Why $\pi \neq 4$, ever!

Def: Arc Length for arc, $y = f(x) \rightarrow \; L = \int ds$, where $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$.

Def: Equation for unit circle: $a^2 = y^2 + x^2, \; x, y \in \mathbb{R}$. Simple trigonometry can be used to show that this equation can be transformed into a parametric equation $a^2 = a^2[\cos^2(t) + sin^2(t)]$, where $t \in \{0, 2\pi\}$.

I. Show the derivation of the arc-length (perimeter) of a circle (from the equation of a circle).

Note: For a circle with diameter (d=1) centered on the origin, the radius = ½ ($\therefore \; a = \frac{1}{2}$).

Consider only the top hemisphere of a circle for simplicity: $f_{topHemisphere}(x) = \sqrt{a^2 - x^2}$, such that the entire circle's perimeter is $2 * f_{topHemisphere}(x)$.

Next calculate the arc length of the top hemisphere: $dy = \frac{-x}{\sqrt{a^2-x^2}} dx \rightarrow \frac{dy}{dx} = \frac{-x}{\sqrt{a^2-x^2}} \rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{a^2-x^2}$.

Finishing the integral: $ds = \sqrt{1 + \frac{x^2}{a^2-x^2}} dx \rightarrow L = \int ds = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{1 + \frac{x^2}{a^2-x^2}} dx = a * \tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)\Big|_{-\frac{1}{2}}^{\frac{1}{2}}$.

Note this is an improper integral, so you need to do: $\lim_{t \rightarrow \frac{1}{2}, -\frac{1}{2}} \left[\frac{1}{2} * \tan^{-1}\left(\frac{x}{\sqrt{\frac{1}{4} - x^2}}\right)\right]$, which equals

$2\frac{\pi}{4} = \frac{\pi}{2}$ (as expected).

II. Show that the area of the step-wise approximation converges to πr^2.

To be a bit lazy, we will show that an area larger than the step-wise triangles, which are said to approximate the perimeter of the inscribed circle, go to zero (as $n \rightarrow \infty$). If this is the case, then the "excess" area between the circle and this bizarre shape goes to zero, and the area enclosed by the step-wise shape goes to πr^2.

Let's look at the triangle formed between the center diameter and the (step-wise) bottom fold:

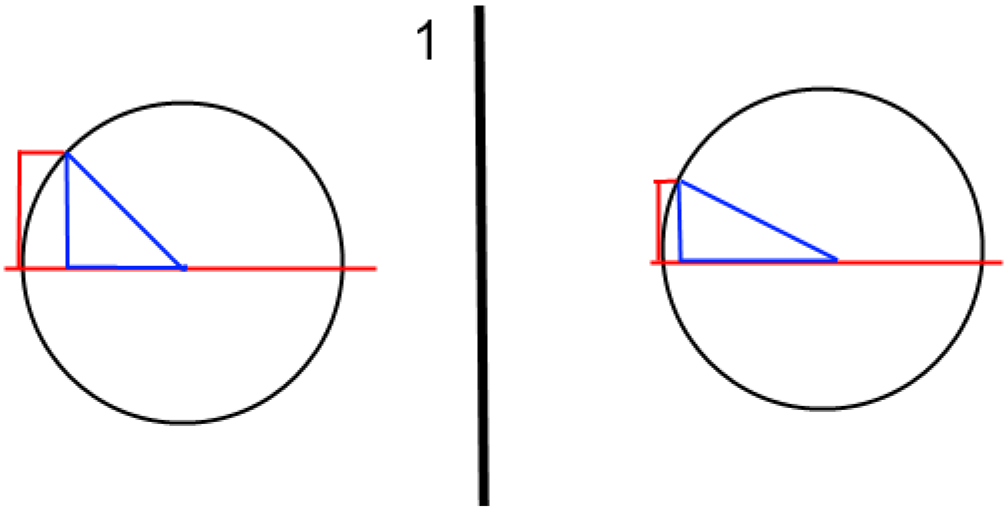

Figure 1: Projections -- Forgive the terrible scale

The height of the first step (vertical) can be approximated (exactly in the first iteration) as $(radius) * \sin(angle)$. This would be the opposite (in triangle terms) side of the blue triangle times its hypotenuse (which will always be the circle's radius). In the first iteration, there are exactly two of these triangles per quadrant (forth of the circle), in the second, there are two (and two smaller triangles). Using the fact that each triangle in between the two (side and top) triangles will be shorter (in terms of base and height), what I'd like to propose is that we can approximate the "excess area" or area between the step-wise approximation, as follows:

$$Area(triangle) = \frac{1}{2}(base)(height) = \frac{1}{2}(radius) * \left[\sin\left(\frac{\pi}{2^{n+1}}\right)\right]^2$$

