2310H Sequences and series

We want to use limit processes to extend our reach from the familiar to the unfamiliar, by approximating some exotic functions and numbers in terms of more familiar ones. E.g. we will approximate irrational numbers like e and π, by simple rational numbers. And we will approximate exotic functions like e^x, $\ln(1+x)$, $\sin(x)$, and $\arctan(x)$, by polynomials. Remember, to say some irrational number is a limit of rational numbers just means we can approximate the irrational number as closely as we like by rational numbers. Also to say a function is a limit of polynomials means we can approximate the given function as closely as we like by polynomials. For this we must decide what it means for two functions to be near each other. Does it mean all their values are uniformly near? I.e. that there is not much distance between their graphs?" Or does it mean there is not much area between their graphs? Or something else? We will discuss the various choices below.

Definition: If S is any set, a sequence with values in S is simply a function a:N-->S where N is the "natural numbers", i.e. the positive integers. We denote the value a(n) by a_n and display the whole function as its sequence of values: $a_1,a_2,....,a_n,.....$

Remark: It is not essential that the sequence begin with 1; it could begin with 0, or -4, or 1000. The important thing is that it begin somewhere and go on up to infinity. I.e. it is only infinite in one direction, upwards.
We often use letters to denote the values of a sequence that remind us of the nature of the elements of the set S.

Example 1: If S is the real numbers we might call the elements x_n, and write the sequence as $\{x_n\}$ or $x_1,x_2,x_3,....,x_n,....$

Example 2: If the set S is the Euclidean plane R^2, we might write $\{p_n\}$ or $p_1,p_2,p_3,....,p_n,....$ for a sequence of points in the plane.

Example 3: If the set S is the set of continuous functions on the interval [a,b], we might write $f_1,f_2,....,f_n,....$ or $\{f_n\}$ for the sequence of functions.

In all three of our examples, we can add, subtract, and multiply our elements by real numbers. We want to define next a notion of "size" or "length" or "absolute value" for our elements.

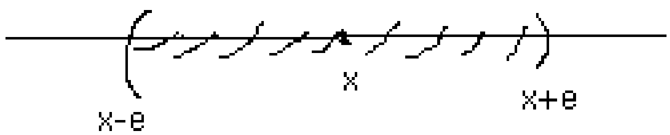
ex. 1: For real numbers define the absolute value of a number x to be $|x|$ = its absolute value.

ex.2. For a point p =(x,y) in the plane, define its absolute value to be the Euclidean distance from the origin, $|p| = \text{sqrt}(x^2+y^2)$.

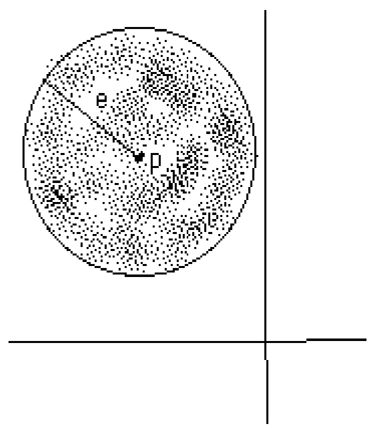
ex. 3. For a function f on [a,b] there are several natural choices, which yield different results. The one suited to our present purposes is called the "sup norm", which is the maximum of all the absolute values $|f(x)|$, i.e. $||f||$ = the global maximum of the function $|f|$. Thus $||f||$ is the maximum of the absolute values $|f(x)|$ of f evaluated at every point x in [a,b].

Thus ||f|| is the height of the highest point of the graph of the function y = |f(x)|, over the interval [a,b]. We know from a big theorem in my 2300H notes, that there exists such a maximum. These notions of length lead a notion of distance between two objects, and hence of a notion of "epsilon neighborhood" centered at one object:

ex.1: Given two real numbers x,y, their distance apart is |x-y|. For e > 0, the e - neighborhood of x, is the open interval (x-e, x+e) of all real numbers closer to x than e.

