Proof that light rays can follow a purely radial line in FLRW

The FLRW metric centred at S gives the following formula for the line element:

$ds^2 = dt^2 - a(t)^2[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta\ d\phi^2)]$ **[7]**

t, r, θ and ϕ are the 'comoving coordinates', corresponding to indices 0,1,2,3. r is often denoted by χ, but here we use r. $a(t)$ is the cosmic scale parameter at cosmic time t. The values of the metric components in this coordinate system are:

$g_{00} = 1;\ g_{11} = -\frac{a(t)^2}{1-kr^2};\ g_{22} = -a(t)^2 r^2;\ g_{33} = -a(t)^2 r^2 sin^2\theta$ **[7a]**

All other components (the off-diagonal ones) are zero. We will only need to use the first two of the four nonzero components. Since the metric is diagonal in the comoving coordinates, we also have the following inverse components:

$g_{00} = 1;\ g_{11} = -\frac{1-kr^2}{a(t)^2};\ g_{22}$ and g_{33} are not used. **[7b]**

Let S be the spacetime location $[0, 0, ,]$ in the FLRW coordinate system. Let Λ be a curve starting at S and extending radially in a way that is lightlike along its length, and which has constant θ and ϕ. We postulate that there is a geodesic satisfying these criteria, which will be a lightlike geodesic and hence can be the path of a light ray emanating from S.

Parameterise Λ by $\lambda : R \to \Lambda$.

Along Λ we have, from [7], since θ and ϕ do not change, $ds = 0$, and the curve is lightlike: $\frac{dr}{\sqrt{1-kr^2}} = \frac{dt}{a(t)}$ **[10]**

The sign is positive because r increases with t (recall that r is radial distance from S).

Now we need to find the equation for λ, if any, that satisfies the requirements for a geodesic, ie the geodesic equation, written in terms of a parameter u.

The geodesic equation is: $\frac{d^2x^i}{du^2} + \Gamma^i{}_{kl}\ \frac{dx^k}{du}\ \frac{dx^l}{du} = 0$ [**11** - see Schutz 6.51]

And we must show that, for some function λ, the geodesic equation is satisfied for each of the four cases $i = 0, 1, 2, 3$. We will use the case $i = 0$ to identify the function and then show that that function satisfies the geodesic equations for $i = 1, 2, 3$.

First consider the case $i = 0$.

We calculate the Christoffel symbol's value for i=0 as follows:

$$\begin{aligned}
\Gamma^0{}_{kl} &= \frac{1}{2}g^{0\beta}(g_{\beta k,l} + g_{\beta l,k} - g_{kl,\beta}) \quad [\textbf{11a} \text{ - see Schutz 6.32}] \\
&= \frac{1}{2}g^{00}(g_{0k,l} + g_{0l,k} - g_{kl,0}) \quad [\text{since } g_{0\beta} = 0 \text{ unless } \beta = 0] \\
&= \frac{1}{2}g^{00}(g_{00,l}\ \delta^0_k + g_{00,k}\ \delta^0_l - g_{kl,0}) \\
&= -\frac{1}{2}g_{kl,0}[\text{since } g_{00} = g^{00} = 1, \text{ which is constant}]
\end{aligned}$$

Hence [11] for i=0 becomes:

