Intrinsic anomalous Hall conductivity in non-uniform electric field

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We study how the intrinsic anomalous Hall conductivity is modified in two-dimensional crystals with broken time-reversal symmetry due to weak inhomogeneity of the applied electric field. Focusing on a clean non-interacting two-band system without band crossings, we derive the general expression for the Hall conductivity at small finite wavevector q to order q^2, which governs the Hall response to the second gradient of the electric field. Using the Kubo formula, we show that the answer can be expressed through the Berry curvature, Fubini-Study quantum metric, and the rank-3 symmetric tensor which is related to the quantum geometric connection and physically corresponds to the gauge-invariant part of the third cumulant of the position operator. We further compare our results with the predictions made within the semiclassical approach. By deriving the semiclassical equations of motion, we reproduce the result obtained from the Kubo formula in some limits. We also find, however, that the naïve semiclassical description in terms of the definite position and momentum of the electron is not fully consistent because of singular terms originating from the Heisenberg uncertainty principle.

Introduction. — One of the most spectacular manifestations of quantum mechanics in solids is the effect of band geometry and topology on transport coefficients. Examples include the anomalous Hall effect [1], Chern insulators [2], topological insulators [3], and topological semimetals [4]. Remarkably, some intriguing transport properties of these materials can be solely explained by the peculiarities of the band structure [5]. For instance, the intrinsic anomalous Hall effect in magnetic materials can be elegantly described in terms of the Berry curvature on the Brillouin zone [1, 5–8]. Another interesting example is the explanation of natural optical activity at low frequencies, also known as gyrotropic magnetic effect, via the intrinsic magnetic moment of the Bloch electrons on the Fermi surface [9, 10].

Recently, attention has been drawn to the conducting properties of metals and insulators in an inhomogeneous electric field. One set of studies focused on the momentum-dependent part of the Hall conductivity in magnetic field and its profound connection to the Hall viscosity in Galilean invariant systems [11–14]. Moreover, it was shown that the Hall viscosity determines the size-dependent part of the Hall resistance and can be used as an indication of the hydrodynamic flow of the electron fluid [13, 14], which was recently probed experimentally [15]. Other works studied the modifications of the semiclassical equations of motion in inhomogeneous field due to the non-zero Fubini-Study quantum metric and the manifestation of these modifications in transport and optical measurements [16, 17].

In this work, we study the intrinsic contribution to the anomalous Hall current in non-uniform electric field. Instead of external magnetic field, we assume that time-reversal symmetry in a crystal is broken by, e.g., magnetic order, leading to the finite Berry curvature. The correction due to the electric field inhomogeneity is captured by the momentum dependence of the Hall conductivity, which at small momenta q can be expanded as (choosing vector $\mathbf{q}$ to be along the $x-$axis)

$$\sigma_{AH}(q) = \sigma_{AH}^{(0)} + q^2 \sigma_{AH}^{(2)} + \dots \qquad (1)$$

Here we explicitly defined the antisymmetric part of the conductivity tensor as $\sigma_{AH}(\mathbf{q}) \equiv [\sigma_{xy}(\mathbf{q}) - \sigma_{yx}(\mathbf{q})]/2$, and the subscript "$AH$" stands for "anomalous Hall". We calculate $\sigma_{AH}^{(2)}$ for a generic clean two-dimensional two-band system (without any external magnetic field) and show that it is expressed through three gauge-invariant objects defined on the Brillouin zone: Berry curvature, quantum metric, and the fully symmetric rank-3 tensor defined through the symplectic connection [18]. The latter object determines the gauge-invariant part of the third cumulant of the position operator (analogous to how quantum metric determines the second cumulant of the position operator) and was recently shown to enter the answer for the shift photocurrent in Weyl semimetals [19, 20]. We assume that the two bands are separated by a finite energy gap everywhere in the Brillouin zone. We use the Kubo formula to obtain the most general microscopic answer.

We further compare our result with the answer obtained within the semiclassical approach. To do that, we derive the semiclassical equations of motion in a non-uniform electric field up to the second order in the field gradients. We find that semiclassics reproduces terms defining $\sigma_{AH}^{(2)}$ in the insulating regime in the limit when the two bands are well separated. However, we also show that the Heisenberg uncertainty principle does not allow for a reliable semiclassical description when dealing with the second gradients of the electric field. In particular, we find that the equations of motion for the electron wave packet contain some terms which formally become divergent in case when the wave packet is narrow in momentum space (i.e., corresponds to the state with a well-defined momentum). The origin of these terms is clear: semiclassical Boltzmann approach relies on the assumption that the particles are described by the well-defined momentum and coordinate, which is inherently incompatible with the basics of quantum mechanics. This contradiction, however, does not cause any problems when dealing with the uniform part of the electric field or with its first gradient, and we also show that the semiclassics well agrees with the Kubo formula in some limits once these wavepacket-dependent terms are discarded.

Kubo formula result. — The most straightforward way to calculate the intrinsic Hall conductivity from the microscopic band structure is to use the Kubo formula. For simplicity, we consider generic two-band Hamiltonian $\hat{H}_0(\mathbf{k})$ defined in the two-dimensional quasimomentum space, which is diagonal in the basis of Bloch wavefunctions $|u_{V,C}(\mathbf{k})\rangle$ with the spectrum $\varepsilon_{V,C}(\mathbf{k})$, $\hat{H}_0(\mathbf{k})|u_{V,C}(\mathbf{k})\rangle = \varepsilon_{V,C}(\mathbf{k})|u_{V,C}(\mathbf{k})\rangle$. Indices "$V$" and "$C$" stand for the "valence" and "conduction" band, respectively.

The Hall conductivity is given by the antisymmetric part of the conductivity tensor $\sigma_{\alpha\beta}(i\omega_n,\mathbf{q})$ which is related to the current-current correlation function, $K_{\alpha\beta}(i\omega_n,\mathbf{q}) = \langle \hat{j}^\alpha(i\omega_n,\mathbf{q})\hat{j}^\beta(-i\omega_n,-\mathbf{q})\rangle$, as $\sigma_{\alpha\beta}(i\omega_n,\mathbf{q}) = -K_{\alpha\beta}(i\omega_n,\mathbf{q})/\omega_n$ [21]. Strictly speaking, the correlator $K_{\alpha\beta}$ only describes the paramagnetic contribution to conductivity and, in principle, the diamagnetic term should also be included. The latter, however, does not contribute to the (antisymmetric) Hall component of conductivity and will be omitted hereafter.

We find that the simplest and most physically intuitive result is obtained in case when chemical potential lies within the band gap, i.e., when the system is in the insulating state. The uniform part of the anomalous Hall conductivity is then quantized and given by $\sigma_{AH}^{(0)} = -\frac{e^2}{\hbar}\frac{1}{S}\sum_{\mathbf{k}}\Omega_{xy}(\mathbf{k}) = \frac{e^2}{h}C$, where S is the total area of the system, integer C is the Chern number, and $\Omega_{ij}(\mathbf{k}) = -2\,\mathrm{Im}\langle\partial_{k_i}u_V|\partial_{k_j}u_V\rangle$ is the Berry curvature of the valence band [1]. As for the q^2 component of the Hall conductivity, we find that in the static limit $\omega\to 0$ it equals to

$$\sigma_{AH}^{(2)} = \frac{e^2}{2\hbar}\frac{1}{S}\sum_{\mathbf{k}} g_{xx}\Omega_{xy} - \frac{\hbar}{\varepsilon_C-\varepsilon_V}\left[\frac{v_{Cx}-v_{Vx}}{3}\frac{\partial\Omega_{xy}}{\partial k_x} + \frac{v_{Cx}-v_{Vx}}{2}T_{xxy} - \frac{v_{Cy}-v_{Vy}}{2}T_{xxx}\right] - \frac{2\hbar^2 v_{Cx}v_{Vx}}{(\varepsilon_C-\varepsilon_V)^2}\Omega_{xy}, \quad (2)$$

where the summation is over the states in the completely filled valence band and we fix our coordinate system such that $\mathbf{q}$ is along the $x-$axis. We also suppressed the indices $\mathbf{k}$ in the above expression for brevity. Band velocities $v_{V(C)}$ and the quantum metric tensor of the valence band g_{ij} are defined as $v_{V(C)i}(\mathbf{k}) = \partial_{k_i}\varepsilon_{V(C)}(\mathbf{k})/\hbar$ and $g_{ij}(\mathbf{k}) = \mathrm{Re}\left[\langle\partial_{k_i}u_V|\partial_{k_j}u_V\rangle - \langle\partial_{k_i}u_V|u_V\rangle\langle u_V|\partial_{k_j}u_V\rangle\right]$, respectively. As is clear from Eq. (2), the answer for $\sigma_{AH}^{(2)}$ also contains the components of a fully symmetric tensor

$$T_{ijl} = \frac{1}{3}\,\mathrm{Im}(c_{ijl} + c_{jli} + c_{lij}), \quad (3)$$

where

$$c_{ijl} = \langle u_V|(\partial_{k_i}\partial_{k_j}P_C)(\partial_{k_l}P_C)|u_V\rangle \quad (4)$$

is the quantum geometric connection of the valence band [20] and $P_C = |u_C\rangle\langle u_C| = 1 - |u_V\rangle\langle u_V|$ is the projector onto the conduction band. The real part of the tensor c_{ijl} is identified with the Christoffel symbols, while the imaginary part is known as the symplectic connection [18] (also called symplectic Christoffel symbols in Ref. [20]). All the geometric quantities that determine the answer in Eq. (2), Ω_{ij}, g_{ij}, and T_{ijl}, are invariant under the gauge transformation $|u(\mathbf{k})\rangle \to e^{i\phi_{\mathbf{k}}}|u(\mathbf{k})\rangle$.

Equation (2) is one of the main results of the present work. We see that in case when the bandwidth is much smaller than the band gap, i.e., when bands are nearly flat, $\sigma_{AH}^{(2)}$ is mainly determined by the first term, involving product of the Berry phase Ω_{xy} and quantum metric g_{xx}. That this term indeed dominates σ_{AH} in this limit has been verified numerically for a Haldane model with flattened bands [22]. This result can be qualitatively understood as follows. The size of the maximally localized Wannier orbital is known to be given by the quantum metric tensor g_{ij} [23, 24]. The effective electric field averaged over the size of the wave packet that the particle experiences in a slowly varying electric field can be estimated as $\mathbf{E}(0) + [\partial^2\mathbf{E}(0)/\partial x^2]g_{xx}$. The correction to the anomalous velocity then equals $\delta v_j \sim \delta E_i\,\Omega_{ij} \sim (\partial^2 E_i/\partial x^2)g_{xx}\Omega_{ij}$, which in Fourier space gives exactly $\sigma_{AH}^{(2)} \propto q^2 g_{xx}\Omega_{xy}$.

Since the tensor T_{ijl} is not well known in the condensed matter context, we briefly comment on the significance of this object. Defined as the fully symmetric part of the symplectic Christoffel symbols, T_{ijl} encapsulates the additional geometric information about the band structure that is not captured by either the Berry curvature or quantum metric [25]. Physically, it determines the gauge-invariant part of the third cumulant (skewness) of the position operator averaged over the electron configuration. This observation reconciles the results of Refs. [19] and [20] which computed the circular shift photocurrent in topological semimetals and expressed the answers in terms of the third cumulant and the symplectic Christoffel symbols, correspondingly. We also note that the real part of the quantum geometric connection, the Christoffel symbols, was shown to determine the linear shift photocurrent [20].