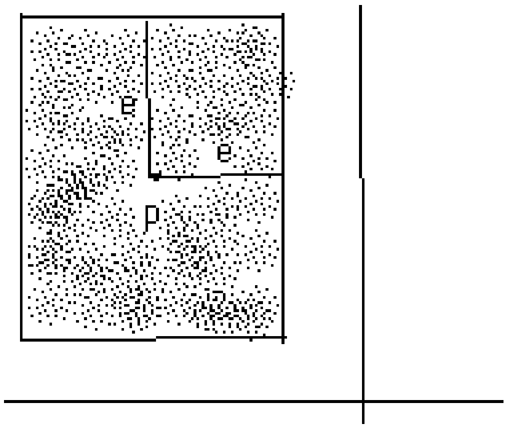


ex.2: Given two points in the plane $p_1 = (x_1,y_1)$, and $p_2 = (x_2,y_2)$, their distance apart is $|p_1-p_2| = \mathrm{sqrt}([x_1-x_2]^2 + [y_1-y_2]^2)$. Given e > 0, the e - neighborhood of p, is the open disc of radius e, centered at p, of all real points in the plane closer to p than e, in the usual "Euclidean norm".

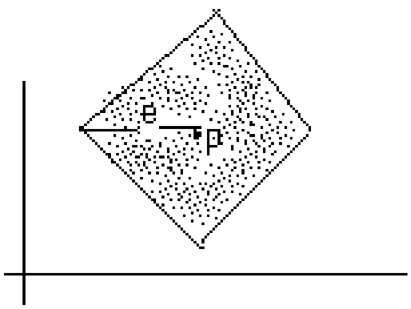


Remark: We get the same notion of convergence in the plane, but not exactly the same notion of distance, by saying that the distance between two points $p_1 = (x_1,y_1)$, and $p_2 = (x_2,y_2)$, is the maximum of $|x_1-x_2|$, or $|y_1-y_2|$. I.e. by seeing how far apart their x and their y coordinates are, and taking the larger difference as the distance between the points.

Then given e > 0, the e - neighborhood of p, would be the open square of radius e, centered at p, all real points in the plane closer to p than e, in the "maximum norm".



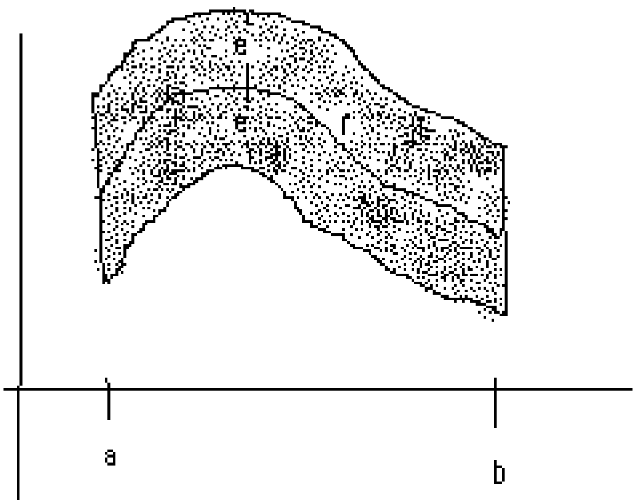
There is a third natural notion of length and distance for points in the plane, called the "sum norm", where the length of p = (x,y) = |x|+|y| is the sum of the absolute values of the coordinates. Then the distance from $p_1 = (x_1,y_1)$, to $p_2 = (x_2,y_2)$ is $|x_1-x_2|+|y_1-y_2|$, and the e - nbhd of p, is a "diamond" of radius e, centered at p:



ex.3: The three definitions of "length" we discussed in the plane all have generalizations to "size" of functions. The Euclidean norm generalizes to the "L_2-norm" where a function has size $= (\int_a^b f^2)^{1/2}$, the square root of the integral of its square. If we think of a function as a "vector" with an infinite number of components, this definition yields a related definition of "dot product" $f \bullet g = \int_a^b f(x)g(x)dx$ which allows one to talk about the "angle" between two functions and perpendicularity of functions. This particularly use in approximating functions by sines and cosines, called the theory of "Fourier series".

The sum norm generalizes to $\int_a^b |f|$, the integral of the absolute value. This was Matt's suggestion, and it is very useful in extending the notion of integrability of functions to more general functions than the ones Riemann's definitions works for. Convergence using this notion of length, the "L_1-norm", leads to the theory of "Lebesgue integration".

For our purpose of approximating functions by polynomials, it is useful to choose the generalization of the "max norm" we defined above. Thus the distance between two functions f,g in the max norm, is defined as $||f-g||$ = maximum of all differences $|f(x)-g(x)|$, for all x in [a,b]. For given e>0, the resulting e - nbhd of f, is represented by a strip extending a distance e both above and below the graph of f. I.e. a function g is within a distance e of f if and only if its graph lies entirely in that strip.



Remarks: All our notions of length satisfy these basic properties:

(i) "triangle inequality" $|a+b| \leq |a| + |b|$, $|a-b| \geq |a| - |b|$.