$$\begin{aligned}
0 &= \frac{d^2t}{du^2} - \frac{1}{2}g_{kl,0}\frac{dx^k}{du}\frac{dx^l}{du} \\
&= \frac{d^2t}{du^2} - \frac{1}{2}g_{00,0}(\frac{dt}{du})^2 - \frac{1}{2}g_{11,0}(\frac{dr}{du})^2 \text{ [since } \frac{d\theta}{du} \text{ and } \frac{d\phi}{du} \text{ must be zero]} \\
&= \frac{d^2t}{du^2} - \frac{1}{2}\frac{\partial(\frac{-a(t)^2}{1-kr^2})}{\partial t}(\frac{dr}{du})^2 \text{ [since } g_{00} \text{ is constant at 1]} \\
&= \frac{d^2t}{du^2} + (\frac{dr}{du})^2\frac{a(t)a'(t)}{1-kr^2} \qquad \mathbf{[12]} \\
&= \frac{d^2t}{du^2} + (\frac{dr}{dt})^2(\frac{dt}{du})^2\frac{a(t)a'(t)}{1-kr^2} \\
&= \frac{d^2t}{du^2} + \frac{1-kr^2}{a(t)^2}(\frac{dt}{du})^2\frac{a(t)a'(t)}{1-kr^2} \text{ [by 10]} \\
&= \frac{d^2t}{du^2} + (\frac{dt}{du})^2\frac{a'(t)}{a(t)} \\
&= \frac{1}{a(t)}\frac{d(a(t)\frac{dt}{du})}{du}
\end{aligned}$$

Hence, as $a(t) \neq 0$ we have $0 = \frac{d(a(t)\frac{dt}{du})}{du}$, whence:
$a(t)\frac{dt}{du} = A$ for some constant A. **[13]**
Hence:

$$\begin{aligned}
\frac{dr}{du} &= \frac{dr}{dt}\frac{dt}{du} \\
&= \frac{A}{a(t)}\frac{\sqrt{1-kr^2}}{a(t)} \quad \text{[by 13 and 10]} \\
&= \frac{A\sqrt{1-kr^2}}{a(t)^2} \quad \mathbf{[14]}
\end{aligned}$$

This enables u to be determined as a function of r (as the light moves from S to O) and hence of t and, by inverting these, we get a definition of λ as a function of u.
We can choose any positive real value we like for A. Choosing $A = 1$ would be simplest, as it would make it disappear from the following equations. But we'll leave it in there for now, to preserve generality. For any given value of A, we can define a point $O = \lambda(1)$. Then, if we (without loss of generality) set $\lambda(0) = S$ and integrate [13], we obtain:
$A = \int_0^1 A\ du = \int_{t(S)}^{t(O)} a(t)dt$ **[15]**
Writing the above equations for r and t in terms of λ to make things clearer, we get:
$\frac{d\lambda^0}{du} = \frac{A}{a(t)}, \frac{d\lambda^1}{du} = \frac{A\sqrt{1-kr^2}}{a(t)^2}, \frac{d\lambda^2}{du} = \frac{d\lambda^3}{du} = 0$

Having identified the function λ as the solution to the geodesic equation [11] for

$i = 0$, we now need to check that it satisfies the geodesic equation for $i = 1, 2, 3$. First, consider the case $i = 1$ (r).

We calculate the Christoffel symbol's value for i=1 as follows:

$$\begin{aligned}
\Gamma^1{}_{kl} &= \frac{1}{2} g^{1\beta}(g_{\beta k,l} + g_{\beta l,k} - g_{kl,\beta}) \quad [\textbf{11a} \text{ - see Schutz 6.32}] \\
&= \frac{1}{2} g^{11}(g_{1k,l} + g_{1l,k} - g_{kl,1}) \quad [\text{since } g_{1\beta} = 1 \text{ unless } \beta = 1] \\
&= \frac{1}{2} g^{11}(g_{11,l}\ \delta^1_k + g_{11,k}\ \delta^1_l - g_{kl,1})
\end{aligned}$$

Hence [11] for i=1 becomes:

$0 = \frac{d^2r}{du^2} + \frac{1}{2} g^{11}(g_{11,l}\ \delta^1_k + g_{11,k}\ \delta^1_l - g_{kl,1}) \frac{dx^k}{du} \frac{dx^l}{du}$

$= \frac{d^2r}{du^2} + \frac{1}{2} g^{11}(2g_{11,k} \frac{dr}{du} \frac{dx^k}{du} - g_{00,1}(\frac{dt}{du})^2 - g_{11,1}(\frac{dr}{du})^2)$ [since $\frac{d\theta}{du}$ and $\frac{d\phi}{du}$ must be zero]