As an example, we consider the case of a massive Dirac fermion described by the Hamiltonian $\hat{H}_0(\mathbf{k}) = \hbar v_F(k_x\sigma_x + k_y\sigma_y) + \Delta\sigma_z$, where σ_i are Pauli matrices. We find that its Hall conductivity is given by

$$\sigma_{AH}^{\mathrm{Dirac}}(q) \approx -\frac{e^2}{2h}\left(1 - \frac{\hbar^2 v_F^2 q^2}{12\Delta^2}\right). \tag{5}$$

An extra prefactor 1/2 appears due to the fact that the realistic band structure (e.g., a Haldane model) always contains an even number of Dirac points. We further emphasize that this result as well as Eq. (2) are obtained for a system with the chemical potential inside the band gap (an insulator). In principle, one can generalize the answer for a metallic system with the chemical potential residing in a partially filled band. In this case, the final result also contains the contributions from the vicinity of the Fermi surface which are rather complicated and require extra care. In particular, these contributions are very sensitive to the order in which frequency ω and wavevector q are taken to zero and demonstrate singular dependence on the electron's density in the clean limit. These and related questions are discussed in more detail in the Supplemental Materials [25].

It is known that, for the Galilean invariant quantum Hall states, $\sigma_{AH}^{(2)}$ is determined by the Hall viscosity at large magnetic fields [11–13]. However, the direct comparison of our result for the anomalous Hall conductivity, Eq. (2), and Hall viscosity for the lattice systems found in Ref. [26] does not reveal any obvious connection between these two quantities. This is not surprising since a generic crystal does not possess Galilean invariance.

Semiclassical description. — To get more intuition about the answer obtained within the Kubo formula, we now apply the semiclassical approach to the same problem. While we show that this approach is useful for obtaining certain insight into the origin of the most relevant terms in some limits, it still has a number of limitation which do not allow for an accurate quantitative description. The most restrictive limitation is imposed by the uncertainty principle. This principle forbids a quantum particle to have a definite position and momentum simultaneously, which, in turn, is the key assumption of the semiclassical Boltzmann formalism.

We consider an electron moving in a periodic potential of a lattice with the Hamiltonian $\hat{H}_0$ in an inhomogeneous static electric field $\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r})$, such that the full Hamiltonian is given by

$$\hat{H} = \hat{H}_0 - e\varphi(\hat{\mathbf{r}}). \tag{6}$$

Hamiltonian $\hat{H}_0(\mathbf{k})$ used in the Kubo-formula derivation is the second-quantized version of $\hat{H}_0$, written in momentum space. In our further derivation, we closely follow the approach of Ref. [16]. In particular, we assume that the periodic part of the Hamiltonian (without the electrostatic potential), $\hat{H}_0$, is diagonal in the basis of the Bloch wavefunctions $|\psi_{\mathbf{k}}\rangle = e^{i\mathbf{k}\cdot\hat{\mathbf{r}}}|u(\mathbf{k})\rangle$, $\hat{H}_0|\psi_{\mathbf{k}}\rangle = \varepsilon_{\mathbf{k}}|\psi_{\mathbf{k}}\rangle$, where $\hbar\mathbf{k}$ is quasimomentum and the function $u_{\mathbf{k}}(\mathbf{r}) \equiv \langle\mathbf{r}|u(\mathbf{k})\rangle$ has the periodicity of the crystal in real space, with the normalization condition $\langle\psi_{\mathbf{k}'}|\psi_{\mathbf{k}}\rangle = \delta(\mathbf{k}-\mathbf{k}')$.

The goal now is to derive the corrections to the semiclassical equations of motion due to the final gradients of the electric field. More specifically, we are interested in the second gradient, which is equivalent to the q^2 term in the Hall conductivity calculated above. To obtain the correction, we consider the dynamics of the wave packet constructed of the states within the same band and defined as $|\Psi(t)\rangle = \int d\mathbf{k}\, a(\mathbf{k},t)|\psi_{\mathbf{k}}\rangle$. Within this single-band approximation, the Schrödinger equation $i\hbar(\partial/\partial t)|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle$ determines the dynamics of $a(\mathbf{k},t)$ as

$$i\hbar\frac{\partial a(\mathbf{k},t)}{\partial t} = \varepsilon_{\mathbf{k}}a(\mathbf{k},t) - e\int d\mathbf{k}'\, a(\mathbf{k}',t)\langle\psi_{\mathbf{k}}|\varphi(\hat{\mathbf{r}})|\psi_{\mathbf{k}'}\rangle, \tag{7}$$

with the normalization condition $\int d\mathbf{k}|a(\mathbf{k},t)|^2 = 1$ [16].

To take into account weak inhomogeneity of the electric field, we expand the electrostatic potential near $\mathbf{r} = 0$ as $\varphi(\mathbf{r}) = -E^\mu r_\mu - \frac{1}{2}E^{\mu\nu}r_\mu r_\nu - \frac{1}{6}E^{\mu\nu\xi}r_\mu r_\nu r_\xi - \ldots$, where $E^{\mu\nu\xi}$, $E^{\mu\nu}$, and E^μ are fully symmetric tensors that do not depend on $\mathbf{r}$, and the summation over the repeated indices is implied. The electric field near $\mathbf{r} = 0$ is then given by $E^\mu(\mathbf{r}) \approx E^\mu + E^{\mu\nu}r_\nu + \frac{1}{2}E^{\mu\nu\xi}r_\nu r_\xi$. The correction to the electron's velocity proportional to $E^{\mu\nu\xi}$ determines the q^2 term in the Hall conductivity, $\sigma_{AH}^{(2)}$.

To derive the semiclassical expression for the wave packet velocity, we define its position $R_\alpha(t)$ and momentum $K_\alpha(t)$ as

$$\begin{aligned} R_\alpha(t) &\equiv \langle\Psi(t)|\hat{r}_\alpha|\Psi(t)\rangle, \\ K_\alpha(t) &\equiv \langle\Psi(t)|\hat{k}_\alpha|\Psi(t)\rangle, \end{aligned} \tag{8}$$

where $\hbar\hat{k}_\alpha$ is the quasimomentum operator satisfying $\hat{k}_\alpha|\psi_{\mathbf{k}}\rangle = k_\alpha|\psi_{\mathbf{k}}\rangle$. Parameterizing function $a(\mathbf{k},t)$ as $a(\mathbf{k},t) = |a(\mathbf{k},t)|e^{-i\gamma(\mathbf{k},t)}$, one easily finds that

$$\begin{aligned} R_\alpha(t) &= \int d\mathbf{k}\, \tilde{R}_\alpha(\mathbf{k},t)|a(\mathbf{k},t)|^2, \\ K_\alpha(t) &= \int d\mathbf{k}\, k_\alpha|a(\mathbf{k},t)|^2, \end{aligned} \tag{9}$$

with

$$\tilde{R}_\alpha(\mathbf{k},t) \equiv \frac{\partial\gamma(\mathbf{k},t)}{\partial k_\alpha} + A_\alpha(\mathbf{k}), \tag{10}$$

and $A_\alpha(\mathbf{k}) = i\langle u(\mathbf{k})|\partial_{k_\alpha} u(\mathbf{k})\rangle$ is the Berry connection. If the wave packet is strongly peaked at momentum $\mathbf{K}$, $|a(\mathbf{k},t)|^2 \approx \delta(\mathbf{k}-\mathbf{K})$, semiclassical coordinate of the wave packet becomes simply $R_\alpha \approx \tilde{R}_\alpha(\mathbf{K})$.

Thus far, our semiclassical analysis was similar to that of Ref. [16]. In what follows, however, we are mostly interested in the second-order gradient correction to the semiclassical equations of motion which, to the best of our knowledge, has never been studied before. Assuming that the wavepacked is narrowly peaked in the momentum space, we find up to the order $d^2\mathbf{E}(\mathbf{R})/dR_\mu dR_\nu$ (equivalently, to the order $E^{\mu\nu\xi}$):

$$\begin{aligned}
\dot{K}^\alpha(t) &= -\frac{e}{\hbar}E^\alpha(\mathbf{R}) - \frac{e}{2\hbar}E^{\alpha\mu\nu}\left[g_{\mu\nu}(\mathbf{K}) + f_{\mu\nu}\{|a(\mathbf{k},t)|\}\right], \\
\dot{R}_\alpha(t) &= \frac{1}{\hbar}\frac{\partial\varepsilon_{\mathbf{k}}}{\partial k_\alpha} - \frac{e}{\hbar}E^\mu(\mathbf{R})\Omega_{\mu\alpha} + \frac{e}{2\hbar}\frac{\partial E^\mu(\mathbf{R})}{\partial R_\nu}\frac{\partial g_{\mu\nu}}{\partial K_\alpha} - \\
&\quad - \frac{e}{6\hbar}E^{\mu\nu\xi}\left(3g_{\mu\nu}\Omega_{\xi\alpha} - \frac{\partial T_{\mu\nu\xi}}{\partial K_\alpha} - \frac{\partial^2\Omega_{\xi\alpha}}{\partial K_\mu \partial K_\nu}\right) - \\
&\quad - \frac{e}{2\hbar}E^{\mu\nu\xi}\tilde{f}_{\mu\nu\xi\alpha}\{|a(\mathbf{k},t)|\},
\end{aligned} \tag{11}$$

with functionals $f_{\mu\nu}$ and $\tilde{f}_{\mu\nu\xi\alpha}$ given by

$$\begin{aligned}
f_{\mu\nu}\{|a(\mathbf{k},t)|\} &\equiv \int d\mathbf{k}\, \frac{\partial|a(\mathbf{k},t)|}{\partial k_\mu}\frac{\partial|a(\mathbf{k},t)|}{\partial k_\nu}, \\
\tilde{f}_{\mu\nu\xi\alpha}\{|a(\mathbf{k},t)|\} &\equiv \int d\mathbf{k}\, \frac{\partial|a(\mathbf{k},t)|}{\partial k_\mu}\frac{\partial|a(\mathbf{k},t)|}{\partial k_\nu}\Omega_{\xi\alpha}(\mathbf{k}).
\end{aligned} \tag{12}$$

Equations (11) represent the second main result of the present work. The derivation is straightforward but tedious, so we delegate it to the Supplemental Materials [25]. The first gradient correction, $E^{\mu\nu}\partial g_{\mu\nu}/\partial K_\alpha$, has been obtained and discussed in Refs. [16, 17], and our answer agrees with it. The second-order gradient term containing $E^{\mu\nu\xi}$ is a new result that deserves further discussion.

Within the kinetic equation approach, current density equals $j_\alpha = -(e/S)\sum_{\mathbf{K}} \dot{R}_\alpha f_{\mathbf{K}}$, where $f_{\mathbf{K}}$ is the distribution function. To the leading order, $f_{\mathbf{K}}$ is given by the Fermi-Dirac distribution, and the current is given by $-e\dot{R}_\alpha$ summed over the filled states, thus allowing for a simple interpretation of all the terms in Eq. (11).

One can easily show that the term with $g_{\mu\nu}\Omega_{\xi\alpha}$ exactly reproduces the first term in Eq. (2). The remaining terms in Eq. (2) contain first or second powers of the bandgap $(\varepsilon_C - \varepsilon_V)$ in the denominator and could in principle be perturbatively captured by the semiclassical approach [17, 27]; we, however, do not present such analysis in this work. Other two terms in Eq. (11) contain full derivatives $\partial T_{\mu\nu\xi}/\partial K_\alpha$ and $\partial^2\Omega_{\xi\alpha}/\partial K_\mu\partial K_\nu$, consequently, their contribution to the current vanishes in the case of a completely filled band, so they do not appear in Eq. (2) [28].

Finally, there are terms $f_{\mu\nu}$ and $\tilde{f}_{\mu\nu\xi\alpha}$ in Eq. (11) which pose the main problem for the semiclassical description of the wave packet dynamics to the second order in the electric field gradients. These terms are given by Eq. (12) and are very non-universal in a sense that they strongly depend on the shape of the wave packet, i.e., function $a(\mathbf{k},t)$. To estimate the magnitude of these terms, we may assume that $a(\mathbf{k},t)$ has form of the Gaussian distribution with the width Δk. It is clear then that $f_{\mu\nu} \propto 1/(\Delta k)^2$ and $\tilde{f}_{\mu\nu\xi\alpha} \propto \Omega_{\xi\alpha}/(\Delta k)^2$, thus diverging as $\Delta k \to 0$, which corresponds to the limit of well-defined quasiparticles in momentum space. These terms originate from the correlators $\langle\Psi(t)|\hat{r}_\mu\hat{r}_\nu|\Psi(t)\rangle$ (and higher moments) and clearly represent the Heisenberg uncertainty principle, which implies that the wavefunctions strongly localized in momentum space experience large variation with the position. While this fundamental principle is not an obstacle for the quaiclassical description at the zeroth and first order in field gradients, it clearly manifests itself at the second order. We see, however, that once the terms $f_{\mu\nu}$ and $\tilde{f}_{\mu\nu\xi\alpha}$ are neglected, our semiclassical answer well agrees with the Kubo formula calculation for an insulating case in the limit when the band separation is much larger than the bandwidth.

