And because in three dimensions,

$$-\frac{\hbar^2}{2\bar{m}}\nabla^2\Psi(r) = \frac{2\pi\hbar^2}{\bar{m}}\delta(\mathbf{r}), \tag{2.23}$$

the Schrödinger equation is satisfied. This interaction potential is known as Fermi's pseudo-potential [77]. Furthermore, one can show that the energy of the low-energy bound state can also be reproduced.

Renomalizable Contact Potential. A pseudo-potential model can nicely reproduce the low-energy physics. However, it has a shortcoming that the operator is not Hermitian. Thus, it is not very convenient to use the pseudo-potential in many circumstances, in particular, when a second-quantized form of a many-body Hamiltonian is needed. For studying many-body physics, it is still convenient to use a delta-function contact potential as $V(\mathbf{r}) = g\delta(\mathbf{r})$. Though we have already known that it will cause a divergent problem at short distance, nevertheless, let us proceed further and see how serious the problem is and whether there are ways to fix the problem.

Here we consider spin-1/2 fermions with this delta-function interaction potential as an example. With a delta-function interaction potential, the second-quantized Hamiltonian for spin-1/2 fermions can be written as

$$\hat{\mathcal{H}} = \int d^3\mathbf{r}\left(\sum_{\sigma}\hat{\Psi}^{\dagger}_{\sigma}(\mathbf{r})\left(-\frac{\hbar^2}{2m}\nabla^2\right)\hat{\Psi}_{\sigma}(\mathbf{r}) + g\hat{\Psi}^{\dagger}_{\uparrow}(\mathbf{r})\hat{\Psi}^{\dagger}_{\downarrow}(\mathbf{r})\hat{\Psi}_{\downarrow}(\mathbf{r})\hat{\Psi}_{\uparrow}(\mathbf{r})\right), \tag{2.24}$$

where $\hat{\Psi}^{\dagger}_{\sigma}(\mathbf{r})$ and $\hat{\Psi}_{\sigma}(\mathbf{r})$ ($\sigma =\uparrow,\downarrow$) are creation and annihilation operators for fermions at position $\mathbf{r}$. In the momentum space, this Hamiltonian is given by

$$\hat{\mathcal{H}} = \sum_{\mathbf{k}\sigma}\frac{\hbar^2\mathbf{k}^2}{2m}\Psi^{\dagger}_{\mathbf{k}\sigma}\Psi_{\mathbf{k}\sigma} + \frac{g}{V}\sum_{\mathbf{k},\mathbf{k_1},\mathbf{k_2}}\hat{\Psi}^{\dagger}_{\frac{\mathbf{k}}{2}+\mathbf{k_1},\uparrow}\hat{\Psi}^{\dagger}_{\frac{\mathbf{k}}{2}-\mathbf{k_1},\downarrow}\hat{\Psi}_{\frac{\mathbf{k}}{2}-\mathbf{k_2},\downarrow}\hat{\Psi}_{\frac{\mathbf{k}}{2}+\mathbf{k_2},\uparrow}, \tag{2.25}$$

where V is the volume of the system. Here the second term represents scattering between atoms, with the center-of-mass momentum $\mathbf{k}$ conserved and the relative momenta changing from $\mathbf{k_2}$ to $\mathbf{k_1}$.

We first compute a two-body scattering T-matrix with Hamiltonian equation 2.25. We consider an on-shell scattering process with both incoming and outgoing states having the same energy E and the center-of-mass momentum equaling zero. Since the interaction vertex g is now a constant independent of momentum, the leading order diagram is a direct scattering from the incoming state to the outgoing state, whose contribution is g, as shown in Figure 2.3(a). The next-order diagram involves intermediate states, and the relative momentum $\mathbf{p}$ of the intermediate state can be taken at any momentum. Its contribution can be computed by the second-order processes as

$$\frac{1}{V}\sum_{\mathbf{p}} g\frac{1}{E - \frac{\hbar^2\mathbf{p}^2}{m} + i0^+}g, \tag{2.26}$$

where $i0^+$ is a mathematical technicality necessary for the calculation of the integrals and is also a consequence of causality. Furthermore, one can systematically consider all the

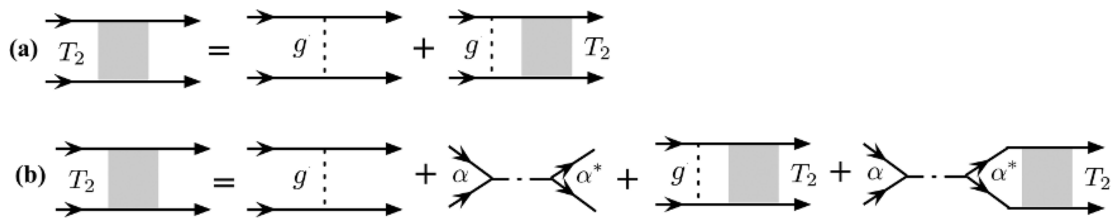

Figure 2.3 T-matrix for two-body scattering. Ladder diagrams for two-body T-matrix (a) of the renormalizable contact potential model Eq. 2.24 and (b) for the two-channel model Eq. 2.66. The two-channel model will be discussed in Section 2.4.

higher order contributions by including more intermediate states, as illustrated by the so-called ladder diagram shown in Figure 2.3(a). It turns out that for two-body problems, unlike the many-body situation to be discussed in later chapters, the summation of the ladder diagram is an exact solution. The summation of the ladder diagram leads to the so-called *Schwinger–Dyson equation* given by

$$\begin{aligned} T_2(E) &= g + \frac{1}{V}\sum_{\mathbf{p}} g \frac{1}{E - \frac{\hbar^2 \mathbf{p}^2}{m} + i0^+} g + \dots \\ &= g + \frac{g}{V}\sum_{\mathbf{p}} \frac{1}{E - \frac{\hbar^2 \mathbf{p}^2}{m} + i0^+} T_2(E), \end{aligned} \tag{2.27}$$

and thus

$$T_2(E) = \frac{g}{1 - \frac{g}{V}\sum_{\mathbf{p}} \frac{1}{E - \frac{\hbar^2 \mathbf{p}^2}{m} + i0^+}}. \tag{2.28}$$

Here it is important to notice that the summation over momentum in Eq. 2.28 behaves as $\int d^3\mathbf{p}(1/p^2)$ at large momentum and diverges at large momentum in three dimensions. This divergence comes from the upper limit of the energy integration and is called the *ultraviolet divergence*. As we discussed in Box 2.2, such an ultraviolet divergence means the short-range physics is not treated properly. Here, it means nothing but that the short-range $1/r$ behavior of the free wave function should not be taken to the $r \to 0$ limit, and the δ-function contact potential is not appropriate.

This divergence can also be viewed from the Hamiltonian in momentum space equation 2.25, where the scattering vertex is taken as independent of the momentum transfer, because the Fourier transformation of a δ-function potential is a constant. However, this is unphysical because in any physical model with finite range r_0, this scattering vertex always decays toward zero when the transferred momentum is much larger than $\hbar/r_0$. By taking this momentum dependence of the scattering vertex into account, the large momentum divergence in the summation of Eq. 2.28 can be avoided. Nevertheless, the momentum dependence of the scattering vertex at large momentum comes from the short-range structure of the microscopic potential, which is the nonuniversal physics that we do not want to include.

Hence, we encounter a dilemma. On one hand, we understand that the zero-range δ-function potential, or equivalently saying, a momentum-independent scattering vertex at large momentum, is unphysical, which causes ultraviolet divergence. On the other hand,