Where 'n' is the iteration number, the radius is $\frac{1}{2}$, and the 2^{n+1} term comes in from the fact we start in at $\frac{\pi}{4}$ (45°), then split that into two sections (by "folding"), which becomes $\frac{\pi}{8}$ (22.5°), and so on. For each iteration, we have a total of 2^n triangles to sum for the "extra area", so the full formula for 'n' iterations is:

$$Excess\ Area = \frac{1}{4}(2^n) \cdot \sin^2\left(\frac{\pi}{2^{n+1}}\right)$$

And this is going to be greater than the actual area of the intermediate step triangles, since they are smaller (on every iteration) than the side and top triangle. It turns out that as $n \to \infty$, for an infinite number of iterations:

$$\lim_{n\to\infty} \frac{1}{4}(2^n) \cdot \sin^2\left(\frac{\pi}{2^{n+1}}\right) = 0 \text{ !!}$$

And this approximate extra area goes to zero (meaning the area inside the step-wise approximation goes to πr^2), and since:

$$Approximate\ Area > Actual\ Area\ \&\ Approximate\ Area \to 0, \therefore Actual\ Area \to 0$$

To prove to you that the approximate area is larger than the actual area of the triangles, consider the “inner” and “step” triangles constructed in the approximation:

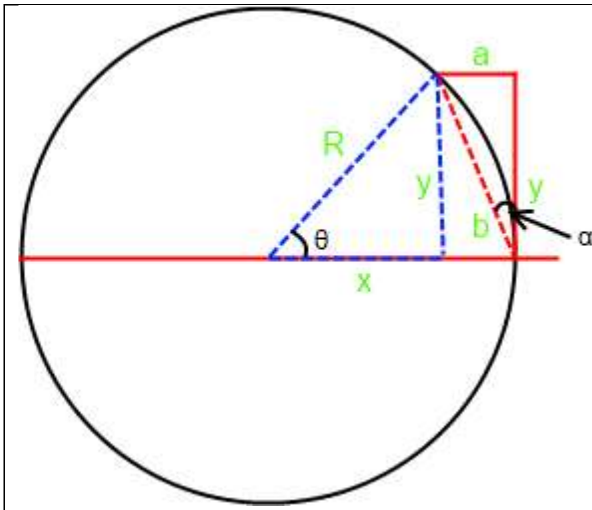

Figure 2: The triangles in consideration

$$y = R \cdot sin(\theta)$$
$$a = R - \mathrm{R} \cdot \cos(\theta)$$
$$b^2 = y^2 + a^2 = R^2 \cdot \sin^2(\theta) + (R - \mathrm{R} \cdot \cos(\theta))^2$$
$$b^2 = 4R^2 \cdot \sin^2\left(\frac{\theta}{2}\right), b = 2R \cdot sin\left(\frac{\theta}{2}\right)$$
$$\alpha = \frac{\pi}{2} - \theta \rightarrow \theta = \frac{\pi}{2} - \alpha$$
$$a = 2R \cdot \sin\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) \cdot \sin(\alpha)$$
$$R = \operatorname{atan}\left(\frac{y}{x}\right) = \operatorname{atan}\left(\frac{r \cdot \sin(\theta)}{r - r \cdot \cos(\theta)}\right)_{as\ r \to R}$$

If we consider summing up “step” triangle perimeters ($a + y$ from the diagram above), we get:

$$Perimeter(step_s) = a + y = R \cdot \sin(\theta) + R - R \cdot \cos(\theta) = R(1 + \sin(\theta) - \cos(\theta))$$

As we start adding more steps into successive approximations, ‘R’ doesn’t change, but the number of steps and ‘θ’ do. Note that $\theta(1,2,3, \dots, n) = \frac{\pi}{4}, \frac{\pi}{8}, \frac{\pi}{16}, \dots, \frac{\pi}{2^{n+1}}$.. Summing up an infinite number of steps as per the approximation, we have:

$$\lim_{n\to\infty} \sum_{i=0}^{n} R \cdot \left[1 + \sin\left(\frac{\pi}{2^{n+1}}\right) - \cos\left(\frac{\pi}{2^{n+1}}\right)\right] \Delta n = R \cdot \int_0^\infty \left[1 + \sin\left(\frac{\pi}{2^{n+1}}\right) - \cos\left(\frac{\pi}{2^{n+1}}\right)\right] dn$$

