.124}$$

and therefore

$$R_0 = \frac{2f(R_0)}{f'(R_0) - 1}. \tag{2.125}$$

If we want to find viable black hole solutions of Eq. (2.120), some conditions must be imposed in order to make $f(R)$ theories consistent with known gravitational and cosmological facts. These conditions are (Cembranos et al. 2011):

1. $f''(R) \geq 0$ for $R \gg f''(R)$. This is the stability requirement for a high-curvature classical regime and that of the existence of a matter dominated era in cosmological evolution. A simple physical interpretation can be given to this condition: if an effective gravitational constant $G_{\text{eff}} \equiv G/(1 + f'(R))$ is defined, then the sign of its variation with respect to R, dG_{eff}/dR, is uniquely determined by the sign of $f''(R)$, so in case $f''(R) < 0$, G_{eff} would grow as R does, because R generates more and more curvature. This mechanism would destabilize the metric field, since it would not have a fundamental state because any small curvature would grow to infinity. Instead, if $f''(R) \geq 0$, a counter reaction mechanism operates to compensate this R growth and stabilize the system.
2. $1 + f'(R) > 0$. This conditions ensures that the effective gravitational constant is positive, as it can be checked from the previous definition of G_{eff}.
3. $f'(R) < 0$. Keeping in mind the strong restrictions of Big Bang nucleosynthesis and cosmic microwave background, this condition ensures that the expected behavior be recovered at early times, that is, $f(R)/R \to 0$ and $f'(R) \to 0$ as $R \to \infty$. Conditions 1 and 2 together demand $f(R)$ to be a monotonously increasing function between the values $-1 < f'(R) < 0$.
4. $f'(R)$ must be small in recent epochs. This condition is mandatory in order to satisfy imposed restrictions by local (solar and galactic) gravity tests.

In looking for constant curvature R_0 vacuum solutions for fields generated by massive charged objects we follow Cembranos et al. (2011). The action (in units of $G = c = \hbar = k = 1$) is:

$$S = \frac{1}{16\pi}\int d^4x\sqrt{|g|}\big(R + f(R) - F_{\mu\nu}F^{\mu\nu}\big), \tag{2.126}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and A_μ the electromagnetic potential. This action leads to the field equations:

$$\begin{aligned} R_{\mu\nu}\big(1+f'(R_0)\big) - \frac{1}{2} g_{\mu\nu}\big(R_0 + f(R_0)\big) \\ -2\left(F_{\mu\alpha}F^{\alpha}_{\ \nu} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta}F^{\alpha\beta}\right) = 0. \end{aligned} \tag{2.127}$$

If we take the trace of the previous equation, then (2.124) is recovered due to the fact that $F^{\mu}_{\ \mu} = 0$.

The axisymmetric, stationary and constant curvature R_0 solution that describes a black hole with mass, electric charge, and angular momentum was originally found by Carter (1973). In Boyer-Lindquist coordinates, the metric describing with no coordinate singularities the spacetime exterior to the black hole and interior to the cosmological horizon (provided it exists), is:

$$\begin{aligned} ds^2 = \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 + \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left[a\frac{dt}{\Xi} - \big(r^2+a^2\big)\frac{d\phi}{\Xi}\right]^2 \\ \frac{\Delta_r}{\rho^2}\left(\frac{dt}{\Xi} - a\sin^2\theta \frac{d\phi}{\Xi}\right)^2, \end{aligned} \tag{2.128}$$

with

$$\begin{aligned} a\Delta_r &\equiv \big(r^2+a^2\big)\left(1 - \frac{R_0}{12} r^2\right) - 2Mr + \frac{q^2}{(1+f'(R_0))}, \\ \rho^2 &\equiv r^2 + a^2\cos^2\theta, \\ \Delta_\theta &\equiv 1 + \frac{R_0}{12} a^2 \cos^2\theta, \\ \Xi &\equiv 1 + \frac{R_0}{12} a^2, \end{aligned} \tag{2.129}$$

where M, a and q, as before, denote the mass, spin, and electric charge parameters, respectively.

