EM Waves phase proof

Statement Prove that the EF and the MF are in phase far away from an oscilating dipole

Proof Let A,B be the poles (point charges $q_a = q(t), q_b = -q_a$) of an AB dipole of length ℓ. The oscilations are the same with an LC circuit where

$$q(t) = q_0 \sin \omega t, \quad i(t) = \partial_t q = i_0 \cos \omega t$$

We will focus on the Ox axis (perpendicular to the axis that passes from A,B) where O is the midpoint of AB. Let $M(x)$ be a point on Ox such that $(OM) = x$. The magnetic field due to the dipole is given as a function of position and time (its algebraic value) as

$$B_D(x,t) = \frac{\mu_0 i(t)}{2\pi x^2} = \frac{\mu_0 i_0}{2\pi} \frac{\cos \omega t}{x}$$

. with amplitude $b(x) = \frac{\mu_0 i_0}{2\pi x}$

Furthermore let $(AM) = (BM) = r(x)$ and $\theta = \angle AMB$. Then the magnitude of the electric field from each point A,B respectively are given as:

$$\|E_{A,D}\|(x,t) = \|E_{B,D}\| = \frac{k_c|q(t)|}{r^2(x)} = \frac{4kq_0|\sin \omega t|}{\ell^2 + 4x^2}$$

Since $q_a = -q_b$ and $\mathbf{E}_D = \mathbf{E}_{A,D} + \mathbf{E}_{B,D}$ then it turns out to be that

$$E_D^2 = E_{A,D}^2 + E_{B,D}^2 + 2\|E_{A,D}E_{B,D}\| \cos(\pi - \theta)$$

. Using the cosine law:

$$1 - \cos \theta = \frac{2\ell^2}{\ell^2 + 4x^2}$$

and that the algebraic value of the EF is given as a function of position and time:

$$E_D(x,t) = \frac{8kq_0\ell \sin \omega t}{(\ell^2 + 4x^2)^{3/2}}$$

with an amplitude of $\epsilon(x) = \frac{8kq_0\ell}{(\ell^2+4x^2)^{3/2}}$.

We are now going to focus on the set of points $S = \bigcup_{j=0}^{n} \{P_j\}$ with x-coordinates $x_1, \ldots, x_n$ such that $x_1 < \ldots < x_n$. We also partition the set to n-parts such that we define Δx as

$$x_{j+1} - x_j = \frac{x_n - x_0}{n} \;, j = 0, \ldots, n$$

. For simplicity reasons, let $b_j = b(x_j)$ and $\epsilon_j = \epsilon(x_j)$

As $x \to \infty$

$$b_\infty = \epsilon_\infty = 0$$

.

Now, consider that P_0 is does not interfere with previous points $\notin S$ and that the point P_n has an electric field:

$$E_n = \epsilon_n \sin \omega t + \sum_{j=0}^{n-1} \epsilon'_j \sin(\omega t - k(n-j)\Delta x)$$

due to the waves travelling to P_n from P_0 up to P_{n-1}

and a magnetic field

$$B_n = b_n \cos \omega t + \sum_{j=0}^{n-1} b'_j \cos(\omega t - k(n-j)\Delta x)$$

where ϵ'_j, b'_j are the "post-interference"- scaled - amplitudes that satisfy the relationship $b'_j \geq b_j$ and $\epsilon'_j \geq \epsilon_j$

Let $n, x_n, x \to \infty$ and then

$$B_\infty = \sum_{j=0}^{\infty} b'_j \cos(\omega t - k(n-j)\Delta x)$$

$$E_\infty = \sum_{j=0}^{\infty} \epsilon'_j \sin(\omega t - k(n-j)\Delta x)$$

and

$$\frac{n-j}{n} \to 1$$

leading to

$$B_\infty = \sum_{j=0}^{\infty} b'_j \cos(\omega t - k(x_n - x_0))$$

$$E_\infty = \sum_{j=0}^{\infty} \epsilon'_j \sin(\omega t - k(x_n - x_0))$$

Define the phases as $\varphi_E(x,t) = \pi/2 + \omega t - k(x - x_0)$ and $\varphi_B(x,t) = \omega t - k(x - x_0)$. Therefore:

$$\lim_{x\to\infty} \frac{\varphi_E}{\varphi_B} = 1$$

so $\varphi_E \approx \varphi_B$ for large x

Henceforth, the equations become

$$B(x,t) = B_0 \cos\varphi(x,t) E(x,t) = E_0 \cos\varphi(x,t)$$

where there exist amplitude scaling factors $\tilde{p}_{\epsilon j}, \tilde{p_{b j}}$ defined from some arbitrary functions / transformations such that the infinite sum of the scaled amplitudes converge to B_0, E_0 respectively and $c = E_0/B_0$