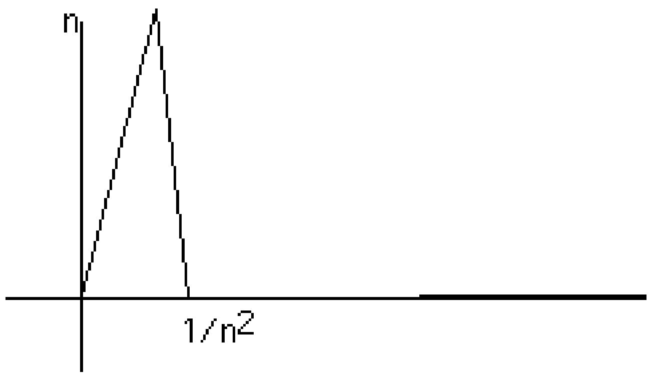
(ii) homogeneity: $|ca| = |c||a|$, where c is a real number.
e.g. $\|cf\| = |c|\ \|f\|$, for a function f and a constant c.

(iii) non degeneracy: $|a| = 0$ if and only if $a = 0$.

Although all three norms in the plane give the same notion of convergence, this is not true for their generalizations to functions. Here the sup norm is more restrictive than the L_1 or L_2 norms.

Exercise: If two continuous functions on [a,b] functions are everywhere within e of each other then their integrals are also within e(b-a) of each other hence also close. [Hint: Recall the monotonicity property of integrals, that $f(x) \leq g(x)$ for all x in [a,b], implies $\int_a^b f \leq \int_a^b g$.]

In particular a function which is everywhere close to zero, has integral which is also close to zero. I.e. if a function is small in the sup norm, it is also small in the L_1 norm. On the other hand a function can have integral very close to zero and yet can have some very large values. Hence a function can be small in the L_1 norm and yet be very large in the sup norm. Here is one such:



This function has sup norm equal to n, and yet has integral 1/(2n). So the sup norm approaches infinity while the L_1 norm approaches zero. Thus convergence is different in these two norms.

Thus it is harder for functions to approximate other functions in the sup norm, which means that the limit function will retain more properties of the approximating functions. This suits us since we are interested in approximating very good functions like sin and e^x, which have the same good properties of continuity and differentiability as the approximating functions we will use, the polynomials. (If on the other hand we wanted to define the notion of integral for functions with lots of discontinuities, we would use a norm like the integral norm which allows very continuous functions to approximate very discontinuous ones.)

Definition: A sequence $\{s_n\}$ in S, (where S is one of our three sets equipped with the appropriate distance), converges to an element s_∞ of S, if and only if, for every e>0, there exists a positive integer N, such that whenever $n \geq N$, then $|s_n-s_\infty| < e$.

Remark: This means that no matter how small an e- neighborhood we describe around our limit point s_∞, after a certain element s_N, all the rest of the sequence lies in that neighborhood. In particular if a sequence converges to s_∞, and we form a new sequence by throwing away the first billion elements of our old sequence, the new sequence also converges to s_∞. Thus whether or not a sequence converges, and what the limit is, is unaffected by any given finite number of elements of the sequence.

In particular, if a sequence converges to s_∞, then the new sequence formed by adding in a billion or so 1's at the beginning of the sequence, still converges to the same limit. Thus there is no reason to expect to able to guess the limit of a sequence just by looking at the first hundred trillion elements or so.

Remark: Because all our notions of length satisfy the triangle inequality, it follows that the sum of two convergent sequences converges to the sum of the limits, and homogeneity implies that multiplying the elements of a sequence by a constant multiplies the limit by that constant. Non degeneracy implies that the limit of a sequence is unique, i.e. the same sequence cannot converge to two different limits. Of course these are intuitive properties we might expect. And they are indeed true.

Lemma: A convergent sequence is bounded. I.e. if $\{s_n\}$ converges, then there is some positive number K such that for all n, $|s_n| \leq K$.

proof: By definition of convergence, if $\{s_n\}$ converges to s, then given say e = 1, there is an N such that all elements after s_N are within a distance 1 of s, so that for all $n \geq N$, we have $|s_n| \leq |s| + 1$. Hence if we let K be the maximum of the numbers $|s_1|, |s_2|, \ldots, |s_{N-1}|, |s_N|+1$, then for all n, we have $|s_n| \leq K$. **QED.**

Remark: The converse does not hold, since the sequence
1,-1,1,-1,1,-1,.... is bounded but not convergent.