The potential vector and electromagnetic field tensor in Eq. (2.127) for metric (2.128) are:

$$\begin{aligned} A &= -\frac{qr}{\rho^2}\left(\frac{dt}{\Xi} - a\sin^2\theta\frac{d\phi}{\Xi}\right), \\ F &= -\frac{q(r^2 - a^2\cos^2\theta)}{\rho^4}\left(\frac{dt}{\Xi} - a\sin^2\theta\frac{d\phi}{\Xi}\right) \wedge dr \\ &\quad - \frac{2qra\cos\theta\sin\theta}{\rho^4} d\theta \wedge \left[a\frac{dt}{\Xi} - \big(r^2+a^2\big)\frac{d\phi}{\Xi}\right]. \end{aligned} \tag{2.130}$$

We adopt $Q^2 \equiv q^2/(1 + f'(R_0))$ in what follows to refer to the electric charge parameter of the black hole.

If we take the limits $M \to 0$, $Q \to 0$, $a \to 0$ to the metric, we obtain a constant curvature spacetime metric:

$$ds^2 = -\left(1 - \frac{R_0 r^2}{12}\right)dt^2 + \frac{1}{(1 - \frac{R_0 r^2}{12})}dr^2 + r^2 d\Omega^2, \tag{2.131}$$

that corresponds to either a de Sitter or an anti-de Sitter space-time depending on the sign of R_0. Clearly, when $a \to 0$ and $Q \to 0$, Schwarzschild black hole is recovered.

Calculating $R^{\mu\nu\sigma\rho}R_{\mu\nu\sigma\rho}$, only $\rho = 0$ happens to be an intrinsic singularity, and considering the definition of ρ in (2.129), such singularity is given by:

$$r = 0 \quad \text{and} \quad \theta = \pi/2. \tag{2.132}$$

Keeping in mind that we are working with Boyer-Lindquist coordinates, the set of points given by $r = 0$ and $\theta = \pi/2$ represent a ring in the equatorial plane of radius a centered on the rotation axis of the black hole, just as in Kerr black holes.

The horizons are found from $g^{rr} = 0$, i.e. their location is given by the roots of the equation $\Delta_r = 0$:

$$r^4 + \left(a^2 - \frac{12}{R_0}\right)r^2 + \frac{24M}{R_0}r - \frac{12}{R_0}\left(a^2 + Q^2\right) = 0. \tag{2.133}$$

This is a fourth order equation that can be rewritten as:

$$(r - r_-)(r - r_{\text{int}})(r - r_{\text{ext}})(r - r_{\text{cosm}}) = 0, \tag{2.134}$$

where r_- is always a negative solution with no physical meaning, r_{int} and r_{ext} are the interior and exterior horizons respectively, and r_{cosm} represents the cosmological event horizon for observers between r_{ext} and r_{cosm}. This horizon divides the region that the observer could see from the region she/he could never see if she/he waited long enough. The existence of real solutions for this equation is given by a factor h, called *horizon parameter* (Cembranos et al. 2011):

$$\begin{aligned} h \equiv & \left[\frac{4}{R_0}\left(1 - \frac{R_0}{12}a^2\right)^2 - 4\left(a^2 + Q^2\right)\right]^3 \\ & + \frac{4}{R_0}\left\{\left(1 - \frac{R_0}{12}a^2\right)\left[\frac{4}{R_0}\left(1 - \frac{R_0}{12}a^2\right)^2 \right.\right. \\ & \left.\left. + 12\left(a^2 + Q^2\right)\right] - 18M^2\right\}^2. \end{aligned} \tag{2.135}$$

For a negative scalar curvature R_0, three options may be considered: (i) $h > 0$: there are only two real solutions, r_{int} and r_{ext}, lacking this configuration a cosmological horizon, as it is expected for an anti-de Sitter like Universe. (ii) $h = 0$: there is only a degenerated root, particular case of an extremal black hole, whose interior and exterior horizons have merged into one single horizon with a null surface gravity. (iii) $h < 0$: there is no real solution to (2.135), which means absence of horizons and then a naked singularity.

For a positive curvature R_0, there are also several configurations depending on the value of h: (i) $h < 0$: both r_{int}, r_{ext}, and r_{cosm} are positive and real, thus the black hole possesses a well-defined horizon structure in an Universe with a cosmological horizon. (ii) $h = 0$: two different cases may be described, either r_{int} and r_{ext} become degenerated solutions, or r_{ext} and r_{cosm} do so. The first case represents an extremal black hole. The second case can be understood as the cosmological limit for which a black hole preserves its exterior horizon without being "torn apart" by the relative recession speed between two radially separated points induced by the cosmic expansion in an Universe described by a constant positive curvature. (iii) $h > 0$: there is only one positive root, that may be either r_{int} or r_{cosm}. In the first case, the mass of the black hole has exceeded the limit imposed by the cosmology ($h = 0$), and there are neither exterior nor cosmological horizon. This situation just leaves the interior horizon to cover the singularity (*marginal naked singularity case*). If the root corresponds to r_{cosm}, there is a naked singularity with a cosmological horizon. From a certain positive value of the curvature R_0^{crit} onward, the h factor goes to zero for two values of a, i.e., apart from the usual $a_{\max}$ for which the black hole turns extremal, there is now a spin lower bound $a_{\min}$, below which the black hole turns into a *marginally extremal* black hole. Therefore