It is also instructive to demonstrate an alternative derivation of Eq. (11), which is less rigorous but more physically intuitive. The first equation is simply the Newton's law stating that the rate of the momentum change equals the external force: $\hbar\dot{\mathbf{K}} = \langle\Psi(t)|\mathbf{E}(\hat{\mathbf{r}})|\Psi(t)\rangle$. To derive the second equation, we introduce the effective quasiparticle energy $\varepsilon_{eff}(\mathbf{R},\mathbf{K}) = \langle\Psi(t)|\hat{H}_0 - e\varphi(\hat{\mathbf{r}})|\Psi(t)\rangle$, where $\hbar\mathbf{K}$ is the momentum of the wave packet. The equation for the effective velocity then reads as $\hbar\dot{R}_\alpha \approx (\partial\varepsilon_{eff}/\partial K_\alpha) - \hbar\Omega_{\alpha\mu}\dot{K}_\mu$, while the Newton's law can be rewritten as $\hbar\dot{\mathbf{K}} \approx -\partial\varepsilon_{eff}/\partial\mathbf{R}$ [16]. It is straightforward to check that the resulting equations are equivalent to Eq. (11). The only subtle difference originates from the singular terms analogous to Eq. (12), which we discuss in more detail in the Supplemental Materials [25].

This approach has the further advantage of elucidating the physical meaning and origin of different terms. For example, the singular terms $f_{\mu\nu}$ and $\tilde{f}_{\mu\nu\xi\alpha}$ originate from the correlator $\langle\Psi|\hat{r}_\mu\hat{r}_\nu|\Psi\rangle$, which determines the real-space width of the state and appears in the expression for $\hbar\dot{\mathbf{K}}$. While these terms are singular for the wave packets narrowly peaked in momentum space, they vanish in case of maximally localized Wannier functions, $|a(\mathbf{k})| = \text{const}$. In the latter case, the correlator can be roughly estimated by the averaged quantum metric $g_{\mu\nu}$ [23, 24]. In fact, there is a well-established procedure for how to define the width in such a way that the corresponding cumulant averaged over the filled band does not suffer from any divergencies and is given exactly by the quantum metric averaged over the same filled band [29, 30]. If for some reason further development of the semiclassical approach is necessary, it seems likely that this approach would allow for a formulation which is free of any singularities and completely agrees with the Kubo formula results found in this work. Finally, when calculating ε_{eff}, we notice that the tensor $T_{\mu\nu\xi}$ determines the gauge-invariant part of the third cumulant of the position operator $\hat{\mathbf{r}}$, $T_{\mu\nu\xi} \approx \langle\Psi|\delta\hat{r}_\mu\delta\hat{r}_\nu\delta\hat{r}_\xi|\Psi\rangle_{\text{g.-i.}}$, where $\delta\hat{r}_\mu \equiv \hat{r}_\mu - \langle\hat{r}_\mu\rangle$ [25].

Conclusions. — In conclusion, we have calculated the q^2 correction to the intrinsic anomalous Hall conductivity in the inhomogeneous electric field in clean crystals without time-reversal symmetry. To do that, we have applied the Kubo formula to a generic two-band model and then compared the results with the predictions obtained from the semiclassical approach. We showed that the two approaches agree with each other in some limits once the uncertainty principle limitations of the semiclassics are neglected. As a next step, it would be interesting to relate the newly found q^2 correction to the Hall current to possible experiments revealing the hydrodynamics of electrons in solids [31]. In particular, it is interesting to study whether $\sigma_{AH}^{(2)}$ determines the finite size correction to the Hall resistivity in the crystals with broken time-reversal symmetry, analogous to how the Hall viscosity η_{xy} does it in the narrow channel or Corbino geometry experiments in the Galilean-invariant systems in strong external magnetic field [13, 14]. We leave the study of these questions for future work.

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Supplemental Materials for "Intrinsic anomalous Hall conductivity in non-uniform electric field"

This Supplemental Material consists of two section. In the first Section, we provide the details of the intrinsic Hall conductivity calculation using the Kubo formula. The second Section is dedicated to the consideration within the semiclassical approximation and comparison between the two approaches.

I. KUBO FORMULA CALCULATION

Conductivity in a clean non-interacting system is given by the Kubo formula $\sigma_{\alpha\beta}(i\omega_n,\mathbf{q}) = -K_{\alpha\beta}(i\omega_n,\mathbf{q})/\omega_n$ [21], with the current-current correlation function $K_{\alpha\beta}(i\omega_n,\mathbf{q}) = \langle \hat{j}^\alpha(i\omega_n,\mathbf{q})\hat{j}^\beta(-i\omega_n,-\mathbf{q})\rangle$ that equals to

$$K_{\alpha\beta}(i\omega,\mathbf{q}) = -\frac{1}{S}\sum_{m,n,\mathbf{k}}\frac{n_F[\varepsilon_n(\mathbf{k})]-n_F[\varepsilon_m(\mathbf{k}+\mathbf{q})]}{\varepsilon_n(\mathbf{k})-\varepsilon_m(\mathbf{k}+\mathbf{q})+i\hbar\omega}F^{nm}_{\alpha\beta}(\mathbf{k},\mathbf{q}), \tag{S1}$$

where

$$F^{nm}_{\alpha\beta}(\mathbf{k},\mathbf{q}) \equiv \langle u_n(\mathbf{k})|\hat{j}^\alpha_{\mathbf{k}+\frac{\mathbf{q}}{2}}|u_m(\mathbf{k}+\mathbf{q})\rangle\langle u_m(\mathbf{k}+\mathbf{q})|\hat{j}^\beta_{\mathbf{k}+\frac{\mathbf{q}}{2}}|u_n(\mathbf{k})\rangle, \qquad \hat{j}^\alpha_{\mathbf{k}} \equiv \frac{e}{\hbar}\frac{\partial \hat{H}_0(\mathbf{k})}{\partial k_\alpha}. \tag{S2}$$

Here $\varepsilon_n(\mathbf{k})$ define the energy bands of the crystal, S is the total area of the system, and $n_F[\varepsilon]$ is the Fermi-Dirac distribution function. An extra minus sign in Eq. (S1) comes from the fermion loop [21]. At zero temperature, $n_F[\varepsilon]$ becomes simply the Heaviside step function, $n_F[\varepsilon] = \Theta(\varepsilon_F - \varepsilon)$, where ε_F is the Fermi energy. We emphasize that the correlator $K_{\alpha\beta}(i\omega,\mathbf{q})$ only determines the paramagnetic contribution to the total current and, in principle, the diamagnetic term should also be added. The latter, however, is purely symmetric; hence, it does not contribute to the antisymmetric (Hall) part of conductivity and will be neglected hereafter. The summation over $\mathbf{k}$ should be understood as $\frac{1}{S}\sum_{\mathbf{k}} \to \int \frac{d\mathbf{k}}{(2\pi)^2}$.

We focus on a two-level system in two dimensions without band crossings and assume that the Fermi level resides in the valence band for definiteness. Then, we separate the total response function into the sum of the interband and intraband contributions, $K_{\alpha\beta}(i\omega,\mathbf{q}) = K^{\text{inter}}_{\alpha\beta}(i\omega,\mathbf{q}) + K^{\text{intra}}_{\alpha\beta}(i\omega,\mathbf{q})$:

$$\begin{aligned} K^{\text{inter}}_{\alpha\beta}(i\omega,\mathbf{q}) &= -\frac{1}{S}\sum_{\mathbf{k}\in\text{occ.}}\frac{F^{VC}_{\alpha\beta}(\mathbf{k},\mathbf{q})}{\varepsilon_V(\mathbf{k})-\varepsilon_C(\mathbf{k}+\mathbf{q})+i\hbar\omega} + \frac{F^{VC}_{\beta\alpha}(\mathbf{k},-\mathbf{q})}{\varepsilon_V(\mathbf{k})-\varepsilon_C(\mathbf{k}-\mathbf{q})-i\hbar\omega}, \\ K^{\text{intra}}_{\alpha\beta}(i\omega,\mathbf{q}) &= -\frac{1}{S}\sum_{\mathbf{k}\in\text{occ.}}\frac{F^{VV}_{\alpha\beta}(\mathbf{k},\mathbf{q})}{\varepsilon_V(\mathbf{k})-\varepsilon_V(\mathbf{k}+\mathbf{q})+i\hbar\omega} + \frac{F^{VV}_{\beta\alpha}(\mathbf{k},-\mathbf{q})}{\varepsilon_V(\mathbf{k})-\varepsilon_V(\mathbf{k}-\mathbf{q})-i\hbar\omega}, \end{aligned} \tag{S3}$$

where indices "V" and "C" stand for "valence" and "conduction", respectively, and the summation is over the occupied states only. Below, we calculate these contributions separately, so the total Hall conductivity $\sigma_{AH}(\omega,\mathbf{q}) \equiv [\sigma_{xy}(\omega,\mathbf{q}) - \sigma_{yx}(\omega,\mathbf{q})]/2$ is given by the sum $\sigma_{AH}(\omega,\mathbf{q}) = \sigma^{\text{inter}}_{AH}(\omega,\mathbf{q}) + \sigma^{\text{intra}}_{AH}(\omega,\mathbf{q})$. The subscript "$AH$" stands for "anomalous Hall".

I.A Geometry of the band structure

Before continuing the calculation, we introduce the Berry curvature $\Omega_{\alpha\beta}(\mathbf{k})$, quantum metric $g_{\alpha\beta}(\mathbf{k})$, and the (real) symmetric rank-3 tensor $T_{\alpha\beta\gamma}(\mathbf{k})$ of the valence band as

$$\begin{aligned}
\Omega_{\alpha\beta}(\mathbf{k}) &= \frac{\partial A_\beta(\mathbf{k})}{\partial k_\alpha} - \frac{\partial A_\alpha(\mathbf{k})}{\partial k_\beta} = i\left(\left\langle \frac{\partial u_V(\mathbf{k})}{\partial k_\alpha}\middle|\frac{\partial u_V(\mathbf{k})}{\partial k_\beta}\right\rangle - \left\langle \frac{\partial u_V(\mathbf{k})}{\partial k_\beta}\middle|\frac{\partial u_V(\mathbf{k})}{\partial k_\alpha}\right\rangle\right) = -2\mathrm{Im}\left\langle \frac{\partial u_V(\mathbf{k})}{\partial k_\alpha}\middle|\frac{\partial u_V(\mathbf{k})}{\partial k_\beta}\right\rangle, \\
g_{\alpha\beta}(\mathbf{k}) &= \frac{1}{2}\left\{\left\langle \frac{\partial u_V(\mathbf{k})}{\partial k_\alpha}\middle|\frac{\partial u_V(\mathbf{k})}{\partial k_\beta}\right\rangle - \left\langle \frac{\partial u_V(\mathbf{k})}{\partial k_\alpha}\middle| u_V(\mathbf{k})\right\rangle \left\langle u_V(\mathbf{k})\middle|\frac{\partial u_V(\mathbf{k})}{\partial k_\beta}\right\rangle + (\alpha\leftrightarrow\beta)\right\}, \\
T_{\alpha\beta\gamma}(\mathbf{k}) &\equiv -\left(A_\gamma g_{\alpha\beta} + A_\alpha g_{\gamma\beta} + A_\beta g_{\alpha\gamma}\right) - A_\alpha A_\beta A_\gamma + \frac{1}{3}\left(\frac{\partial^2 A_\gamma}{\partial k_\beta \partial k_\alpha} + \frac{\partial^2 A_\alpha}{\partial k_\beta \partial k_\gamma} + \frac{\partial^2 A_\beta}{\partial k_\gamma \partial k_\alpha}\right) - \\
&\quad - \frac{i}{2}\left[2\left\langle u_V\middle|\frac{\partial^3 u_V}{\partial k_\alpha \partial k_\beta \partial k_\gamma}\right\rangle + \frac{\partial(g_{\alpha\beta}+A_\alpha A_\beta)}{\partial k_\gamma} + \frac{\partial(g_{\alpha\gamma}+A_\alpha A_\gamma)}{\partial k_\beta} + \frac{\partial(g_{\beta\gamma}+A_\beta A_\gamma)}{\partial k_\alpha}\right],
\end{aligned} \tag{S4}$$