$= \frac{d^2r}{du^2} + \frac{1}{2} g^{11}(2g_{11,0} \frac{dr}{du} \frac{dt}{du} + 2g_{11,1}(\frac{dr}{du})^2 - g_{00,1}(\frac{dt}{du})^2 - g_{11,1}(\frac{dr}{du})^2)$

$= \frac{d^2r}{du^2} + \frac{1}{2} g^{11}(2g_{11,0} \frac{dr}{du} \frac{dt}{du} + g_{11,1}(\frac{dr}{du})^2)$ [since $g_{00} = 1$ is constant]

$= \frac{d}{du}(Aa(u)^{-2}\sqrt{1 - kr(u)^2}) + g^{11} g_{11,0} \frac{dr}{du} \frac{dt}{du} + \frac{1}{2} g^{11} g_{11,1}(\frac{dr}{du})^2$

$= (-2Aa'(t) \frac{dt}{du} a^{-3}\sqrt{1 - kr(u)^2} + \frac{1}{2} \frac{-2krAa^{-2}}{\sqrt{1-kr^2}} \frac{dr}{du}) - \frac{1-kr^2}{a^2}(\frac{\partial}{\partial t}(\frac{-a(t)^2}{1-kr^2}) \frac{dr}{du} \frac{dt}{du} + \frac{1}{2} \frac{\partial}{\partial r}(\frac{-a(t)^2}{1-kr^2})(\frac{dr}{du})^2)$

$= -\frac{A\sqrt{1-kr^2}}{a^3}(2a'(t) \frac{dt}{du} + \frac{kra}{1-kr^2} \frac{dr}{du}) + \frac{1-kr^2}{a^2}(\frac{2a(t)a'(t)}{1-kr^2} \frac{dr}{du} \frac{dt}{du} + \frac{1}{2}(\frac{-2kr(-1)a^2}{(1-kr^2)^2})(\frac{dr}{du})^2)$

$= -\frac{A\sqrt{1-kr^2}}{a^3}(2a'(t) \frac{A}{a} + \frac{kra}{1-kr^2} \frac{A\sqrt{1-kr^2}}{a^2}) + \frac{1}{a^2}(2a(t)a'(t) \frac{A\sqrt{1-kr^2}}{a^2} \frac{A}{a} + \frac{kra^2}{1-kr^2}(\frac{A\sqrt{1-kr^2}}{a^2})^2)$

$= -\frac{A^2\sqrt{1-kr^2}}{a^4}(2a'(t) + \frac{kr}{\sqrt{1-kr^2}} - 2a'(t) - \frac{kr}{\sqrt{1-kr^2}})$

$= 0$ as required. So the geodesic equation for $i = 1$ is satisfied by the λ we chose.

Now check for $i = 2$ (θ)

We calculate the Christoffel symbol's value for i=2 as follows:

$$\begin{aligned}
\Gamma^2{}_{kl} &= \frac{1}{2} g^{2\beta}(g_{\beta k,l} + g_{\beta l,k} - g_{kl,\beta}) \quad [\textbf{11a} \text{ - see Schutz 6.32}] \\
&= \frac{1}{2} g^{22}(g_{2k,l} + g_{2l,k} - g_{kl,2}) \quad [\text{since } g_{1\beta} = 1 \text{ unless } \beta = 1] \\
&= \frac{1}{2} g^{22}(g_{22,l}\ \delta^2_k + g_{22,k}\ \delta^2_l - g_{kl,2})
\end{aligned}$$

Now if $\{k, l\} \subset \{0, 1\}$ this is zero because $\delta^2_k = \delta^2_l = 0$ and g_{kl} does not depend on θ.

