2210 Exponential and log functions

Exponential functions, even simple ones like 10^x, or 2^x, are relatively difficult to describe and to calculate because they involve taking high roots of integers, and we do not know much even about computing square roots, much less cube roots or fifth roots, or 29th roots, etc. Lets review the "standard" description of exponential functions, and then see the additional problems involved in trying to compute their derivatives. Lets start with an easy base like 2.

Positive Integer exponents

We want to define 2^x for all real numbers as a continuous function. We start by saying that $2^1 = 2$, and $2^n =$ a product of n factors of 2, for any positive integer n. I.e. $2^2 = 2(2)$, $2^3 = 2(2)(2)$, $2^4 = 2(2)(2)(2)$, and so on.

Negative integer exponents

But what next? How do we define negative powers of 2? or 2^0? Notice a very important property of the exponential function, it satisfies $2^{(n+m)} = 2^n 2^m$ for all positive integers n,m. I.e. to get a product of n+m factors of 2, just multiply a product with n factors by a product with m factors. Altogether, there will be n+m factors of 2. This is such a useful property that we would like it to continue to hold for all values of the exponential function. But that demand limits how we can define the exponential function very much. I.e. if we want to have $2^m = 2^{(0+m)} = 2^0 2^m$, then we must have $2^0 = 1$. And then if we want to have $1 = 2^0 = 2^{(n+(-n))} = 2^n 2^{-n}$, then we must have $2^{-n} = 1/2^n$. Thus we have no choice about how to define negative and zero powers of 2.

Fractional exponents

What about rational powers? If we want to have $2 = 2^1 = 2^{(1/2 + 1/2)} = (2^{1/2})(2^{1/2})$, then $(2^{1/2})$ must be a number which gives 2 when multiplied by itself, i.e. we must have $(2^{1/2}) =$ sqrt(2). Similarly, we must have $2^{1/3} =$ cuberoot(2), and $2^{1/n} =$ nth root(2). For $2^{m/n}$ we must have $2^{m/n} =$

$2^{(1/n + 1/n +....+1/n)}$ (m terms) =

$(2^{1/n})(2^{1/n})(.........)(2^{1/n})$ (m factors) =

[nth root(2)]^m = nth root(2^m).

As before than we must have $2^{-(n/m)} = 1/(2^{n/m}) =$

1/[nth root(2^m)]. Thus we are forced to make this definition of a rational power of 2, just by the definition for positive integer powers, plus the basic law $2^{(x+y)} = 2^x\, 2^y$. This completely determines the exponential function on all rational numbers.

Irrational exponents and continuity

Then what about irrational numbers? This extension uses continuity, and a complete proof would take more care and time than we wish to devote to it. But we can state it easily as follows. Note that 2^x is an increasing function on rational numbers, since for every positive n and m, $2^{m/n}$ = [nth root(2m)] is greater than one. Hence for any rational numbers r < s, we have

(s-r)= a positive rational number, so $2^s = 2^{r+(s-r)} = 2^r\, 2^{s-r}$ where 2^{s-r} is greater than 1. Since we get 2^s from 2^r by multiplying 2^r by a number greater than 1, $2^s > 2^r$, i.e. 2^x is an increasing function. Then we extend it to irrational values just by keeping it increasing. I.e. define for any irrational number x, 2^x to be the smallest real number not smaller than 2^r for any rational number $r < x$. Put another way, choose an infinite decimal expansion for x. Then for each n, taking an approximation by only using the first n digits, gives us a sequence of approximations to x from below, by rational numbers. If we exponentiate each of these rational numbers, we get a bounded increasing sequence of real numbers which therefore have a limit, and we call this limit 2^x.
One can prove with some work, that with this definition, 2^x is a continuous increasing function, defined for all real numbers, and that it still satisfies the relation $2^{(x+y)} = 2^x\, 2^y$ for all real numbers x,y.

Differentiating exponential functions

We see that it is not a trivial matter even to evaluate an exponential function at a rational number, since we must extract a root, and often a rather high order one. What about computing the derivative? Is the function 2^x differentiable? If it is what is the derivative? Suppose we start from the definition, the only place to ever start such an investigation. Then 2^x has a derivative if the limit $(2^{x+h} - 2^x)/h$ exists as h approaches 0. We can simplify this using the law $2^{(x+h)} = 2^x\, 2^h$. Thus we ask whether $2^x\, (2^h-1)/h$ has a limit as h-->0. Since 2^x is constant in h, this is true if and only if $(2^h-1)/h$ has a limit, i.e. since $1 = 2^0$, 2^x has a derivative at x, if and only if it has a derivative at 0.

It is not easy to see whether or not the limit $(2^h-1)/h$ exists as h-->0, and if it does exist, to see what it is equal to. Lets assume that it does exist, and call the limit K. Then from the calculation above, it follows that the derivative of 2^x equals $K(2^x)$, i.e. the derivative of 2^x, if it exists, is a constant multiple of 2^x. Even if we assume this limit exists, we need some way to calculate this constant.