where we have also defined the Berry connection $A_\alpha(\mathbf{k}) \equiv i\left\langle u_V(\mathbf{k})\middle|\frac{\partial u_V(\mathbf{k})}{\partial k_\alpha}\right\rangle$ (again, we suppress argument $\mathbf{k}$ in the expression for $T_{\alpha\beta\gamma}$ for brevity). All the quantities are calculated in the valence band since we assumed that the Fermi energy lies within the valence band. We point out that $\Omega_{\alpha\beta}$, $g_{\alpha\beta}$, and $T_{\alpha\beta\gamma}$ are invariant under the gauge transformations $|u(\mathbf{k})\rangle \to e^{i\phi_{\mathbf{k}}}|u(\mathbf{k})\rangle$. To emphasize this fact, we note that these quantities may be rewritten through other gauge-invariant objects, quantum geometric tensor $c_{\alpha\beta}$ and the quantum geometric connection $c_{\alpha\beta\gamma}$:

$$c_{\alpha\beta} = \langle u_V|(\partial_{k_\alpha}P_C)(\partial_{k_\beta}P_C)|u_V\rangle, \qquad c_{\alpha\beta\gamma} = \langle u_V|(\partial_{k_\alpha}\partial_{k_\beta}P_C)(\partial_{k_\gamma}P_C)|u_V\rangle, \tag{S5}$$

where $P_C = |u_C\rangle\langle u_C| = 1 - |u_V\rangle\langle u_V|$ is the projector onto the conduction band. Then, it is straightforward to check that

$$\begin{aligned}
c_{\alpha\beta} &= \left(g_{\alpha\beta} - \frac{i}{2}\Omega_{\alpha\beta}\right), \qquad c_{\alpha\beta\gamma} = -i(A_\alpha g_{\beta\gamma} + A_\beta g_{\alpha\gamma} + A_\gamma g_{\alpha\beta}) - \frac{1}{2}\left(A_\alpha\Omega_{\beta\gamma} + A_\beta\Omega_{\alpha\gamma} - A_\gamma\Omega_{\alpha\beta}\right) + i\frac{\partial^2 A_\gamma}{\partial k_\alpha \partial k_\beta} + \\
&+ \left\langle u_V\middle|\frac{\partial^3 u_V}{\partial k_\alpha\partial k_\beta\partial k_\gamma}\right\rangle + \frac{\partial}{\partial k_\beta}\left(g_{\alpha\gamma} + A_\alpha A_\gamma - \frac{i}{2}\Omega_{\alpha\gamma}\right) + \frac{\partial}{\partial k_\alpha}\left(g_{\beta\gamma} + A_\beta A_\gamma - \frac{i}{2}\Omega_{\beta\gamma}\right) - iA_\alpha A_\beta A_\gamma - A_\gamma\frac{\partial A_\beta}{\partial k_\alpha},
\end{aligned} \tag{S6}$$

so one easily finds

$$\Omega_{\alpha\beta} = -2\,\mathrm{Im}\,c_{\alpha\beta}, \qquad g_{\alpha\beta} = \mathrm{Re}\,c_{\alpha\beta}, \qquad T_{\alpha\beta\gamma} = \frac{1}{3}\,\mathrm{Im}\,(c_{\alpha\beta\gamma} + c_{\beta\gamma\alpha} + c_{\gamma\alpha\beta}), \tag{S7}$$

where in the last equality we also used

$$\frac{1}{3}(c_{\alpha\beta\gamma} + c_{\beta\gamma\alpha} + c_{\gamma\alpha\beta}) = iT_{\alpha\beta\gamma} + \frac{1}{6}\left(\frac{\partial g_{\beta\gamma}}{\partial k_\alpha} + \frac{\partial g_{\alpha\gamma}}{\partial k_\beta} + \frac{\partial g_{\alpha\beta}}{\partial k_\gamma}\right). \tag{S8}$$

The real part of $c_{\alpha\beta\gamma}$ is related to the quantum metric as

$$\mathrm{Re}\,c_{\alpha\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\beta\gamma}}{\partial k_\alpha} + \frac{\partial g_{\alpha\gamma}}{\partial k_\beta} - \frac{\partial g_{\alpha\beta}}{\partial k_\gamma}\right). \tag{S9}$$

It is the same expression that gives the Christoffel symbols of the Levi-Civita connection in terms of the corresponding metric. The geometric interpretation of the imaginary part of $c_{\alpha\beta\gamma}$ follows from the identity $\partial_{k_\gamma}\Omega_{\alpha\beta} = 2\,\mathrm{Im}\,[c_{\gamma\beta\alpha} - c_{\gamma\alpha\beta}]$ which can be rewritten as

$$\nabla_\gamma\Omega_{\alpha\beta} = \frac{\partial\Omega_{\alpha\beta}}{\partial k_\gamma} + \tilde{\Gamma}_{\alpha\gamma\beta} - \tilde{\Gamma}_{\beta\gamma\alpha} = \frac{\partial\Omega_{\alpha\beta}}{\partial k_\gamma} - \tilde{\Gamma}^\delta_{\alpha\gamma}\Omega_{\delta\beta} - \tilde{\Gamma}^\delta_{\beta\gamma}\Omega_{\alpha\delta} = 0, \tag{S10}$$

where we have defined $\Omega_{\gamma\delta}\tilde{\Gamma}^\delta_{\alpha\beta} = \tilde{\Gamma}_{\alpha\beta\gamma} = 2\,\mathrm{Im}\,c_{\alpha\beta\gamma}$, and ∇_γ is a covariant derivative. Geometrically, Eq. (S10) shows that $\tilde{\Gamma}_{\alpha\beta\gamma}$ defines a connection (a prescription for parallel transport) on the Brillouin zone that preserves the Berry curvature two-form, i.e., a symplectic connection [18]. Unlike the Levi-Civita connection that preserves the metric tensor, a symplectic connection is not uniquely determined by the 2-form. As was shown in Ref. [18], any symplectic connection can be represented in the form $\tilde{\Gamma}_{\alpha\beta\gamma} = \frac{1}{3}(\partial_\beta\Omega_{\gamma\alpha} + \partial_\alpha\Omega_{\gamma\beta}) + t_{\alpha\beta\gamma}$, where $t_{\alpha\beta\gamma}$ can be any fully symmetric rank-3 tensor. In our case, $\tilde{\Gamma}_{\alpha\beta\gamma}$ and $t_{\alpha\beta\gamma}$ are fixed by the band structure such that this relation takes form

$$\operatorname{Im} c_{\alpha\beta\gamma} = -\frac{1}{6}(\partial_\alpha\Omega_{\beta\gamma} + \partial_\beta\Omega_{\alpha\gamma}) + T_{\alpha\beta\gamma}. \tag{S11}$$

Since tensor $T_{\alpha\beta\gamma}$ is not expressed solely through the Berry curvature and quantum metric, it is a new gauge-invariant geometric object that also describes physical properties of Bloch states. As we show below in Section II, it determines the gauge-invariant part of the third cumulant of the position operator, analogous to how the quantum metric $g_{\alpha\beta}$ determines the gauge-invariant part of the second cumulant [23, 24, 29, 30].

I.B Interband contribution

The most involved part of the calculation is the evaluation of the matrix elements $F^{VC}_{\alpha\beta}(\mathbf{k},\mathbf{q})$ and $F^{VV}_{\alpha\beta}(\mathbf{k},\mathbf{q})$. In this subsection, we focus on $F^{VC}_{\alpha\beta}(\mathbf{k},\mathbf{q})$ (interband contribution). Choosing vector $\mathbf{q}$ to be along the x-axis, $\mathbf{q} = q\hat{x}$, one can expand it into the Taylor series:

$$F^{VC}_{\alpha\beta}(\mathbf{k},\mathbf{q}) = \frac{e^2}{\hbar^2}\left[f^{VC}_{0\alpha\beta}(\mathbf{k}) + f^{VC}_{1\alpha\beta}(\mathbf{k})q + f^{VC}_{2\alpha\beta}(\mathbf{k})q^2 + \dots\right]. \tag{S12}$$

The calculation of $f^{VC}_{i\alpha\beta}(\mathbf{k})$ is tedious but straightforward, and we find:

$$\begin{aligned}
f^{VC}_{0\alpha\beta}(\mathbf{k}) &= (\varepsilon_C - \varepsilon_V)^2 c_{\alpha\beta},\\
f^{VC}_{1\alpha\beta}(\mathbf{k}) &= \left[\frac{(\varepsilon_C - \varepsilon_V)^2}{2}c_{\alpha\beta}\right]' + \frac{(\varepsilon_C - \varepsilon_V)}{2}\hbar\left[(v_{C\alpha} + v_{V\alpha})c_{x\beta} + (v_{C\beta} + v_{V\beta})c_{\alpha x}\right],\\
f^{VC}_{2\alpha\beta}(\mathbf{k}) &= \left[\frac{(\varepsilon_C - \varepsilon_V)^2}{8}c_{\alpha\beta}\right]'' + \frac{(\varepsilon_C - \varepsilon_V)^2}{4}\left\{c_{x\alpha}c_{x\beta} + c_{\alpha x}c_{\beta x} - 2c_{xx}c_{\alpha\beta}\right\} +\\
&\quad + \frac{\varepsilon_C - \varepsilon_V}{4}\hbar\left\{\left[(v_{C\alpha} + v_{V\alpha})c_{x\beta} + (v_{C\beta} + v_{V\beta})c_{\alpha x}\right]' - \frac{v_{C\alpha} - v_{V\alpha}}{2}c_{xx\beta} - \frac{v_{C\beta} - v_{V\beta}}{2}\bar{c}_{xx\alpha}\right\} +\\
&\quad + \frac{\hbar^2}{4}\left\{(v_{C\alpha} + v_{V\alpha})(v_{C\beta} + v_{V\beta})c_{xx} + (v_{Cx} - v_{Vx})(v_{C\alpha} + v_{V\alpha})c_{x\beta} + (v_{Cx} - v_{Vx})(v_{C\beta} + v_{V\beta})c_{\alpha x}\right\},
\end{aligned} \tag{S13}$$

where we suppressed argument $\mathbf{k}$ on the right hand side of all the equalities for brevity, and tensors $c_{\alpha\beta}$ and $c_{\alpha\beta\gamma}$ are defined in Eqs. (S5)-(S6). Symbol "$'$" implies the derivative with the respect to k_x, i.e., $f'(\mathbf{k}) \equiv \partial f(\mathbf{k})/\partial k_x$, and we have defined the band velocities $v_{V(C)\alpha} \equiv \partial_{k_\alpha}\varepsilon_{V(C)}/\hbar$. When deriving the above expressions, we found it useful to exploit the relation

$$\left\langle u_n(\mathbf{k})\middle|\frac{\partial \hat{H}_0(\mathbf{k})}{\partial k_\alpha}\middle|u_m(\mathbf{k})\right\rangle = \delta_{nm}\hbar v_{n\alpha}(\mathbf{k}) + \left[\varepsilon_m(\mathbf{k}) - \varepsilon_n(\mathbf{k})\right]\left\langle u_n(\mathbf{k})\middle|\frac{\partial u_m(\mathbf{k})}{\partial k_\alpha}\right\rangle, \qquad \langle u_n(\mathbf{k})|u_m(\mathbf{k})\rangle = \delta_{nm}, \tag{S14}$$

where n and m label bands.