Hence [11] for i=2 becomes:

$\frac{d^2\theta}{du^2} = -\Gamma^2{}_{kl} \frac{dx^k}{du} \frac{dx^l}{du}$

and at least one of k and l must be in $\{2, 3\}$ but then at least one of $\frac{dx^k}{du}$ and $\frac{dx^l}{du}$ must be zero. So the geodesic requirement for θ becomes $\frac{d^2\theta}{du^2} = 0$, which is satisfied because θ is constant along Λ by definition.

To check for $i = 3$ we can use the same argument as we did for $i = 2$ to show that the geodesic requirement for ϕ is satisfied.

Hence, there exists a geodesic that follows the path Λ and is lightlike. Hence a light ray can traverse that path.

Corollary - Parallel Transport along the radial ray

Now we want to prove a more general result, that any vector $\vec{V_S} \in T_S M$ with zero components V_S^θ and V_S^ϕ will maintain those zero components upon parallel transport along the lightlike radial geodesic Λ.
The parallel transport equation is:
$\frac{dx^k}{du}\frac{\partial v^i}{\partial x^k} + \Gamma^i{}_{kl}\ \frac{dx^k}{du}\ \frac{dv^l}{du} = 0$ [**30** - see Schutz 6.48]
From the above, we know that $\frac{dx^2}{du} = \frac{dx^3}{du} = 0$ so for $i = 2$ this equation becomes:
$\frac{dt}{du}\frac{\partial V^2}{dt} + \frac{dr}{du}\frac{\partial V^2}{dr} + \Gamma^2_{0l}\ \frac{dt}{du}\ \frac{dv^l}{du} + \Gamma^2_{1l}\ \frac{dr}{du}\ \frac{dv^l}{du} = 0$
Next we use the fact that $\Gamma^2_{00} = \Gamma^2_{01} = 0$ to transform this equation to:
$\frac{dV^2}{du} + \Gamma^2_{02}\ \frac{dt}{du}\ \frac{dV^2}{du} + \Gamma^2_{03}\ \frac{dt}{du}\ \frac{dV^3}{du} + \Gamma^2_{12}\ \frac{dr}{du}\ \frac{dV^2}{du} + \Gamma^2_{13}\ \frac{dr}{du}\ \frac{dV^3}{du} = 0$
It is easy to find the following values of Christoffel symbols: $\Gamma^\theta_{t\phi} = \Gamma^\theta_{r\phi} = 0, \Gamma^\theta_{t\theta} = \frac{a'(t)}{a}, \Gamma^\theta_{r\theta} = \frac{1}{r}$. Inserting these in the parallel transport equation for θ we get $\frac{dV^2}{du}(1 + \frac{a'(t)}{a}\frac{dt}{du} + \frac{1}{r}\frac{dr}{du}) = 0$.

Substituting in this for $\frac{dt}{du}$ and $\frac{dr}{du}$ gives:
$\frac{dV^2}{du}(1 + \frac{Aa'(t)}{a^2} + \frac{A\sqrt{1-kr^2}}{a^2 r}) = 0$. All components of the second factor are positive with the possible exception of $a'(t)$, which is positive as long as the universe is expanding. So in an expanding universe it must be the case that $\frac{dV^2}{du} = 0$. This result is probably also true in a non-expanding universe, but proving that is a little more work, which we do not undertake here.

We need to prove the same result for $\frac{dV^3}{du}$. The results for Christoffel symbols are analogous:
$\Gamma^\phi_{t\theta} = \Gamma^\phi_{r\theta} = 0, \Gamma^\phi_{t\phi} = \frac{a'(t)}{a}, \Gamma^\phi_{r\phi} = \frac{1}{r}$. We can then run through the same argument as we used for $\frac{dV^2}{du}$ but with θ and ϕ everywhere swapped. That gives the result that $\frac{dV^3}{du} = 0$.

Hence we can conclude that the θ and ϕ components of $\vec{V_S}$ remain zero as it is parallel transported along the radial null geodesic.