We take an indirect approach below. We will work backwards, by finding a function we know to be differentiable, which we then show equals 2^x. For this we must have a way to recognize the exponential function. This follows from the same reasoning used above to predict the values of an exponential function from a small number of them.

Theorem: If f(x) is a continuous function defined on all reals such that
1) f(0) = 1,
2) f(x+y) = f(x) f(y), for all reals x,y,
then $f(1) = a > 0$, and $f(x) = a^x$ for all real x.
Proof: Since f(1) = f(1/2 + 1/2) = f(1/2) f(1/2), f(1) = a is a square hence non negative. Since 1 = f(0) = f(1+ (-1)) = f(1) f(-1), f(1) cannot be 0, so $f(1) = a > 0$. Reasoning as above we see that $f(r) = a^r$ for all rational r. Then we conclude that $f(x) = a^x$ for all real x, by continuity as above.
QED.

Corollary: Since a differentiable function is also continuous, if we can find a differentiable function f(x) satisfying f(0) = 1, and f(x+y) = f(x) f(y), for all reals x,y, then f must be an exponential function. If we can find one with f(1) = 2, then $f(x) = 2^x$, and we will have proven that 2^x is differentiable.

Logarithms

First we discuss the inverse function of an exponential function, the so called logarithm function. It follows from arguments like those above that an exponential function a^x with a >0, is increasing if and only if a > 1, and is decreasing if a < 1. The exponential function 1^x is neither increasing nor decreasing, but a constant equal to 1, and has no inverse. However for all a > 0 and a ≠ 1, the function a^x has an inverse function called the "log base a", written $\log_a(x)$. To find the domain of the log function we must determine the range of values of the exponential function, so we assume for simplicity that a > 1. Then notice that if say a = 1+h where h > 0, then for all positive integers n, we have $a^n = (1+h)^n = 1 + nh + \ldots$.., with all terms positive, so $a^n > 1+nh$. Since the right hand side grows to infinity as n does, we conclude that a^x gets arbitrarily large for large x, i.e. the limit of a^x is $+\infty$ as x approaches $+\infty$. Since $a^{-x} = 1/a^x$, we see that a^x approaches zero from above as x approaches $-\infty$. Thus a^x assumes all positive values as x ranges over all reals, so the domain of the inverse function $\log_a(x)$ is the positive reals. Moreover $\log_a$ is also continuous and satisfies the law $\log_a(xy) = \log_a(x) + \log_a(y)$ for all positive reals x,y, opposite to the law for the exponential function. Since a function determines its inverse, we have the analogous theorem to recognize a log function.

Theorem: If g is a continuous function defined for all positive reals, satisfying
1) g(1)= 0, and g(b) ≠ 0 for some b > 0,
2) g(xy) = g(x) + g(y), for all positive x,y,
then there is a unique a > 0 with g(a) = 1, and $g(x) = \log_a(x)$, for all positive x.

[**Note:** the function g(x) = 0 for all x > 0. satisfies hypotheses 1), 2), except the second part of 1), and does not satisfy the conclusion.]

Now all we have to do is find a differentiable function g satisfying the conditions in the theorem, and then it must be a log function. If we find one with g(2) = 1, it will prove that $\log_2(x)$ is differentiable. Moreover by the inverse function theorem it will follow that 2^x is also differentiable, and we will have accomplished by a very indirect route, the goal of proving this fact, which we temporarily gave up on earlier.

The only tool we have for constructing differentiable functions is the fundamental theorem of calculus, which allows us to construct a function with any given continuous derivative. Thus in order to construct a log function we need to know the derivative of a log function. Using the chain rule we have $a^{\log_a(x)} = x$, so $Ka^{\log_a(x)} \log_a'(x) = 1$, so $\log_a'(x) = 1/K(a^{\log_a(x)}) = 1/Kx$. Thus assuming it is differentiable, a log function must have derivative equal to 1/Kx, for some K≠0. Now with this information, we can construct a differentiable function with derivative 1/Kx, and ask whether it is a log function, which we have every right to expect to be true. Moreover the simplest choice for K is obviously 1, so we begin from that choice.

Define $L(x) = \int_1^x \frac{dt}{t}$. We claim L(x) is a log function. To check this we must show that L(1) = 0, but L is not everywhere zero (which is "obvious"), then that L is continuous, which is always true for a function defined by an integral, indeed by the FTC this one is even differentiable with derivative 1/x, and finally that L(xy) = L(x) + L(y) for all x,y > 0. To prove this last formula we use the MVT. I.e. fix y > 0 and let g(x) = L(xy). Then g'(x) = yL'(xy) = y(1/xy) = 1/x = L'(x). Thus L and g have the same derivative so must differ by a constant according to the MVT. I.e. g(x) = L(xy) = L(x) + c for some c. To evaluate c, set x = 1 and get L(y) = L(1) + c = c. So c = L(y) and thus L(xy) = L(x) + L(y) as claimed.