There is a class of bounded which does always converge, namely bounded and monotone sequences. These converge almost by definition of the real numbers. To see it, recall the following axiom, which is satisfied by the real numbers, either by assumption, or by proving directly that it holds for the set of infinite decimals.

Completeness axiom: A non empty set of real numbers which has an upper bound has a least upper bound. I.e. if some number is $\geq$ than all numbers in the non empty set S, then there is some <u>smallest</u> number which is still $\geq$ all numbers in S.

Corollary: A weakly monotone sequence converges if and only if it is bounded.

Proof: We know a convergent sequence is bounded. Conversely, if the sequence is both bounded and monotone, say monotone increasing, then let K be the least upper bound of the sequence. I.e. let K be the smallest number such that for all n, we have $s_n \leq K$. Then we claim the sequence $\{s_n\}$ converges to K. Let e>0 be given. Then since K is the smallest number which is $\geq$ all elements of the sequence, the number K-e must be less than some element of the sequence. Suppose $s_N > K-e$. Then for all $n \geq N$, we have $sN \leq s_n$, by monotonicity. But since K is an upper bound of the entire sequence we also have $K-e < s_n \leq K < K+e$, for all $n \geq N$. Hence the sequence $\{s_n\}$ converges to K. **QED.**

Remark: Notice this gives a way to tell a sequence is convergent without finding the limit. Just find any upper bound for a weakly increasing sequence and you know the sequence converges even if you cannot determine what is the least upper bound, i.e. the limit.

Example: The sequence $\{1/n\}$ for n a positive integer, thus converges to something, since it is monotone decreasing and bounded below by 0. But to see it actually converges to 0, we must show that 0 is the greatest lower bound of the sequence. If there were a positive number e>0 smaller than all fractions 1/n, then the reciprocal 1/e would be larger than all the positive integers n. So we can show that 1/n converges to 0, by showing equivalently that the sequence $\{n\}$ is unbounded above.

This is obvious if you believe that the real numbers are represented by infinite decimals, as we assumed in this course. If you are in a more axiomatic course where all you know is that the reals obey the completeness axiom above, then you can prove this as follows: if the integers <u>are</u> bounded above, then they have a least upper bound say K. Then K-1 is not an upper bound so there is a positive integer N such that N>K-1. But then N+1 > K, contradicts the assumption that K is an upper bound of all integers. **QED.**

This idea can be generalized as follows.
Definition: A sequence $\{s_n\}$ is called Cauchy, if and only if for every $e>0$, there is some N, such that, for all $n,m \geq N$, we have $|s_n - s_m| < e$.

Exercise: Any convergent sequence is Cauchy.

Remark: In our three examples, the converse is true, i.e. in the real numbers, in the plane, and in the space of continuous functions on [a,b] with the sup norm, any Cauchy sequence converges to an element of the same space.

Intuitively to say a sequence is Cauchy, means the elements of the sequence are bunching up, but they might not converge unless there actually is a point of our space at the place where they are bunching. For example, if our space were the real numbers, except for zero, then the sequence $\{1/n\}$ would still be Cauchy, but would not converge in our space simply because we had removed the previous limit. Since lots of sequences of rational numbers have irrational limits, Cauchy sequences of rationals would not always converge in the space of rationals. E.g. the sequence 3, 3.1, 3.14, 3.141, 3.1415, 1.14159,.... of rationals, which converges to π, (if the decimals are chosen appropriately), would be Cauchy in the rationals, but would not converge in the space of rationals. I.e. some space have “holes” in them, and a sequence could head towards a hole in the space, and be Cauchy, but not have a limit in the space, just because the limit is missing from the space.

None of our examples have holes in them, as stated by the next theorem.

Theorem: In all three of our examples, the real numbers, in the plane, and in the space of continuous functions on [a,b] with the sup norm, every Cauchy sequence $\{s_i\}$ converges to some limit in the given space.
proof: This is a real theorem, on the level of an analysis course so we will only sketch the proof. You may take this theorem for granted.
Example (i) For those interested, we do the case of real numbers first: define for each n, a_n = the greatest lower bound of the elements s_i in the sequence such that $i \geq n$. Define b_n = least upper bound of those elements s_i with $i \geq n$. Then $\{a_n\}$ is a weakly increasing sequence and $\{b_n\}$ is a weakly decreasing sequence, since the Cauchyness of the sequence $\{s_i\}$ implies that $|a_n-b_n|$ converges to zero. Thus both sequences $\{a_n\}$ and $\{b_n\}$ converge by the previous corollary, in fact to the same limit K, which is also the limit of the sequence $\{s_i\}$.