Since we are interested in the Hall conductivity, we only need the antisymmetric parts of functions $f^{VC}_{i\alpha\beta}(\mathbf{k})$. After antisymmetrization, we find for functions $\tilde{f}^{VC}_{i\alpha\beta} \equiv f^{VC}_{i\alpha\beta} - f^{VC}_{i\beta\alpha}$:

$$\begin{aligned}
\tilde{f}^{VC}_{0\alpha\beta}(\mathbf{k}) &= -i(\varepsilon_C - \varepsilon_V)^2\Omega_{\alpha\beta},\\
\tilde{f}^{VC}_{1\alpha\beta}(\mathbf{k}) &= -\frac{i}{2}\left[(\varepsilon_C - \varepsilon_V)^2\Omega_{\alpha\beta}\right]' + \frac{i}{2}\hbar(\varepsilon_C - \varepsilon_V)\left[(v_{C\beta} + v_{V\beta})\Omega_{x\alpha} - (v_{C\alpha} + v_{V\alpha})\Omega_{x\beta}\right],\\
\tilde{f}^{VC}_{2\alpha\beta}(\mathbf{k}) &= -\frac{i}{16}\left[(\varepsilon_C - \varepsilon_V)^2\Omega_{\alpha\beta}\right]'' + \frac{i}{4}\hbar\left[(\varepsilon_C - \varepsilon_V)(v_{C\beta} + v_{V\beta})\Omega_{x\alpha}\right]' + \frac{i}{4}(\varepsilon_C - \varepsilon_V)^2 g_{xx}\Omega_{\alpha\beta} +\\
&\quad + \frac{i}{4}\hbar(\varepsilon_C - \varepsilon_V)(v_{C\alpha} - v_{V\alpha})\left(\frac{1}{3}\Omega'_{x\beta} - T_{xx\beta}\right) - (\alpha \leftrightarrow \beta).
\end{aligned} \tag{S15}$$

Expanding Eq. (S3) in small ω and q up to the order $O(\omega q^2, \omega^2 q, \omega^3)$, we find for the antisymmetrized interband contribution to the current-current correlation function, $\tilde{K}^{\rm inter}_{\alpha\beta}(i\omega, q) \equiv [K^{\rm inter}_{\alpha\beta}(i\omega, q) - K^{\rm inter}_{\beta\alpha}(i\omega, q)]/2$:

$$
\begin{aligned}
&\tilde{K}^{\rm inter}_{xy}(i\omega, q) \approx -\frac{e^2}{\hbar^2}\frac{1}{S}\sum_{\mathbf{k}\in \rm occ.} \frac{i}{2}q\left[(\varepsilon_C - \varepsilon_V)\Omega_{xy}\right]' - \hbar\omega\Omega_{xy} + \frac{\hbar\omega q^2}{2}\left\{g_{xx}\Omega_{xy} - \frac{\Omega''_{xy}}{4} + \right.\\
&+\frac{\hbar}{\varepsilon_C - \varepsilon_V}\left[(v_{Cx}\Omega_{xy})' - \frac{v_{Cx} - v_{Vx}}{3}\Omega'_{xy} - \frac{v_{Cx} - v_{Vx}}{2}T_{xxy} + \frac{v_{Cy} - v_{Vy}}{2}T_{xxx}\right] - \left.\hbar^2\frac{v_{Cx}(v_{Cx} + v_{Vx})}{(\varepsilon_C - \varepsilon_V)^2}\Omega_{xy}\right\} - \\
&- i\hbar^2\omega^2 q\left\{\frac{1}{2}\left(\frac{\Omega_{xy}}{\varepsilon_C - \varepsilon_V}\right)' - \hbar\frac{v_{Cx} + v_{Vx}}{(\varepsilon_C - \varepsilon_V)^2}\Omega_{xy}\right\} + \hbar^3\omega^3\frac{\Omega_{xy}}{(\varepsilon_C - \varepsilon_V)^2}.
\end{aligned}
\tag{S16}
$$

We note that that we have neglected the q^3 contribution here. The interband contribution to the Hall conductivity is determined as $\sigma^{\rm inter}_{AH}(i\omega_n, \mathbf{q}) \equiv [\sigma^{\rm inter}_{xy}(i\omega_n, \mathbf{q}) - \sigma^{\rm inter}_{yx}(i\omega_n, \mathbf{q})]/2 = -\tilde{K}^{\rm inter}_{xy}(i\omega_n, \mathbf{q})/\omega_n$, which after the analytical continuation $i\omega \to \omega + i\delta$ becomes (δ is an infinitesimal positive number which physically corresponds to the single-particle scattering rate)

$$
\begin{aligned}
&\sigma^{\rm inter}_{AH}(\omega, q) \approx -\frac{e^2}{\hbar}\frac{1}{S}\sum_{\mathbf{k}\in \rm occ.} \Omega_{xy} + \frac{q}{2\hbar\omega}\left[(\varepsilon_C - \varepsilon_V)\Omega_{xy}\right]' - \frac{q^2}{2}\left\{g_{xx}\Omega_{xy} - \frac{\Omega''_{xy}}{4} + \right.\\
&+\frac{\hbar}{\varepsilon_C - \varepsilon_V}\left[(v_{Cx}\Omega_{xy})' - \frac{v_{Cx} - v_{Vx}}{3}\Omega'_{xy} - \frac{v_{Cx} - v_{Vx}}{2}T_{xxy} + \frac{v_{Cy} - v_{Vy}}{2}T_{xxx}\right] - \left.\hbar^2\frac{v_{Cx}(v_{Cx} + v_{Vx})}{(\varepsilon_C - \varepsilon_V)^2}\Omega_{xy}\right\} + \\
&+ \hbar\omega q\left\{\frac{1}{2}\left(\frac{\Omega_{xy}}{\varepsilon_C - \varepsilon_V}\right)' - \hbar\frac{v_{Cx} + v_{Vx}}{(\varepsilon_C - \varepsilon_V)^2}\Omega_{xy}\right\} + \hbar^2\omega^2\frac{\Omega_{xy}}{(\varepsilon_C - \varepsilon_V)^2}.
\end{aligned}
\tag{S17}
$$

The first term in this expression, Ω_{xy}, is the conventional (uniform) intrinsic contribution to the Hall conductivity. The second term is proportional to q/ω and, in principle, can be larger than the q^2 contribution, which is the main focus of this work. However, this term vanishes in the insulating state after summation over $\mathbf{k}$ (since it is proportional to a full derivative). Also, as we show below, this term exactly cancels the corresponding q/ω term from the intraband contribution in the static limit, $\omega \ll v_F q$, where v_F is some typical Fermi velocity at the Fermi level. In the optical limit, $\omega \gg v_F q$, this term must be taken into account in the metallic regime.

The third term is proportional to q^2 and in the insulating regime (when all full derivatives can be discarded) exactly reproduces Eq. (2) of the main text. Finally, the terms proportional to ωq and ω^2 can be neglected in the static limit or when the bandgap, $\varepsilon_C - \varepsilon_V$, is much larger than the bandwidth. Otherwise, these terms must be taken into account as well.

I.C Intraband contribution

The calculation of the intraband contribution is similar but much more lengthy. Since there is no bandgap $\varepsilon_C - \varepsilon_V$ in the denominator of the intraband term in Eq. (S3), we need to expand $F^{VV}_{\alpha\beta}(\mathbf{k}, \mathbf{q})$ up to the fourth order in $\mathbf{q}$. Assuming again that $\mathbf{q}$ is along the x-axis, one can write

$$
F^{VV}_{\alpha\beta}(\mathbf{k}, \mathbf{q}) = \frac{e^2}{\hbar^2}\left[f^{VV}_{0\alpha\beta}(\mathbf{k}) + f^{VV}_{1\alpha\beta}(\mathbf{k})q + f^{VV}_{2\alpha\beta}(\mathbf{k})q^2 + f^{VV}_{3\alpha\beta}(\mathbf{k})q^3 + f^{VV}_{4\alpha\beta}(\mathbf{k})q^4 + \ldots\right].
\tag{S18}
$$

As we are interested in the antisymmetric part only, we directly calculate $\tilde{f}^{VV}_{i\alpha\beta}(\mathbf{k}) \equiv f^{VV}_{i\alpha\beta}(\mathbf{k}) - f^{VV}_{i\beta\alpha}(\mathbf{k})$. After lengthy calculation, we find

$$
\begin{aligned}
&\tilde{f}^{VV}_{0\alpha\beta}(\mathbf{k}) = 0, \qquad \tilde{f}^{VV}_{1\alpha\beta}(\mathbf{k}) = i\hbar[\varepsilon_C(\mathbf{k}) - \varepsilon_V(\mathbf{k})][v_{V\alpha}(\mathbf{k})\Omega_{x\beta}(\mathbf{k}) - v_{V\beta}(\mathbf{k})\Omega_{x\alpha}(\mathbf{k})], \qquad \tilde{f}^{VV}_{2\alpha\beta}(\mathbf{k}) = \frac{1}{2}\left[\tilde{f}^{VV}_{1\alpha\beta}(\mathbf{k})\right]',\\
&\tilde{f}^{VV}_{3\alpha\beta}(\mathbf{k}) = \frac{i}{8}\hbar\left[(\varepsilon_C - \varepsilon_V)\Omega_{x\beta}v_{V\alpha}\right]'' - \frac{i}{4}\hbar(\varepsilon_C - \varepsilon_V)(v_{V\alpha} + v_{C\alpha})g_{xx}\Omega_{x\beta} - \frac{i}{8}(\varepsilon_C - \varepsilon_V)^2\Omega_{x\alpha}\left(\frac{\partial g_{xx}}{\partial k_\beta} - 2g'_{x\beta}\right) - \\
&- \frac{i}{4}\hbar^2 v_{V\alpha}v_{C\beta}T_{xxx} + \frac{i(\varepsilon_C - \varepsilon_V)v_{V\alpha}}{4}\hbar\left[-g_{xx}\Omega_{x\beta} + \frac{1}{12}\Omega''_{x\beta} - \frac{T'_{xx\beta}}{2} + \frac{1}{6}\frac{\partial T_{xxx}}{\partial k_\beta}\right] - (\alpha \leftrightarrow \beta),\\
&\tilde{f}^{VV}_{4\alpha\beta}(\mathbf{k}) = \frac{1}{2}\left[\tilde{f}^{VV}_{3\alpha\beta}(\mathbf{k})\right]' - \frac{1}{24}\left[\tilde{f}^{VV}_{1\alpha\beta}(\mathbf{k})\right]^{(3)}.
\end{aligned}
\tag{S19}
$$

Unlike the interband contribution which could be expanded in small ω and q simultaneously, the intraband term is very sensitive to the order of limits, i.e., whether the system is in the optical or static limit. In the optical limit, $\omega \gg v_F q$, we find to the leading order for $\tilde{K}^{\rm intra}_{xy}(\omega, q) \equiv [K^{\rm intra}_{xy}(\omega, q) - K^{\rm intra}_{yx}(\omega, q)]/2$

$$\tilde{K}^{\rm intra}_{xy}(\omega \gg v_F q) = -\frac{e^2}{\hbar^2}\frac{q^2}{\hbar\omega}\frac{1}{S}\sum_{\mathbf{k}\in\text{occ.}} \tilde{f}^{VV}_{2xy}(\mathbf{k}) = \frac{e^2}{\hbar^2}\frac{q^2}{2i\omega}\frac{1}{S}\sum_{\mathbf{k}\in\text{occ.}} [(\varepsilon_C - \varepsilon_V)v_{Vx}\Omega_{xy}]' \,, \tag{S20}$$

which contributes to the Hall conductivity term

$$\sigma^{\rm intra}_{\rm Hall}(\omega \gg v_F q) = -\frac{e^2}{\hbar^2}\frac{q^2}{2\omega^2}\frac{1}{S}\sum_{\mathbf{k}\in\text{occ.}} [(\varepsilon_C - \varepsilon_V)v_{Vx}\Omega_{xy}]' \,. \tag{S21}$$

This term is of the order q^2/ω^2 and hence can be neglected compared to the $O(q/\omega)$ interband contribution (though it still can be larger than the $O(q^2)$ interband contribution).