To see what the base is we must find the unique positive number a such that L(a) = 1. This is not so immediate, but one can show using approximations of the integral that the base is a number we shall call e such that 2.71828 < e < 2.71829. Indeed since the midpoint estimate is an underestimate for a function like 1/x which is concave up, using the midpoint estimate on the interval [1,3] we get L(3) ≥ 2(1/2) = 1, so e ≤ 3. Subdividing into [1,2] and [2,2.8] we have midpoint estimate 2/3 + 5/12 = 13/12 ≤ L(2.8), so e < 2.8. Then using the trapezoidal upper estimate for the subdivision [1,1.4], [1.4,1.8] and [1.8,2.2], [2.2,2.6] gives
(1/5)(1 + 10/7 + 10/9 +10/11 + 5/13)
≤ (1/5) (1 + 1.43 + 1.112 + .9091 + .385) ≤ .97.

Thus e > 2.6. Later we will find a better way to estimate e using infinite series.

We see that $L(x) = \log_e(x)$, but it is usual to write it as ln(x). In particular ln(x) is a differentiable function defined on all positive reals, with derivative 1/x, and which takes on all real values. Its inverse is the exponential function e^x whose derivative is also e^x. If K = ln(2) and if we consider the function $g(x) = \int_1^x \frac{dt}{Kt}$, it follws from the same argument as above using MVT, that g is a logarithm function. The base is the unique number a such that g(a) = 1. But since g(x) = (1/K)ln(x) = ln(x)/ln(2), it follows that the number a with g(a) = 1, is a = 2. Hence g(x) = $\log_2(x)$, and we see the derivative of g is 1/(ln(2)x). Using the inverse function theorem, the inverse of g is the differentiable function $f(x) = 2^x$ with derivative $f'(x) = \ln(2)\, 2^x$.

We obtain as well that for all a > 0, the function $\int_1^x \frac{dt}{Kt}$ where K = ln(a), is the differentiable log function $\log_a(x)$. Its inverse is then the differentiable function a^x, with derivative $\ln(a)(a^x)$. In fact, for all a > 0, we can express a^x in terms of e^x, by showing that $a^x = e^{(x \ln(a))}$. To check this we only need show that $e^{(x \ln(a))}$ satisfies the properties that characterize a^x. I.e. $e^{(x \ln(a))}$ is continuous in x, and equals 1 when x = 0, and equals a when x = 1. Thus $e^{(x \ln(a))} = (e^{\ln(a)})^x = a^x$. More generally we have the following theorem.

Theorem: For any real numbers x,y, and a > 0, we have $a^{(xy)} = (a^x)^y$.
Proof: Fix x and consider the function $a^{(xy)}$. It has the value 1 when y = 0, and equals a^x when

$y=1$. Moreover it is continuous in y. Then for all y,z, we have $a^{x(y+z)} = a^{(xy+xz)} = a^{xy} a^{xz}$. Thus a^{xy} does satisfy the properties which characterize the function $(a^x)^y$. **QED.**

Now that we have the definition and the basic properties of exponential and log functions, the other basic properties are their rates of growth. According to the formulas above we can express every exponential function in terms of the simplest one e^x, so we may consider only that one. Similarly we saw above for all $a > 0$, that $\log_a(x) = \ln(x)/\ln(a)$. Basically e^x grows rapidly, faster than any positive power of x, and ln(x) grows slowly, slower than any positive power of x. More precisely, for all $r > 0$, we have $\lim_{x \to \infty} x^r/e^x = 0$, and $\lim_{x \to \infty} \ln(x)/x^r = 0$.
There are several ways to prove this, but the easiest way, and the way which teaches us the most useful fact, is to use

l'Hopital's rule:
Theorem: If f, g are both differentiable functions such that both have limit $\pm\infty$, or if both have limit 0, as x approaches a, or as x approaches ∞, then the limit of f/g equals the limit of f'/g', assuming that latter limit exists.

Using this theorem on x^r/e^x, we can take derivatives until the power of x in the top is non positive, so that then the function in the top is constant or approaches zero as x approaches infinity, and the one in the bottom, still equal to e^x, approaches infinity. Then the limit is $1/\infty$, or $0/\infty$, both = 0. For the limit $\ln(x)/x^r$, one can begin with $\ln(x)/x$ and take one derivative in top and bottom to finish. Then for the general case, one may consider instead the equivalent limit $\ln(x^{1/r})/x = (1/r)\ln(x)/x$, which we have already done. (The only point is that the bottom number should be the rth power of the top number, and that both approach infinity, it does not matter which one we call x.)