Example (ii) For a Cauchy sequence of points $\{p_n\} = \{(x_n,y_n)\}$, in the plane, one can check that both sequences $\{x_n\}$ and $\{y_n\}$ are also Cauchy sequences of real numbers, since $|p_n| \geq$ both $|x_n|$, $|y_n|$. Hence $\{x_n\}$ converges to some x, and $\{y_n\}$ converges to some y, and then $\{p_n\}$ converges to (x,y).

Example (iii) If $\{f_n\}$ is a Cauchy sequence of functions on [a,b], then for each x in [a,b], the definition of the sup norm, forces the sequence of real numbers $\{f_n(x)\}$ to be Cauchy, hence convergent to some number we call $f(x)$. This defines a function f, which we claim is continuous, and is the limit of the sequence $\{f_n\}$.

To see this, let $e>0$ be given. We must find N such that for all $n \geq N$, we have $\|f-f_n| < e$. But we know the sequence $\{f_n\}$ is Cauchy in the sup norm, so for some N, we have $\|f_n-f_m\| < e/3$ for all $n,m \geq N$. Since for all x, $f(x)$ is the limit of the $f_n(x)$, it follows that for all x and all $n \geq N$, we have $|f(x)-f_n(x))| \leq 2e/3$. I.e. given x, there is some $m > N$ such that $|f_m(x)-f(x)| < e/3$. Since for all $n \geq N$, we have $|f_n(x)-f_m(x)| < e/3$, it follows that for all $n \geq N$, $|f(x)-f_n(x)| \leq |f(x)-f_m(x)|+|f_m(x)-f_n(x)| < 2e/3$. Thus for all x, and all $n \geq N$, we have $|f(x)-f_n(x)| < e$. I.e. $\{f_n\}$ converges to f in the sup norm.

Finally we claim the limit function f is continuous on [a,b], hence lies in the space we are working in. To prove this, let z be any point of [a,b]. To show f is continuous there, let $e>0$ be given and try to find $d>0$ such that for all x closer to z than d, we have $|f(x)-f(z)| < e$. This is a classic e/3 proof. I.e. choose N such that for all $n,m \geq N$, we have $\|f_n-f_m\| < e/3$. Then we saw above that also for all $n \geq N$, we have $\|f(x)-f_n(x)\| < e/3$. Now f_N is continuous by hypothesis, so there is a $d>0$ such that for all z closer to x than d, we have $|f_N(z)-f_N(x)|<e/3$. Then just note that
$|f(z)-f(x)| = |f(z)-f_N(z)+f_N(z)-f_N(x)+f_N(x)-f(x)|$

$\leq |f(z)-f_N(z)| + |f_N(z)-f_N(x)| + |f_N(x)-f(x)| < e/3 + e/3 + e/3 = e$.

I.e. $|f(z)-f_N(z)| < e/3$ because f_N is closer than e/3 to f at every point of [a,b]. And $|f_N(x)-f(x)| < e/3$ for the same reason. Then $|f_N(z)-f_N(x)| < e/3$ because f_N is continuous at x, and d was chosen to make this true for f_N since $|z-x| < d$. **QED.**

Remark: Of course this is fairly easy for me now at my age, but I actually gave the e/3 proof of part (iii) above, on a differential equations exam sophomore year in college, to impress my teacher, who had omitted it from the course lectures. (Teachers notice it when you answer more than 100% of what they ask.)

Exercise: (i) If a sequence of functions $\{f_n\}$ converges to f in the sup norm on [a,b], then the integrals also converge, i.e. the sequence of real numbers $\{\int_a^b f_n\}$ converges to the real number $\int_a^b f$.

(ii) In fact the indefinite integrals $G_n = \int_a^x f_n$, which are functions on [a,b], also converge to the function $G = \int_a^x f$, in the sup norm.

Infinite series

Next we want to discuss “infinite sums” i.e. “convergent series”.
Let $\{a_n\}$ be any infinite sequence, and form another sequence of “partial sums” of the original sequence: $s_1 = a_1$, $s_2 = a_1+a_2$, $s_3 = a_1+a_2+a_3$,...,
$s_n= a_1+a_2+...+a_n$,......

Definition: Then we say "the series $\sum_{i=1}^{\infty} a_i$ converges to the limit a_∞", or that "$\sum_{i=1}^{\infty} a_i = a_\infty$", or that "$a_\infty$ is the sum of the series $\sum_{i=1}^{\infty} a_i$", if and only if the sequence $\{s_n\}$ of partial sums converges to a_∞. I.e. if and only if, for every e>0, there is a positive integer N, such that, for all $n \geq N$, we have
$|\sum_{i=1}^{n} a_i - a_\infty| < e$.