Now, to be honest, I don’t like this integral one bit, so I will let Wolfram (http://www.wolframalpha.com) do it for me:

$$let\ f(n) \equiv \frac{\pi}{2^{n+1}} \rightarrow R \cdot \int_0^\infty \left[1 + \sin\left(\frac{\pi}{2^{n+1}}\right) - \cos\left(\frac{\pi}{2^{n+1}}\right)\right] dn = \frac{1}{2}\left[\frac{Ci\big(f(n)\big) - Si\big(f(n)\big) + n \cdot \log(2)}{\log(2)}\right]\Bigg|_{n=0}^{\infty}$$

… $\cong 1.3904405$. Where Ci(x) is the Cosine integral function and Si(x) is the Sine integral function.

Note this is for one hemisphere (1/4th of the total sphere, since we are only counting a+y for $\theta = 0\ to\ \pi/2$). So the total perimeter is $\cong 4 \cdot 1.3904405 \ \cong 5.5618 > 4$.

III. Problems with the approximation.

You may be asking yourself "how does this approximation show that the area of the infinite step approximation asymptotically approaches π, but the perimeter of the approximation does not approach $\pi * d$ (the perimeter of a circle)?!!

This is a valid question, and the basis of this entire article! To answer this question, I will present an argument for why the "Troll Pi" approximation is necessarily incorrect. There are other arguments (including geo-morphic formation/deformation, discontinuity, etc.), but I think this one is easiest to grasp and probably most fundamental to the problem.

My argument is simply that the author of this "approximation" has convinced you that the shortest distance between two points is not a straight line, but rather, simply $|x| + |y|$ (or simply width + height)! Don't see it? Let me elaborate. Consider this staircase:

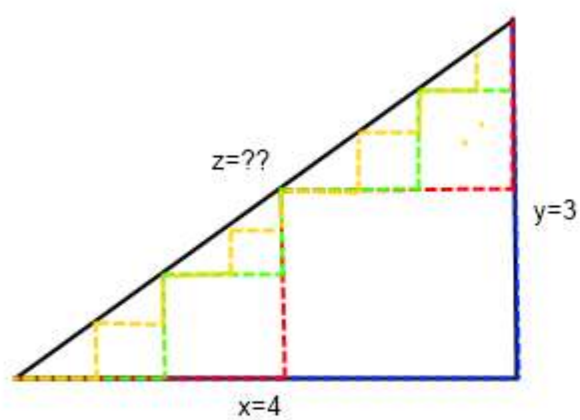

Figure 3: The "fractal" staircase

What is the length of the hypotenuse, "z"? The author of the "Troll Pi" would argue as follows:

- Well, we know that the width of the staircase is 4 units (feet, meters, etc), and the height is 3 units.
- If we start by folding the blue corner into the hypotenuse, we haven't changed the width or the height (we just divided 4 by 2 – counted twice, and then 3 by 2 – counted twice), it's still 4 units wide and 3 units tall.
- If I then fold the red corners into the green, then the green into the yellow, and so on to infinity, I still haven't changed the width (4 units) or the height (3 units). So therefore the length of the hypotenuse (z) must be 4 + 3 = 7 units!

Are you starting to see the flaw? The "Troll Pi" author has argued that the length of the hypotenuse is $|4| + |3| = 7$. You may immediately realize this is wrong from high school trigonometry, but if not, accept it is simply wrong by definition. The shortest distance between two points on the Cartesian plane is a straight line. If we consider the slope of this line (rise/run) and generate a line equation, we have:

$$y = \frac{3}{4}x \to dy = \frac{3}{4}dx \to \frac{dy}{dx} = \frac{3}{4} \to \left(\frac{dy}{dx}\right)^2 = \left(\frac{3}{4}\right)^2 = \left(\frac{9}{16}\right)$$

Let's take that value and plug it into our arc-length equation (used in the first step):

$$ds = \sqrt{1 + \left(\frac{9}{16}\right)}\, dx, L = \int_0^4 ds = \int_0^4 \sqrt{1 + \left(\frac{9}{16}\right)}\, dx = \frac{5}{4}x\Big|_0^4 = 5$$

And I posit that the length of the hypotenuse is 5 units, not 7! Pythagoras, a few thousand years ago, realized that the hypotenuse of a right triangle is not $z = |x| + |y|$, but rather that $z^2 = x^2 + y^2$! Remember that formula? It says that the length of the staircase is:

$$z^2 = x^2 + y^2 \to z = \sqrt{x^2 + y^2} \to z = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5!$$

Ok, so I purposefully picked the numbers to come out that way, nice and even, but I think you get my point. How does this apply to the "Troll Pi" approximation? Well, if you don't see it, let me help you a bit more.