In the static limit, $\omega \ll v_F q$, the intraband contribution equals (to the linear order in ω and quadratic order in q)

$$\tilde{K}^{\rm intra}_{xy}(\omega \ll v_F q) \approx i\omega\sigma_{\rm intra}(q) + \frac{i}{2}\frac{e^2}{\hbar^2}q\frac{1}{S}\sum_{\mathbf{k}\in\text{occ.}} [(\varepsilon_C - \varepsilon_V)\Omega_{xy}]' \,, \tag{S22}$$

with

$$\begin{aligned}
&\sigma_{\rm intra}(q) \approx \sigma^{(0)}_{\rm intra} + q^2\sigma^{(2)}_{\rm intra}, \\
&\sigma^{(0)}_{\rm intra} = -\frac{i}{2}\frac{e^2}{\hbar^3}\frac{1}{S}\sum_{\mathbf{k}\in\text{occ.}}\left(\frac{\tilde{f}^{VV}_{1xy}}{(v_{Vx} - i\delta)^2}\right)', \qquad \sigma^{(2)}_{\rm intra} = -\frac{i}{2}\frac{e^2}{\hbar^3}\frac{1}{S}\sum_{\mathbf{k}\in\text{occ.}}\left(\frac{\tilde{f}^{VV}_{3xy}}{(v_{Vx} - i\delta)^2}\right)' - \frac{1}{24}\left(\frac{\tilde{f}^{VV\prime}_{1xy}}{(v_{Vx} - i\delta)^2}\right)'' - \\
&-\frac{5}{24}\left(\frac{\tilde{f}^{VV\prime\prime}_{1xy}}{(v_{Vx} - i\delta)^2}\right)' + \frac{1}{4}\left[\frac{1}{v_{Vx} - i\delta}\left(\frac{\tilde{f}^{VV}_{1xy}}{v_{Vx} - i\delta}\right)''\right]' - \frac{1}{12}\left[\frac{1}{v_{Vx} - i\delta}\left(\frac{\tilde{f}^{VV}_{1xy}}{v_{Vx} - i\delta}\right)'\right]'' ,
\end{aligned} \tag{S23}$$

and δ originates from the finite scattering rate. We take $\delta \to +0$ since we consider clean non-interacting systems; it is used to regularize integrals over $\mathbf{k}$.

For the intraband contribution to the Hall conductivity, $\sigma^{\rm intra}_{AH}(\omega, q) \equiv [\sigma^{\rm intra}_{xy}(\omega, q) - \sigma^{\rm intra}_{yx}(\omega, q)]/2$, we then find in the static limit

$$\sigma^{\rm intra}_{AH}(\omega \ll v_F q) \approx \sigma_{\rm intra}(q) + \frac{e^2}{2\hbar^2}\frac{q}{\omega}\frac{1}{S}\sum_{\mathbf{k}\in\text{occ.}} [(\varepsilon_C - \varepsilon_V)\Omega_{xy}]' \,. \tag{S24}$$

The second term exactly cancels the corresponding q/ω term from the interband contribution. As expected, the intraband contribution to the Hall conductivity vanishes when the band is fully filled. Formally, it follows from the fact that the intraband term can be written as a full derivative, hence, it is determined by the vicinity of the Fermi surface.

II. SEMICLASSICAL APPROACH

II.A Semiclassical equations of motion

Now we compare our results obtained from the Kubo formula with the predictions of the semiclassical approach. To do that, we first derive the gradient expansion for the semiclassical equations of motion for an electron in a crystal in the presence of a slowly varying electrical field $\mathbf{E}(\mathbf{r})$ up to the order $\partial^2_{\mathbf{r}}\mathbf{E}(\mathbf{r})$. In our derivation, we closely follow the procedure described in detail by Lapa and Hughes in Ref. 16, making exactly the same assumptions. In particular, we focus on a single band only and neglect the possible contributions of the terms coming from the boundary of the Brillouin zone.

We start with the Hamiltonian for a non-interacting particle in the periodic crystalline field $\hat{H}_0$ in the presence of an external slowly varying electric field $\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\varphi(\mathbf{r})$, which near $\mathbf{r} = 0$ can be expanded into the Taylor series:

$$\hat{H} = \hat{H}_0 - e\varphi(\hat{\mathbf{r}}), \qquad \varphi(\mathbf{r}) = -E^{\mu} r_\mu - \frac{1}{2}E^{\mu\nu} r_\mu r_\nu - \frac{1}{6}E^{\mu\nu\xi} r_\mu r_\nu r_\xi - \dots \tag{S25}$$

Tensors E^{μ}, $E^{\mu\nu}$, and $E^{\mu\nu\xi}$ are fully symmetric and do not depend on $\mathbf{r}$. Hamiltonian $\hat{H}_0$ is the first-quantized version of $\hat{H}_0(\mathbf{k})$ used in the Kubo formula, rewritten in the coordinate space. The eigenstates of $\hat{H}_0$ are labeled by quasimomentum $\hbar\mathbf{k}$ and given by $|\psi_{\mathbf{k}}\rangle$:

$$\hat{H}_0|\psi_{\mathbf{k}}\rangle = \varepsilon_{\mathbf{k}}|\psi_{\mathbf{k}}\rangle, \qquad |\psi_{\mathbf{k}}\rangle = e^{i\mathbf{k}\cdot\hat{\mathbf{r}}}|u_{\mathbf{k}}\rangle, \qquad \langle\psi_{\mathbf{k}}|\psi_{\mathbf{k}'}\rangle = \delta(\mathbf{k}-\mathbf{k}'), \qquad \langle u_{\mathbf{k}}|u_{\mathbf{k}}\rangle = 1, \tag{S26}$$

where $u_{\mathbf{k}}(\mathbf{r}) = \langle\mathbf{r}|u_{\mathbf{k}}\rangle$ has the periodicity of the crystal in the coordinate space. All the inner products involving $|u_{\mathbf{k}}\rangle$ or its derivatives imply the real-space integration over the unit cell multiplied by $(2\pi)^2/S_c$, where S_c is the volume (area) of the unit cell.

The following matrix elements are useful for the further derivation:

$$\begin{aligned}
&\langle\psi_{\mathbf{k}}|\hat{r}_\mu|\psi_{\mathbf{k}'}\rangle = i\frac{\partial}{\partial k_\mu}\delta(\mathbf{k}-\mathbf{k}') + A_\mu(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}'),\\
&\langle\psi_{\mathbf{k}}|\hat{r}_\mu\hat{r}_\nu|\psi_{\mathbf{k}'}\rangle = \frac{\partial^2}{\partial k_\mu \partial k'_\nu}\delta(\mathbf{k}-\mathbf{k}') + iA_\mu(\mathbf{k})\frac{\partial}{\partial k_\nu}\delta(\mathbf{k}-\mathbf{k}') + iA_\nu(\mathbf{k})\frac{\partial}{\partial k_\mu}\delta(\mathbf{k}-\mathbf{k}') - \delta(\mathbf{k}-\mathbf{k}')\left\langle u_{\mathbf{k}}\middle|\frac{\partial^2 u_{\mathbf{k}}}{\partial k_\mu \partial k_\nu}\right\rangle,\\
&\langle\psi_{\mathbf{k}}|\hat{r}_\mu\hat{r}_\nu\hat{r}_\xi|\psi_{\mathbf{k}'}\rangle = -i\frac{\partial^3}{\partial k_\mu \partial k_\nu \partial k_\xi}\delta(\mathbf{k}-\mathbf{k}') - A_\mu(\mathbf{k})\frac{\partial^2\delta(\mathbf{k}-\mathbf{k}')}{\partial k_\nu \partial k_\xi} - A_\nu(\mathbf{k})\frac{\partial^2\delta(\mathbf{k}-\mathbf{k}')}{\partial k_\mu \partial k_\xi} - A_\xi(\mathbf{k})\frac{\partial^2\delta(\mathbf{k}-\mathbf{k}')}{\partial k_\nu \partial k_\mu} -\\
&- i\left\langle u_{\mathbf{k}}\middle|\frac{\partial^2 u_{\mathbf{k}}}{\partial k_\mu \partial k_\nu}\right\rangle\frac{\partial\delta(\mathbf{k}-\mathbf{k}')}{\partial k_\xi} - i\left\langle u_{\mathbf{k}}\middle|\frac{\partial^2 u_{\mathbf{k}}}{\partial k_\mu \partial k_\xi}\right\rangle\frac{\partial\delta(\mathbf{k}-\mathbf{k}')}{\partial k_\nu} - i\left\langle u_{\mathbf{k}}\middle|\frac{\partial^2 u_{\mathbf{k}}}{\partial k_\xi \partial k_\nu}\right\rangle\frac{\partial\delta(\mathbf{k}-\mathbf{k}')}{\partial k_\mu} -\\
&- i\delta(\mathbf{k}-\mathbf{k}')\left\langle u_{\mathbf{k}}\middle|\frac{\partial^3 u_{\mathbf{k}}}{\partial k_\mu \partial k_\nu \partial k_\xi}\right\rangle.
\end{aligned} \tag{S27}$$

Now we study the dynamics of a wave packet $|\Psi(t)\rangle$ parameterized by $a(\mathbf{k},t)$:

$$|\Psi(t)\rangle = \int d\mathbf{k}'a(\mathbf{k}',t)|\psi_{\mathbf{k}'}\rangle. \tag{S28}$$

We emphasize again that the wave packet $|\Psi(t)\rangle$ is assumed to be constructed from the states within a single band. The function $a(\mathbf{k},t)$ satisfies the normalization condition $\int d\mathbf{k}\,|a(\mathbf{k},t)|^2 = 1$ and obeys the usual Schrödinger equation:

$$i\hbar\frac{\partial a(\mathbf{k},t)}{\partial t} = \varepsilon_{\mathbf{k}}a(\mathbf{k},t) - e\int d\mathbf{k}'\langle\psi_{\mathbf{k}}|\varphi(\hat{\mathbf{r}})|\psi_{\mathbf{k}'}\rangle a(\mathbf{k}',t). \tag{S29}$$

Next, we derive the equations of motion for the semiclassical coordinate $R_\mu(t)$ and momentum $K_\mu(t)$ defined as

$$R_\mu(t) \equiv \langle\Psi(t)|\hat{r}_\mu|\Psi(t)\rangle, \qquad K_\mu(t) \equiv \langle\Psi(t)|\hat{k}_\mu|\Psi(t)\rangle, \tag{S30}$$

where we have introduced the quasimomentum operator $\hbar\hat{k}_\mu$ as $\hat{k}_\mu|\psi_{\mathbf{k}}\rangle = k_\mu|\psi_{\mathbf{k}}\rangle$. Rewriting function $a(\mathbf{k},t)$ as $a(\mathbf{k},t) = |a(\mathbf{k},t)|e^{-i\gamma(\mathbf{k},t)}$, we easily find

$$R_\mu(t) = \int d\mathbf{k}\,\tilde{R}_\mu(\mathbf{k},t)|a(\mathbf{k},t)|^2, \qquad K_\mu(t) = \int d\mathbf{k}\,k_\mu|a(\mathbf{k},t)|^2, \tag{S31}$$

where we have also defined

$$\tilde{R}_\mu(\mathbf{k},t) \equiv \frac{i}{2}\left[\frac{1}{a(\mathbf{k},t)}\cdot\frac{\partial a(\mathbf{k},t)}{\partial k_\mu} - \frac{1}{a^*(\mathbf{k},t)}\cdot\frac{\partial a^*(\mathbf{k},t)}{\partial k_\mu}\right] + A_\mu(\mathbf{k}) = \frac{\partial\gamma(\mathbf{k},t)}{\partial k_\alpha} + A_\mu(\mathbf{k}). \tag{S32}$$