The analog of the result about monotone sequences converging says in this setting:

Theorem: If $\{a_n\}$ is any sequence of non negative numbers, the series $\sum_{i=1}^{\infty} a_i$ converges if and only if the partial sums are bounded, i.e. if and only if there is some number K such that for all n, $\sum_{i=1}^{n} a_i \leq K$.

Corollary: If $0<r<1$, and a is any positive real number, then the series $\sum_{n=1}^{\infty} ar^n$ converges.
proof: We need only show that the partial sums are bounded above. But simple multiplication shows that $\sum_{i=1}^{n} a_i = [a - ar^{n+1}]/(1-r)$. Since the denominator is positive, and $0<r<1$, the numerator is less than a, which shows the partial sums are bounded above by $a/(1-r)$. Thus convergence follows.

Remark: We know the series actually converges to the least upper bound of its partial sums, which one can show is also equal to $a/(1-r)$. But we will prove a stronger result, as done in class.

Theorem: If a is any real number and r is any real number with $|r| < 1$, then the series $\sum_{n=1}^{\infty} ar^n$ converges to $a/(1-r)$.
proof: By the proof of the previous result, $|a/(1-r) - \sum_{i=1}^{n} a_i| =$ $|ar^{n+1}/(1-r)|$, so it suffices to show this goes to zero as n goes to infinity. It suffices to show the numerator goes to zero, hence that $|r^n| = |r|^n$ goes to zero, hence that $r^n \to 0$ when $0<r<1$. Equivalently it suffices to show the reciprocal goes to infinity as n goes to infinity, so assume that $s>1$, and then we claim that $s^n \to$ infinity as n goes to infinity. Choose $h>0$ so that $s > 1+h > 1$. Then it suffices to show that $(1+h)^n$ goes to infinity, but by the binomial theorem, $(1+h)^n > 1+nh$, which goes to infinity, because n does, as we proved above. **QED.**

The previous example is the famous "geometric series." It is very useful to have even one example of a convergent series because of the following.

Theorem: If $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ are two series of non negative real numbers, and if $a_i \leq b_i$ for all i, then the convergence of $\sum_{i=1}^{\infty} b_i$ implies the convergence of $\sum_{i=1}^{\infty} a_i$, and hence the non convergence of $\sum_{i=1}^{\infty} a_i$ implies the non convergence of $\sum_{i=1}^{\infty} b_i$.

proof: This follows from an earlier result (two theorems ago) because when the partial sums of one positive series are bounded, so are those of a smaller positive series. **QED.**

Next we state a version of this for our other examples.

Theorem: If $\sum_{i=1}^{\infty} s_i$ is any series of elements from any of our three examples, and if the series of real numbers $\sum_{i=1}^{\infty} |s_i|$ converges, then the original series $\sum_{i=1}^{\infty} s_i$ also converges.

proof: It suffices by the big theorem above, to show that the sequence of partial sums $\{\sum_{i=1}^{n} s_i\}$ is Cauchy. I.e. given e>0 we must show that there is an N such that for $n>m \geq N$, we have $|\sum_{i=m}^{n} s_i| <$ e. But $|\sum_{i=m}^{n} s_i| \leq \sum_{i=m}^{n} |s_i|$, and since the series $\sum_{i=1}^{\infty} |s_i|$ converges, the sequence of partial sums $\{\sum_{i=1}^{n} |s_i|\}$ is Cauchy. Thus we can find N such that for $n,m \geq N$ we have $\sum_{i=m}^{n} |s_i| <$ e. **QED.**

Finally this leads to the famous Weierstrass "M - test" which for many purposes reduces the sup norm convergence of functions to the convergence of positive numbers.

Corollary: If $\{f_n\}$ is a sequence of continuous functions on [a,b], and if there exists a convergent series of positive numbers $\{M_n\}$ such that for all x in [a,b] we have $|f_n(x)| \leq M_n$, then the series $\sum_{i=1}^{\infty} f_n$ converges in the sup norm to a continuous limit function $f = \sum_{i=1}^{\infty} f_n$.

Approximation of transcendental functions by polynomials Finally let's apply these results to study series of polynomials converging to some of our favorite transcendental functions: ln, arctan, sin, e^x.