$$h\big(a_{\max}, M, |R_0| \geq 0, Q\big) = 0$$
$$\Rightarrow \quad a_{\max} \equiv a_{\max}\big(M, |R_0| \geq 0, Q\big), \tag{2.136}$$
$$h\big(a_{\min}, M, R_0 \geq R_0^{\text{crit}} > 0, Q\big) = 0$$
$$\Rightarrow \quad a_{\min} \equiv a_{\min}\big(M, R_0 \geq R_0^{\text{crit}} > 0, Q\big). \tag{2.137}$$

Another interesting feature of Kerr-Newman black holes is the presence of an ergosphere bounded by a Stationary Limit Surface (SLS), given by $g_{tt} = 0$. In Boyer-Lindquist coordinates:

$$\frac{\Delta_\theta \sin^2\theta a^2}{\rho^2 \Xi^2} - \frac{\Delta_r}{\rho^2 \Xi^2} = 0, \tag{2.138}$$

that leads to the fourth order equation

$$r^4 + \left(a^2 - \frac{12}{R_0}\right) r^2 + \frac{24M}{R_0} r - \left(a^2 \cos^2\theta + \frac{12}{R_0}\right) a^2 \sin^2\theta$$
$$- \frac{12}{R_0}\big(a^2 + Q^2\big) = 0, \tag{2.139}$$

which can be rewritten as:

$$(r - r_{S-})(r - r_{S\ \text{int}})(r - r_{S\ \text{ext}})(r - r_{S\ \text{cosm}}) = 0. \tag{2.140}$$

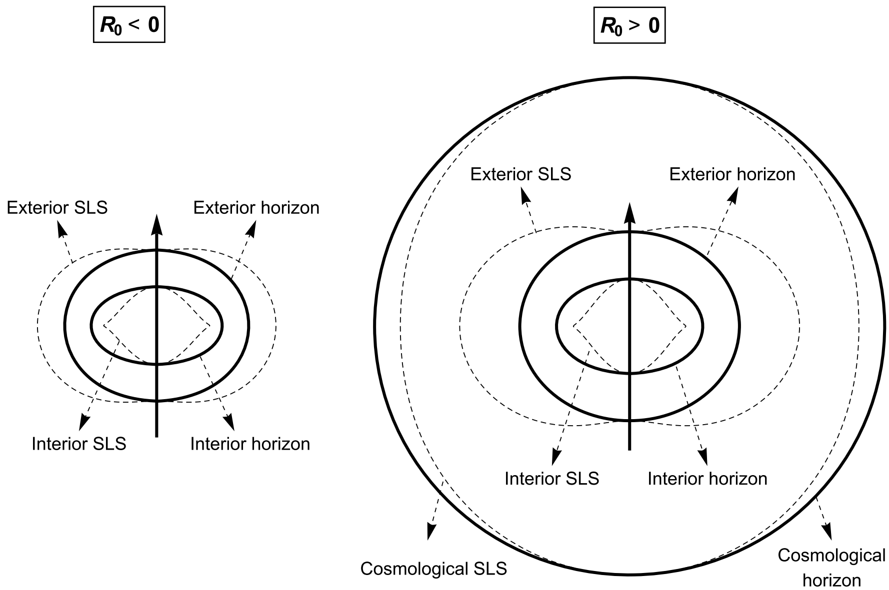

Fig. 2.15 *Left*: Diagram of a Kerr-Newman black hole structure with negative curvature solution $R_0 = -0.4 < 0$, $M = 1$, $a = 0.85$ and $Q = 0.35$ ($h > 0$). *Right*: Black hole structure with positive curvature solution $R_0 = 0.4 > 0$, $M = 1$, $a = 0.9$ and $Q = 0.4$ ($h < 0$). *Dotted surfaces* represent the static limit surfaces (SLS) whereas horizons are shown with *continuous lines*. The rotation axis of the black hole is indicated by the *vertical arrow*. In both types of black hole, the region between the exterior SLS $r_{\mathrm{S\ ext}}$ and its associated exterior horizon r_{ext} is known as *ergoregion*. From Cembranos et al. (2011). Reproduced by permission of the authors

From this equation it follows that each horizon has an "associated" SLS. Both hypersurfaces coincide at $\theta = 0, \pi$ as seen when comparing (2.139) with Eq.(2.133). A scheme of black hole horizons and the corresponding ergospheres is shown in Fig. 2.15 for both signs of R_0.