The equations of motion then read as

$$\begin{aligned}
&\dot{K}_\mu(t) = -\frac{ie}{\hbar}\int d\mathbf{k}d\mathbf{k}'k_\mu\left\{a(\mathbf{k},t)a^*(\mathbf{k}',t)\langle\psi_{\mathbf{k}'}|\varphi(\hat{\mathbf{r}})|\psi_{\mathbf{k}}\rangle - a^*(\mathbf{k},t)a(\mathbf{k}',t)\langle\psi_{\mathbf{k}}|\varphi(\hat{\mathbf{r}})|\psi_{\mathbf{k}'}\rangle\right\},\\
&\dot{R}_\mu(t) = \frac{1}{\hbar}\int d\mathbf{k}\frac{\partial\varepsilon_{\mathbf{k}}}{\partial k_\mu}|a(\mathbf{k},t)|^2 + \frac{e}{\hbar}\int d\mathbf{k}d\mathbf{k}'\left\{\frac{\partial a(\mathbf{k},t)}{\partial k_\mu}a^*(\mathbf{k}',t)\langle\psi_{\mathbf{k}'}|\varphi(\hat{\mathbf{r}})|\psi_{\mathbf{k}}\rangle + \frac{\partial a^*(\mathbf{k},t)}{\partial k_\mu}a(\mathbf{k}',t)\langle\psi_{\mathbf{k}}|\varphi(\hat{\mathbf{r}})|\psi_{\mathbf{k}'}\rangle\right\} -\\
&\qquad - \frac{ie}{\hbar}\int d\mathbf{k}d\mathbf{k}'A_\mu(\mathbf{k})\left\{a(\mathbf{k},t)a^*(\mathbf{k}',t)\langle\psi_{\mathbf{k}'}|\varphi(\hat{\mathbf{r}})|\psi_{\mathbf{k}}\rangle - a^*(\mathbf{k},t)a(\mathbf{k}',t)\langle\psi_{\mathbf{k}}|\varphi(\hat{\mathbf{r}})|\psi_{\mathbf{k}'}\rangle\right\}.
\end{aligned} \tag{S33}$$

Next, we expand $\varphi(\mathbf{r})$ in powers of $\mathbf{r}$ and evaluate the above equations order by order in gradients. We write the answer in the form

$$\dot{R}_\alpha(t) = \dot{R}^{(0)}_\alpha(t) + E^\mu \dot{R}^{(1)}_{\alpha\mu}(t) + \frac{1}{2}E^{\mu\nu}\dot{R}^{(2)}_{\alpha\mu\nu}(t) + \frac{1}{6}E^{\mu\nu\xi}\dot{R}^{(3)}_{\alpha\mu\nu\xi}(t) + \dots$$
$$\dot{K}_\alpha(t) = E^\mu \dot{K}^{(1)}_{\alpha\mu}(t) + \frac{1}{2}E^{\mu\nu}\dot{K}^{(2)}_{\alpha\mu\nu}(t) + \frac{1}{6}E^{\mu\nu\xi}\dot{K}^{(3)}_{\alpha\mu\nu\xi}(t) + \dots \tag{S34}$$

After straightforward calculation, we find

$$\dot{K}^{(1)}_{\alpha\mu}(t) = -\frac{e}{\hbar}\delta_{\alpha\mu}, \qquad \dot{K}^{(2)}_{\alpha\mu\nu}(t) = -\frac{e}{\hbar}\left[R_\mu(t)\delta_{\alpha\nu} + R_\nu(t)\delta_{\alpha\mu}\right],$$
$$\dot{K}^{(3)}_{\alpha\mu\nu\xi}(t) = -\frac{e}{\hbar}\left\{\delta_{\alpha\xi}\langle\Psi(t)|\hat{r}_\mu\hat{r}_\nu|\Psi(t)\rangle + \delta_{\alpha\mu}\langle\Psi(t)|\hat{r}_\xi\hat{r}_\nu|\Psi(t)\rangle + \delta_{\alpha\nu}\langle\Psi(t)|\hat{r}_\mu\hat{r}_\xi|\Psi(t)\rangle\right\}, \tag{S35}$$

where $R_\mu(t)$ is given by Eqs. (S31)-(S32), and

$$\langle\Psi(t)|\hat{r}_\mu\hat{r}_\nu|\Psi(t)\rangle = \int d\mathbf{k}\left[\tilde{R}_\mu(\mathbf{k},t)\tilde{R}_\nu(\mathbf{k},t) + g_{\mu\nu}(\mathbf{k})\right]|a(\mathbf{k},t)|^2 + \frac{1}{4|a(\mathbf{k},t)|^2}\frac{\partial|a(\mathbf{k},t)|^2}{\partial k_\mu}\frac{\partial|a(\mathbf{k},t)|^2}{\partial k_\nu}. \tag{S36}$$

The calculation for $\dot{R}_\alpha(t)$ is also straightforward, but much more tedious. After some work, we find

$$\dot{R}^{(0)}_\alpha(t) = \frac{1}{\hbar}\int d\mathbf{k}\,\frac{\partial\varepsilon_\mathbf{k}}{\partial k_\alpha}|a(\mathbf{k},t)|^2, \qquad \dot{R}^{(1)}_{\alpha\mu}(t) = -\frac{e}{\hbar}\int d\mathbf{k}\,\Omega_{\mu\alpha}(\mathbf{k})|a(\mathbf{k},t)|^2,$$
$$\dot{R}^{(2)}_{\alpha\mu\nu}(t) = -\frac{e}{\hbar}\int d\mathbf{k}\left\{\tilde{R}_\nu(\mathbf{k},t)\Omega_{\mu\alpha}(\mathbf{k}) + \tilde{R}_\mu(\mathbf{k},t)\Omega_{\nu\alpha}(\mathbf{k}) - \frac{\partial g_{\mu\nu}(\mathbf{k})}{\partial k_\alpha}\right\}|a(\mathbf{k},t)|^2,$$
$$\dot{R}^{(3)}_{\alpha\mu\nu\xi}(t) = -\frac{e}{\hbar}\int d\mathbf{k}\left\{\tilde{R}_\nu\tilde{R}_\mu\Omega_{\xi\alpha} + \tilde{R}_\nu\tilde{R}_\xi\Omega_{\mu\alpha} + \tilde{R}_\xi\tilde{R}_\mu\Omega_{\nu\alpha} - \tilde{R}_\xi\frac{\partial g_{\mu\nu}}{\partial k_\alpha} - \tilde{R}_\mu\frac{\partial g_{\xi\nu}}{\partial k_\alpha} - \tilde{R}_\nu\frac{\partial g_{\mu\xi}}{\partial k_\alpha} + T_{\alpha\mu\nu\xi}\right\}|a|^2 +$$
$$+\frac{1}{4|a|^2}\left\{\frac{\partial|a|^2}{\partial k_\mu}\frac{\partial|a|^2}{\partial k_\nu}\Omega_{\xi\alpha} + \frac{\partial|a|^2}{\partial k_\mu}\frac{\partial|a|^2}{\partial k_\xi}\Omega_{\nu\alpha} + \frac{\partial|a|^2}{\partial k_\nu}\frac{\partial|a|^2}{\partial k_\xi}\Omega_{\mu\alpha}\right\}, \tag{S37}$$

and we have suppressed the indices $\mathbf{k}$ and t in the expression for $\dot{R}^{(3)}_{\alpha\mu\nu\xi}(t)$ for brevity. Tensor $T_{\alpha\mu\nu\xi}$ is defined as

$$T_{\alpha\mu\nu\xi} = -\frac{\partial T_{\mu\nu\xi}}{\partial k_\alpha} + g_{\nu\mu}\Omega_{\xi\alpha} + g_{\nu\xi}\Omega_{\mu\alpha} + g_{\xi\mu}\Omega_{\nu\alpha} - \frac{1}{3}\left(\frac{\partial^2\Omega_{\xi\alpha}}{\partial k_\nu\partial k_\mu} + \frac{\partial^2\Omega_{\mu\alpha}}{\partial k_\nu\partial k_\xi} + \frac{\partial^2\Omega_{\nu\alpha}}{\partial k_\xi\partial k_\mu}\right), \tag{S38}$$

and tensor $T_{\mu\nu\xi}$ is defined by Eqs. (S4)-(S7).

The above equations of motion are exact in a sense that they describe the evolution of the correlation functions for any shape of the wave packet $a(\mathbf{k},t)$ (under the assumption that the wave packet is entirely composed of the states within the same band). They allow for the simplest physical interpretation in the limit when the wave packet is sharply peaked in momentum space, i.e., represents a particle with a well-defined momentum and is given by $|a(\mathbf{k},t)|^2 \approx \delta(\mathbf{k}-\mathbf{K})$. The electrical current in this case is simply given by the sum over all occupied states $j_\alpha = -(e/S)\sum_{\mathbf{K}\in\mathrm{occ}}\dot{R}_\alpha(\mathbf{K})f_\mathbf{K}$, where $f_\mathbf{K}$ is the Fermi-Dirac distribution function, and we exactly reproduce Eq. (11) of the main text. In particular, terms that explicitly contain $\mathbf{R}\approx\tilde{\mathbf{R}}(\mathbf{K})$ represent the Taylor expansion for $E^\mu(\mathbf{R})$ and $\partial E^\mu(\mathbf{R})/\partial R_\nu$ near $\mathbf{R}=0$.

We see that the semiclassical approach works best in the insulating regime in the case when energy bands are well-separated. Indeed, in this case, the q^2 component of the Hall conductivity is primarily determined by the term $g_{xx}\Omega_{xy}$, see Eq. (2), which is correctly captured by the semiclassical expressions (11) or (S37). This is not surprising since traditionally semiclassics is designed for a single-band description, hence, not suitable for capturing the terms that explicitly contain the energy gap $\varepsilon_C - \varepsilon_V$. As was demonstrated in Refs. [17, 27], the terms with the inverse powers of the band gap, like those in Eq. (2) apart from $g_{xx}\Omega_{xy}$, can in principle be captured by semiclassics as the perturbative corrections. We do not perform such analysis in the present work, however.

The agreement of the semiclassical approach with the Kubo formula is much less accurate in the metallic regime. While semiclassics captures certain terms which have the form of full derivatives and thus are absent in the insulating case, it generally poorly reproduces the Kubo formula result. The main reason for that is the presence of a non-zero intraband contribution. As is clear from Eqs. (S19)-(S24), the intraband contribution in the static limit contains terms proportional to the bandgap $\varepsilon_C - \varepsilon_V$ or even $(\varepsilon_C - \varepsilon_V)^2$, which clearly could not be captured by semiclassics.

Finally, the semiclassical equations of motion in the non-uniform electric field suffer from the terms originating from the Heisenberg uncertainty principle, $(\partial_{k_\mu}|a|)(\partial_{k_\nu}|a|)$ and $(\partial_{k_\mu}|a|)(\partial_{k_\nu}|a|)\Omega_{\xi\alpha}$. These terms vanish when dealing with the localized Wannier functions, but become divergent in the case of the wave packets narrow in momentum space, which correspond to the particles with well-defined momenta. While these terms have clear physical meaning when considering the time evolution of the correlation functions, it is not clear how to relate them to the physical observables, such as electrical current. We see, however, that once these wave-packet dependent terms are discarded, the semiclassical equations well agree with the microscopic Kubo formulation in the limit where semiclassics is expected to work, i.e., in the insulating regime with the large bandgap.