Claim: e^x: The series $1 + x + x^2/2! + x^3/3! + x^4/4! + \ldots\ldots$ converges to e^x, in the sup norm on any closed bounded interval.

sin(x): The series $x - x^3/3! + x^5/5! - x^7/7! + - \ldots\ldots\ldots$ converges to sin(x), in the sup norm on any closed bounded interval.

ln(1+x): The series $x - x^2/2 + x^3/3 - x^4/4 + x^5/5 - x^6/6 \pm$.......... converges to ln(1+x), in the sup norm on any closed interval strictly contained in the interval (-1,1).

arctan(x): The series $x - x^3/3 + x^5/5 - x^7/7 \pm$...... converges to arctan(x), in the sup norm on any closed interval strictly contained in the interval (-1,1).

To do this we just use the M test to show convergence, then prove that the derivative series also converge term by term to the derivative of the sum, then use uniqueness of these derivatives to equate the series with the known functions.

We can do two of these examples just by integrating term by term, as justified in our exercises above.

Example: ln(1+x):
Consider the formal geometric series expansion of 1/(1+x)

$= 1 - x + x^2 - x^3 + x^4 - +$ ....

In any interval [-r,r] where 0<r<1, the norms of these functions are bounded above by the terms of the series $1+r + r^2 + r^3 +$.... which we know is convergent. Hence also $1 - x + x^2 - x^3 + x^4 - +$ .... converges to some continuous function g(x). But since the partial sum

$1 - x + x^2 - x^3 + ...+(-1)^n x^n = [1/(1+x) - (-1)^n x^n/(1+x)]$, it follows from the fact that r^n-->0, when 0<r<1, as n --> infinity, that the limit of this series is 1/(1+x). Consequently, on any interval [-r,r] with 0<r<1, the series of indefinite integrals (starting at 0) also converges to the indefinite integral of the limit.

So the series $x - x^2/2 + x^3/3 - x^4/4 + x^5/5 - x^6/6 \pm$....... converges also in the sup norm on [-r,r], to $\int_0^x \frac{dt}{1+t} = \ln(1+x)$.

Example: arctan(x): We proceed exactly as above, starting from the series $1 - x^2 + x^4 - x^6 + -$ ....., to show this geometric series converges to $1/(1+x^2)$, and hence that the series of indefinite integrals, starting from 0, namely the series $x - x^3/3 + x^5/5 - x^7/7 \pm$...... converges to $\int_0^x \frac{dt}{1+t^2} =$ arctan(x), in the sup norm on any closed interval strictly contained in the interval (-1,1).

We do the series for e^x almost the same way.
Example: e^x: Consider the series $1 + x + x^2/2! + x^3/3! + x^4/4! +$ ..... We claim this converges on any interval [-r,r] at all. To see this, note that on this interval, the term $x^n/n!$ is bounded in absolute value by $r^n/n!$, so it suffices to show the series $1+r + r^2/2 + r^3/3! +$..... is bounded

above. Just choose $N > r$. Note that certainly the first N terms are bounded since there are only a finite number of them, so it suffices to show the rest of the terms are bounded. So we want to show the series
$r^N/N! + r^{N+1}/(N+1)! + ...$ is bounded. First factor out $r^N/N!$ from all the terms, which leaves
$1 + r/(N+1) + r^2/(N+1)(N+2) + ...$

Now note that this series is smaller than the geometric series
$1 + r/N + r^2/N^2 + r^3/N^3 + ...$ which converges because $0 < (r/N) < 1$.

Hence the original series $1 + x + x^2/2! + x^3/3! + x^4/4! + ...$ converges to some continuous function f, in the sup norm on any interval of form $[-r,r]$ at all. Hence the series of indefinite integrals, starting at 0, which in fact is just the same series, converges to the indefinite integral of f. Since the series equals its own indefinite integral series, the limits are also the same, so $f(x) = \int_0^x f(t)dt$, for all x. But the right side has derivative f(x), so the left side does too, i.e. $f'(x) = f(x)$, for all x. Since also $f(0) = 1$, by looking at the series, we conclude again that $f(x) = e^x$ for all x, and that the convergence is true in the sup norm on any closed bounded interval.

We may already know that any differentiable function f with f'=f and f(0) = 1, is an exponential function, but I recall the proof. Let $g(x) = f(a+x)/f(x)$, and differentiate. Then $g'(x) = [f'(a+x).f(x)-f(a+x).f'(x)]/f^2(x) = [f(a+x).f(x)-f(a+x).f(x)]/f^2(x) = 0$. Thus $g(x) = K$ for some constant K, i.e. so $f(a+x) = K.f(x)$, for some constant K, and since $f(0) = 1$, $K = f(a)$. Thus $f(a+x) = f(a).f(x)$, and f is thus an exponential function.

