If some determined value of the cube root of A is denoted by $\sqrt[3]{A}$, the three possible values of u will be

$$u = \sqrt[3]{A}, \qquad u = \omega\sqrt[3]{A}, \qquad u = \omega^2\sqrt[3]{A}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2}$$

is an imaginary cube root of unity. As to v it will have also three values:

$$v = \sqrt[3]{B}, \qquad v = \omega\sqrt[3]{B}, \qquad v = \omega^2\sqrt[3]{B}$$

but not every one of them can be associated with the three possible values of u, since u and v must satisfy the relation

$$uv = -\frac{p}{3}.$$

If $\sqrt[3]{B}$ stands for that cube root of B which satisfies the relation

$$\sqrt[3]{A} \cdot \sqrt[3]{B} = -\frac{p}{3},$$

then, the values of v that can be associated with

$$u = \sqrt[3]{A}, \qquad u = \omega\sqrt[3]{A}, \qquad u = \omega^2\sqrt[3]{A}$$

will be

$$v = \sqrt[3]{B}, \qquad v = \omega^2\sqrt[3]{B}, \qquad v = \omega\sqrt[3]{B}.$$

Hence, equation (1) will have the following roots:

$$\begin{aligned} y_1 &= \sqrt[3]{A} + \sqrt[3]{B}, \\ y_2 &= \omega\sqrt[3]{A} + \omega^2\sqrt[3]{B}, \\ y_3 &= \omega^2\sqrt[3]{A} + \omega\sqrt[3]{B}. \end{aligned}$$

These formulas are known as Cardan's formulas after the name of the Italian algebraist Cardan (1501–1576), who was the first to publish them. It must be remembered that $\sqrt[3]{A}$ can be taken arbitrarily among the three possible cube roots of A, but $\sqrt[3]{B}$ must be so chosen that

$$\sqrt[3]{A} \cdot \sqrt[3]{B} = -\frac{p}{3}.$$

3. Discussion of Solution. In discussing Cardan's formulas we shall suppose that p and q are real numbers. Then, the nature of the roots will be shown to depend on the function

$$\Delta = 4p^3 + 27q^2$$

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