II.B Intuitive interpretation of the semiclassical result

Now we demonstrate that the semiclassical equations of motion can be obtained from a physically transparent argument. First, we notice that the equation for $\dot{\mathbf{K}}$ represents the Newton's second law and can be rewritten as

$$\dot{K}_\alpha = -\frac{e}{\hbar}\langle\Psi(t)|\mathbf{E}(\hat{\mathbf{r}})|\Psi(t)\rangle. \tag{S39}$$

Second, to derive the equation for $\dot{\mathbf{R}}$, we introduce the effecitve total energy of the wave packet as [16]

$$\varepsilon_{eff} = \langle\Psi(t)|\hat{H}_0 - e\varphi(\hat{\mathbf{r}})|\Psi(t)\rangle = \langle\Psi(t)|\hat{H}_0|\Psi(t)\rangle + e\langle\Psi(t)|E^\mu\hat{r}_\mu + \frac{1}{2}E^{\mu\nu}\hat{r}_\mu\hat{r}_\nu + \frac{1}{6}E^{\mu\nu\xi}\hat{r}_\mu\hat{r}_\nu\hat{r}_\xi + \ldots|\Psi(t)\rangle. \tag{S40}$$

To evaluate this expression up to the second order in the electric field gradients, we use matrix elements given by Eqs. (S30)-(S31) and (S36) as well as the expression for the third moment

$$\begin{aligned}\langle\Psi(t)|\hat{r}_\mu\hat{r}_\nu\hat{r}_\xi|\Psi(t)\rangle = \int d\mathbf{k}\left\{\tilde{R}_\mu\tilde{R}_\nu\tilde{R}_\xi + \tilde{R}_\mu g_{\nu\xi} + \tilde{R}_\nu g_{\xi\mu} + \tilde{R}_\xi g_{\mu\nu} - \frac{1}{3}\left(\frac{\partial^2\tilde{R}_\mu}{\partial k_\nu\partial k_\xi} + \frac{\partial^2\tilde{R}_\nu}{\partial k_\xi\partial k_\mu} + \frac{\partial^2\tilde{R}_\xi}{\partial k_\mu\partial k_\nu}\right) + T_{\mu\nu\xi}\right\}|a|^2 + \\ + \frac{1}{4|a|^2}\frac{\partial|a|^2}{\partial k_\mu}\frac{\partial|a|^2}{\partial k_\nu}\tilde{R}_\xi + \frac{1}{4|a|^2}\frac{\partial|a|^2}{\partial k_\nu}\frac{\partial|a|^2}{\partial k_\xi}\tilde{R}_\mu + \frac{1}{4|a|^2}\frac{\partial|a|^2}{\partial k_\xi}\frac{\partial|a|^2}{\partial k_\mu}\tilde{R}_\nu,\end{aligned} \tag{S41}$$

where we suppressed index $\mathbf{k}$ for brevity and $\tilde{\mathbf{R}}$ is given by Eq. (S32).

Focusing again on the wave packets describing particles with the well-defined momenta, $|a(\mathbf{k})|^2 \approx \delta(\mathbf{k}-\mathbf{K})$, we find

$$\begin{aligned}&\varepsilon_{eff}(\mathbf{R},\mathbf{K}) = \varepsilon_{\mathbf{K}} + eE^\mu R_\mu + \frac{e}{2}E^{\mu\nu}\left(R_\mu R_\nu + g_{\mu\nu}(\mathbf{K}) + \int d\mathbf{k}\frac{\partial|a(\mathbf{k})|}{\partial k_\mu}\cdot\frac{\partial|a(\mathbf{k})|}{\partial k_\nu}\right) + \\ &+ \frac{e}{6}E^{\mu\nu\xi}\left[R_\mu R_\nu R_\xi + R_\mu g_{\nu\xi}(\mathbf{K}) + R_\nu g_{\mu\xi}(\mathbf{K}) + R_\xi g_{\nu\mu}(\mathbf{K}) - \frac{1}{3}\left(\frac{\partial^2 R_\mu}{\partial K_\nu\partial K_\xi} + \frac{\partial^2 R_\nu}{\partial K_\xi\partial K_\mu} + \frac{\partial^2 R_\xi}{\partial K_\mu\partial K_\nu}\right) + T_{\mu\nu\xi}(\mathbf{K}) + \right. \\ &\left.\int d\mathbf{k}\left(\frac{\partial|a(\mathbf{k})|}{\partial k_\mu}\cdot\frac{\partial|a(\mathbf{k})|}{\partial k_\nu}\tilde{R}_\xi(\mathbf{k}) + \frac{\partial|a(\mathbf{k})|}{\partial k_\nu}\cdot\frac{\partial|a(\mathbf{k})|}{\partial k_\xi}\tilde{R}_\mu(\mathbf{k}) + \frac{\partial|a(\mathbf{k})|}{\partial k_\xi}\cdot\frac{\partial|a(\mathbf{k})|}{\partial k_\mu}\tilde{R}_\nu(\mathbf{k})\right)\right],\end{aligned} \tag{S42}$$

where we have used $\langle\Psi(t)|\hat{H}_0|\Psi(t)\rangle = \int d\mathbf{k}\,\varepsilon_{\mathbf{k}}\,|a(\mathbf{k})|^2 \approx \varepsilon_{\mathbf{K}}$ and $\mathbf{R} = \langle\Psi(t)|\hat{\mathbf{r}}|\Psi(t)\rangle \approx \tilde{\mathbf{R}}(\mathbf{K})$. Terms with $E^\mu R_\mu$, $E^{\mu\nu}R_\mu R_\nu$, and $E^{\mu\nu\xi}R_\mu R_\nu R_\xi$ simply sum up into $-\varphi(\mathbf{R})$. Treating then $\mathbf{R}$ and $\mathbf{K}$ as independent variables, we reproduce the Newton's second law as

$$\frac{\partial\varepsilon_{eff}(\mathbf{R},\mathbf{K})}{\partial R_\alpha} \approx eE_\alpha(\mathbf{R}) + \frac{e}{2}E_{\alpha\mu\nu}\left(g^{\mu\nu}(\mathbf{K}) + \int d\mathbf{k}\frac{\partial|a(\mathbf{k})|}{\partial k_\mu}\cdot\frac{\partial|a(\mathbf{k})|}{\partial k_\nu}\right) = -\hbar\dot{K}_\alpha, \tag{S43}$$

where in order to obtain the last term we used the equality

$$\frac{\partial}{\partial R_\alpha}\int d\mathbf{k}\frac{\partial|a(\mathbf{k})|}{\partial k_\mu}\cdot\frac{\partial|a(\mathbf{k})|}{\partial k_\nu}\tilde{R}_\xi(\mathbf{k}) \approx \delta_{\alpha\xi}\int d\mathbf{k}\frac{\partial|a(\mathbf{k})|}{\partial k_\mu}\cdot\frac{\partial|a(\mathbf{k})|}{\partial k_\nu}. \tag{S44}$$

Analogously, we find

$$\frac{\partial \varepsilon_{eff}}{\partial K_\alpha} \approx \frac{\partial \varepsilon_{\mathbf{K}}}{\partial K_\alpha} + \frac{e}{2}\frac{\partial E^\mu(\mathbf{R})}{\partial R^\nu} \cdot \frac{\partial g_{\mu\nu}(\mathbf{K})}{\partial K_\alpha} + \frac{e}{6}E^{\mu\nu\xi}\frac{\partial T_{\mu\nu\xi}(\mathbf{K})}{\partial K_\alpha}, \tag{S45}$$

leading to

$$\begin{aligned}
&\frac{\partial \varepsilon_{eff}(\mathbf{R},\mathbf{K})}{\partial K_\alpha} - \hbar\Omega_{\alpha\xi}\dot{K}_\xi \approx \frac{\partial \varepsilon_{\mathbf{K}}}{\partial K_\alpha} + \frac{e}{2}\frac{\partial E^\mu(\mathbf{R})}{\partial R^\nu} \cdot \frac{\partial g_{\mu\nu}(\mathbf{K})}{\partial K_\alpha} + \frac{e}{6}E^{\mu\nu\xi}\frac{\partial T_{\mu\nu\xi}(\mathbf{K})}{\partial K_\alpha} + \\
&+ \Omega_{\alpha\xi}(\mathbf{K})\left[eE_\xi(\mathbf{R}) + \frac{e}{2}E_{\xi\mu\nu}\left(g^{\mu\nu}(\mathbf{K}) + \int d\mathbf{k}\frac{\partial|a(\mathbf{k})|}{\partial k_\mu}\cdot\frac{\partial|a(\mathbf{k})|}{\partial k_\nu}\right)\right] = \\
&= \hbar\dot{R}_\alpha + \frac{e}{2}E^{\mu\nu\xi}\left\{\frac{1}{3}\frac{\partial^2\Omega_{\alpha\xi}(\mathbf{K})}{\partial K_\mu \partial K_\nu} + \int d\mathbf{k}\frac{\partial|a(\mathbf{k})|}{\partial k_\mu}\cdot\frac{\partial|a(\mathbf{k})|}{\partial k_\nu}[\Omega_{\alpha\xi}(\mathbf{K}) - \Omega_{\alpha\xi}(\mathbf{k})]\right\}.
\end{aligned} \tag{S46}$$

It is straightforward to show that the expression in the brackets of the last line equals to a certain wavepacket-dependent combination of the second derivatives of the Berry curvature and, generally, is non-zero. In fact, the first term can be entirely absorbed by the second one after the proper redefinition of $|a(\mathbf{k})|$. This implies that this term vanishes upon integration over the entire Brillouin zone. At the same time, we recall that semiclassics is expected to agree with the exact answer only in the case when the band is fully filled, provided the wavepacket-dependent terms are neglected. With this notion in mind, the above equation can be somewhat loosely rewritten as

$$\hbar\dot{R}_\alpha \simeq \frac{\partial \varepsilon_{eff}(\mathbf{R},\mathbf{K})}{\partial K_\alpha} - \hbar\Omega_{\alpha\xi}\dot{K}_\xi. \tag{S47}$$

We see that the quasiclassical equations of motion have exactly same form as in the uniform electric field, provided the effective quasiparticle energy is properly defined.

Finally, assuming again that the wavepacket is narrowly peaked in momentum space, $|a(\mathbf{k})|^2 \approx \delta(\mathbf{k}-\mathbf{K})$, we find for the third cumulant

$$\begin{aligned}
&\langle\Psi(t)|\delta\hat{r}_\mu\delta\hat{r}_\nu\delta\hat{r}_\xi|\Psi(t)\rangle \approx T_{\mu\nu\xi}(\mathbf{K}) - \frac{1}{3}\left(\frac{\partial^2\tilde{R}_\xi(\mathbf{K})}{\partial K_\mu\partial K_\nu} + \frac{\partial^2\tilde{R}_\mu(\mathbf{K})}{\partial K_\xi\partial K_\nu} + \frac{\partial^2\tilde{R}_\nu(\mathbf{K})}{\partial K_\mu\partial K_\xi}\right) + \\
&+ \int d\mathbf{k}\left[\frac{\partial|a(\mathbf{k})|}{\partial k_\mu}\cdot\frac{\partial|a(\mathbf{k})|}{\partial k_\nu}(\tilde{R}_\xi - R_\xi) + \frac{\partial|a(\mathbf{k})|}{\partial k_\xi}\cdot\frac{\partial|a(\mathbf{k})|}{\partial k_\nu}(\tilde{R}_\mu - R_\mu) + \frac{\partial|a(\mathbf{k})|}{\partial k_\mu}\cdot\frac{\partial|a(\mathbf{k})|}{\partial k_\xi}(\tilde{R}_\nu - R_\nu)\right],
\end{aligned} \tag{S48}$$

where $\delta\hat{r}_\mu \equiv \hat{r}_\mu - R_\mu = \hat{r}_\mu - \langle\hat{r}_\mu\rangle$. While the cumulant generally depends on the second derivatives of $\tilde{\mathbf{R}}$, its gauge-invariant part (independent of $\tilde{\mathbf{R}}$) is exactly given by $T_{\mu\nu\xi}(\mathbf{K})$.