Technically we have assumed that $f(x) \neq 0$ for this proof, but it is clear that the proof works on any interval where this holds. Since $f'(x) = f(x)$ and $f(0) = 1 > 0$, it follows that f is increasing, hence positive, on R+. Then the formula $f(a+x) = f(a).f(x)$ is true for all $x > 0$. Hence for all $a < 0$, we have $f(1/a) > 0$ and $f(a) = 1/(f(1/a)) > 0$ as well. So f is always positive.

Now to do the series for sin(x), I am a little more challenged as to how to do it. We could of course start from the series $x - x^3/3! + x^5/5! - x^7/7! + - ....$, and show as above that it converges to some function f, on any closed bounded interval, and that the series $x^2/2! - x^4/4! + x^6/6! - + ...$ of indefinite integrals also converges to some function. Hence the series $1 - x^2/2! + x^4/4! - x^6/6! + ...$ also converges to some function g such that $g' = -f$. Integrating again gives a series converging to some function h with $h'' = -f$. However, looking at this last series shows that it again equals $x - x^3/3! + x^5/5! - x^7/7! + - ....$ I.e. that $h = f$. Hence $f'' = -f$. Similarly one sees that $g'' = -g$. Moreover $g(0) = 1$ and $f(0) = 0$.

Now the only problem is to show that these properties can only hold for f = sin and g = cos.

I borrow a proof from a book by Serge Lang.
Lemma: If $f' = g$, $g' = -f$, and $f(0) = 0$, $g(0) = 1$, then $f^2+g^2 = 1$.
proof: Differentiate the lhs. and then set $x = 0$. **QED.**

Now differentiate f.cos –g.sin, and get f'.cos –f.sin -g'.sin –g.cos = 0, hence f.cos –g.sin = a is constant. Similarly f.sin + g.cos = b.

In the meantime, I supply another result, which we have used to treat the last example.

Differentiation of series term by term

We would like to have a criterion telling us when we may differentiate a convergent series term by term, and get a series converging to the derivative of the limit of the original series. The following one is easy to prove and suffices for our purposes.

Theorem: Suppose $\{f_n\}$ is a sequence of C^1 functions on [a,b] (i.e. continuous functions with continuous derivatives), such that both series of norms $\sum_{n=1}^{\infty} ||f_n||$, and $\sum_{n=1}^{\infty} ||f'_n||$ are bounded. Then $\sum_{n=1}^{\infty} f_n(x)$ = f, where f is a continuously differentiable function, and f' = $\sum_{n=1}^{\infty} f'_n(x)$, is the sum of the derivatives of the f_n, (where both series converge in the sup norm).

Remark: For example, by the Weierstrass M test, if we can find convergent series of positive numbers $\sum_{n=1}^{\infty} M_n$, and $\sum_{n=1}^{\infty} K_n$, such that

for all n, and all x in [a,b], we have $|f_n(x)| \le M_n$, and $|f'_n(x)| \le K_n$, then both series of functions converge and the conclusion of the theorem holds.

Proof of theorem: If the hypothesis holds, then by our previous results, the series $\sum_{n=1}^{\infty} f_n$ converges to some continuous limit function f, and also the series of derivatives $\sum_{n=1}^{\infty} f'_n$ converges to some continuous limit function g. We claim f is differentiable and f' = g.

By our theorem on convergence of integrals we know that the series of indefinite integrals $\sum_{n=1}^{\infty} (\int_a^x f'_n)$ converges to the indefinite integral $\int_a^x g$. By the FTC, the indefinite integral $\int_a^x f'_n = f_n(x) - f_n(a)$. Moreover the series $\sum_{n=1}^{\infty} (f_n(x) - f_n(a))$ converges to $\sum_{n=1}^{\infty} f_n(x) - \sum_{n=1}^{\infty} f_n(a)$ = f(x)-f(a).

Since the series $\sum_{n=1}^{\infty} (\int_a^x f'_n)$ and $\sum_{n=1}^{\infty} (f_n(x) - f_n(a))$ are equal, their limits are also equal, so f(x)-f(a) = $\int_a^x g$. Thus also f(x) = $\int_a^x g$ + f(a). Now since the rhs is differentiable by the FTC with derivative g, the same is true of the lhs. Hence f'(x) = g(x) = $\sum_{n=1}^{\infty} f'_n(x)$, as claimed. **QED.**