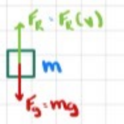
Example 2

Falling Body in Resistive Medium



$$F_g - F_R = ma$$

$$mg - F_R(v) = mv'$$

Linear Air Resistance

$$mg - kv = mv'$$

$$v' + \frac{k}{m}v = g$$

$$A(t) = \int_0^t \frac{k}{m}\,dt = \frac{k}{m}t$$

$$\frac{d}{dt}\left[ve^{\frac{k}{m}t}\right] = ge^{\frac{k}{m}t}$$

$$ve^{\frac{k}{m}t} - v_0 = \int_0^t ge^{\frac{k}{m}t}\,dt = \frac{gm}{k}e^{\frac{k}{m}t} - \frac{gm}{k}$$

$$v(t) = e^{-\frac{k}{m}t}\left(v_0 - \frac{gm}{k}\right) + \frac{gm}{k}$$

$$\lim_{t\to\infty} v(t) = \frac{gm}{k}$$

$$a(t) = \frac{dv(t)}{dt} = -\frac{k}{m}e^{-\frac{k}{m}t}\left(v_0 - \frac{gm}{k}\right) = e^{-\frac{k}{m}t}\left(g - \frac{kv_0}{m}\right)$$

$$\lim_{t\to\infty} a(t) = 0$$

$$\begin{aligned}
s(t) - s(0) = \int_0^t v(t)\,dt &= -\frac{m}{k}e^{-\frac{k}{m}t}\left(v_0 - \frac{gm}{k}\right) + \frac{gm}{k}t + \left[\frac{m}{k}\left(v_0 - \frac{gm}{k}\right)\right] \\
&= v_0\frac{m}{k}\left(1 - e^{-\frac{k}{m}t}\right) + \frac{gm^2}{k^2}e^{-\frac{k}{m}t} + \frac{gm}{k}t - \frac{gm^2}{k^2} \\
&= v_0\frac{m}{k}\left(1 - e^{-\frac{k}{m}t}\right) + \frac{gm}{k}t + \frac{gm^2}{k^2}\left(e^{-\frac{k}{m}t} - 1\right)
\end{aligned}$$

Thus, we have

$$s(t) = v_0\frac{m}{k}\left(1 - e^{-\frac{k}{m}t}\right) + \frac{gm}{k}t + \frac{gm^2}{k^2}\left(e^{-\frac{k}{m}t} - 1\right)$$

$$v(t) = e^{-\frac{k}{m}t}\left(v_0 - \frac{gm}{k}\right) + \frac{gm}{k} = e^{-\frac{t}{b}}(v_0 - c) + c$$

$$a(t) = e^{-\frac{k}{m}t}\left(g - \frac{kv_0}{m}\right)$$

Problem 5

$$\frac{dv}{dt} = \frac{dv}{ds}\cdot\frac{ds}{dt} = v\cdot\frac{dv}{ds}$$

sub into original ODE

$$mg - kv = mv' = mv\frac{dv}{ds}$$

$$\frac{ds}{dv} = \frac{mv}{mg - kv} = \frac{\frac{m}{k}v}{\frac{mg}{k} - v} = \frac{bv}{c - v}$$

$$\int_{v_0}^{v}\frac{ds}{dv}\,dv = \int_{v_0}^{v}\frac{bv}{c - v}\,dv = bc\left(\ln(c) - \ln(c - v)\right) - bv - \left[bc\left(\ln(c) - \ln(c - v_0)\right) - bv_0\right]$$

$$\begin{aligned}
s(v) - s(v_0) &= bc\left(\cancel{\ln(c)} - \ln(c - v)\right) - bv - \left[bc\left(\cancel{\ln(c)} - \ln(c - v_0)\right) - bv_0\right] \\
&= -bc\ln(c - v) + bc\ln(c - v_0) - bv + bv_0 \\
&= bc\ln\left|\frac{c - v_0}{c - v}\right| - bv + bv_0
\end{aligned}$$

$$\begin{aligned}
s(t) - s(0) &= v_0\frac{m}{k}\left(1 - e^{-\frac{k}{m}t}\right) + \frac{gm}{k}t + \frac{gm^2}{k^2}\left(e^{-\frac{k}{m}t} - 1\right) \\
&= v_0 b\left(1 - e^{-\frac{t}{b}}\right) + ct + bc\left(e^{-\frac{t}{b}} - 1\right)
\end{aligned}$$

$$v(t) = e^{-\frac{t}{b}}(v_0 - c) + c$$

$$\frac{v - c}{v_0 - c} = e^{-\frac{t}{b}} \longrightarrow e^{-\frac{t}{b}} - 1 = \frac{v - \cancel{c} - v_0 + \cancel{c}}{v_0 - c} = \frac{v - v_0}{v_0 - c}$$

$$\ln\left|\frac{v - c}{v_0 - c}\right| = -\frac{t}{b}$$

$$t = -b\ln\left|\frac{v - c}{v_0 - c}\right|$$

$$\begin{aligned}
s(t) - s(0) &= v_0 b\left(1 - \frac{v - c}{v_0 - c}\right) - bc\ln\left|\frac{v - c}{v_0 - c}\right| + bc\frac{v - v_0}{v_0 - c} \\
&= bc\ln\left|\frac{v_0 - c}{v - c}\right| + bv_0 + \frac{bc(v - v_0) - v_0 b(v - c)}{v_0 - c}
\end{aligned}$$

$$\frac{bcv - \cancel{bcv_0} - vv_0 b + \cancel{v_0 bc}}{v_0 - c}$$

$$\frac{-bv(v_0 - c)}{v_0 - c}$$

$$= -bv$$

$$= bc\ln\left|\frac{v_0 - c}{v - c}\right| + bv_0 - bv$$

$$= s(v) - s(0)$$