For some general properties of static and spherically symmetric black holes in $f(R)$-theories see Perez-Bergliaffa and Chifarelli de Oliveira Nunes (2011). For Kerr-$f(R)$ black holes and disks around them see Pérez et al. (2013).

2.7.4 Mini Black Holes

In principle, a black hole can have any mass above the Planck mass. A black hole is formed always that there is an energy density large enough as to curve the spacetime forming a null closed surface. A lower possible mass for a black hole is imposed by the Compton wavelength, $\lambda_{\mathrm{C}} = h/Mc$, which represents a limit on the minimum size of the region in which a mass M at rest can be localized. For a sufficiently small M, the reduced Compton wavelength ($\bar{\lambda}_{\mathrm{C}} = \hbar/Mc$) exceeds half the

Schwarzschild radius, and no black hole description exists. The smallest mass for a black hole is thus approximately the Planck mass.

Very low-mass or "mini" black holes evaporate very quickly by emission of Hawking's radiation (see Sect. 3.3). The absence of notable excesses of particles in cosmic radiation—especially in the form of anti-protons—compared with the fluxes expected in a "standard" astrophysical context allows to impose strict constraints on the number density of black holes evaporating in today's Universe. In particular, it can be deduced that their contribution to the total mass of the universe is today no higher than one ten millionth. As these small black holes are likely to have been produced in the early cosmos by the fluctuations in energy density at that time—and with masses that were very low—it is possible to obtain vital information about the universe's degree of inhomogeneity shortly after the period of inflation by imposing upper limits from observations of secondary particles to the possible number of mini black holes.

In addition to these astrophysical and cosmological aspects, there is another route of investigation that is particularly promising for microscopic black holes, namely at particle accelerators. If the center-of-mass energy of two elementary particles is higher than the Planck scale, and their impact parameter is lower than the Schwarzschild radius, a black hole must be produced. If the Planck scale is thus in the TeV range, the 14 TeV center-of-mass energy of the Large Hadron Collider might allow it to become a black-hole factory with a production rate as high as about one per second.

The possible presence of extra compact dimensions (Sect. 1.12) would be beneficial for the production of black holes. The key point is that it allows the Planck scale to be reduced to accessible values, but it also allows the Schwarzschild radius to be significantly increased, thus making the condition for the impact parameter to be smaller than the Schwarzschild radius easier to satisfy. It is important to note that the resulting mini black holes have radii that are much smaller than the size of extra dimensions, and that they can therefore be considered as totally immersed in a D-dimensional space, which has, to a good approximation, a time dimension and $D-1$ non-compact space dimensions. The black hole thus acts like a quasi-selective source of S waves and sees our brane in the same way as the "bulk" associated with the extra dimensions. As the particles residing in the brane greatly outnumber those living in the bulk (essentially gravitons), the black hole evaporates into particles of the Standard Model. Its lifetime is very short (of the order of 10^{-26} s) and its temperature (typically about 100 GeV here) is much lower than it would be with the same mass in a four-dimensional space. The black hole nevertheless retains its characteristic spectrum in the form of a quasi-thermal law peaked around its temperature. From the point of view of detection, it is not too difficult to find a signature for such events: they have a high multiplicity, a large transverse energy, a "democratic" coupling to all particles, and a rapid increase in the production cross-section with energy.

First of all the reconstruction of temperature (determined by the energy spectrum of the particles emitted when the black hole evaporates) as a function of mass (determined by the total energy deposited) allows information to be gained about the dimensionality of space-time. In the case of Planck scales close to the TeV mark, the number of extra dimensions could thus be revealed quite easily by the characteristics of the emitted particles. One can go even further. In particular, quantum gravity effects could be revealed, as behavior during evaporation in the Planck region is sensitive to the details of the gravitational theory used (for more on mini black holes see Frolov and Zelnikov 2011, and references therein).

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