Look at figure 2 – See the red "step" triangle, with sides 'a' and 'y', and hypotenuse 'b'? What the troll has convinced you of is that the "length" of the circle can be approximated by adding up an infinite number of 'a's and 'y's, similarly to the staircase. Again, by definition, this is wrong. Look at the simplest equation for a circle: $x^2 + y^2 = r^2$. It looks like our triangle equation, doesn't it? There is a reason for that – what this equation is saying is that each and every point (of which there are an infinite number of them) must be exactly 'r' distance away from the center of the circle. Another way to think of it is that a circle can be constructed by an infinite number of right triangles, with hypotenuse equal to 'r'.

Now go back to figure 2 – the "correct" length approximation for the circumference (perimeter) is $a^2 + y^2 = b^2$. Let's use the equation defined for the hypotenuse of these "step" iterations and see if that, instead, correctly gives us "π".

From the geometric equation for the "step" triangle's hypotenuse (b): $b = 2R \cdot sin\left(\frac{\theta}{2}\right)$, the question is what happens when we add up an infinite number of these (each of which has an infinitesimal length). Note that for the first triangle (when there are no steps), $\theta = \pi/2$, for the second $\theta = \pi/4$, and so on, so that $\Theta(n) = \frac{\pi}{2^n}$. We also have 2^{n-1} triangles in each successive approximation, so the equation is:

$$Arc - length\ of\ circle\ (top\ right\ quadrent) = \lim_{n\to\infty} 2R \cdot 2^{n-1} \cdot \sin\left(\frac{\pi}{2 * 2^n}\right)$$

$$\lim_{n\to\infty} 2R \cdot 2^{n-1} \cdot \sin\left(\frac{\pi}{2 * 2^n}\right) = 2R \cdot \lim_{n\to\infty} 2^{n-1} \cdot \sin\left(\frac{\pi}{2^{n+1}}\right) = 2\left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right) = \frac{\pi}{4}\ !!$$

And since there are 4 quadrants of the circle, we finally have $Perimeter_{circle} = 4\left(\frac{\pi}{4}\right) = \pi$, and π is saved from the "Troll" and certain miscalculation!

IV. Derivation of the arc-length formula

Just so I don't leave anything out (or you might be wondering where that formula came from), consider again figure 2. What if, instead of knowing 'b' – the hypotenuse, we wanted to know the length of arc of the circle that "curves" through 'a' and 'y' (creating a curved "hypotenuse")? Well we've talked about approximations, so we could approximate it by our formula $arc^2 = a^2 + y^2$, but what if we wanted to do better?

Consider if we shrunk the triangle down to some very tiny size, such that $a \to \Delta a, y \to \Delta y$ (which are both very small, and added up all the hypotenuses of these little triangles such that $\sum_i \Delta a_i = a, \sum_i \Delta y_i = y$. Surely this would be a better approximation, no? Well, using the same logic as the "troll", what if we instead let the small size of these triangles shrink to zero (again, never equaling zero, but approaching zero, as in a limit), and summed up an infinite number of them? We would have:

$$Arc - Length = L, (\Delta L)^2 = (\Delta a)^2 + (\Delta y)^2$$

$$Change\ to\ infinitesimally\ small\ 'dx' : (dL)^2 = (ds)^2 = (da)^2 + (dy)^2$$

$$ds = \sqrt{(da)^2 + (dy)^2} = \sqrt{da^2\left(1 + \frac{(dy)^2}{(da)^2}\right)} = \sqrt{1 + \left(\frac{dy}{da}\right)^2}\, da$$

$$L = \lim_{da \to 0} \sum_0^a \sqrt{1 + \left(\frac{dy}{da}\right)^2}\, da = \int_0^a \sqrt{1 + \left(\frac{dy}{da}\right)^2}\, da = \int ds$$