The sign of $\mathcal{K}^i$ can be determined by comparing the action of its non-relativistic counterpart in an active transformation with the translation operator and using the representation in terms of derivatives:

$$\exp\left[-i(\vec{a}\,\vec{\mathcal{P}} + \vec{v}\,\vec{\mathcal{K}})\right] = \exp\left[-i(\vec{a} + \vec{v}\,t)\vec{\mathcal{P}}\right] \Longrightarrow \mathcal{K}^i = x^0\mathcal{P}^i = -it\nabla^i$$

$$\Longrightarrow [\mathcal{K}^i, \mathcal{H}] = [-it\nabla^i, i\partial_t] = -\nabla^i = -i\mathcal{P}^i$$

The relativistic definition is then

$$\mathcal{M}^{0i} = \mathcal{K}^i = x^0\mathcal{P}^i - x^i\mathcal{P}^0$$

Likewise, as in non-relativistic quantum mechanics,

$$\mathcal{M}^{ij} = \epsilon^{ijk}\mathcal{J}^k = x^i\mathcal{P}^j - x^j\mathcal{P}^i$$

$$\Longrightarrow \mathcal{M}^{\mu\nu} = x^\mu\mathcal{P}^\nu - x^\nu\mathcal{P}^\mu$$

1.2 Polarization vectors

$$\sum_{\lambda'} \epsilon^\mu(\vec{0},\lambda')\vec{\mathcal{J}}^{(j)}_{\lambda'\lambda} = \vec{\mathbb{J}}^\mu{}_\nu \epsilon^\nu(\vec{0},\lambda)$$

$$\Longrightarrow \sum_{\lambda'} \epsilon^0(\vec{0},\lambda')\vec{\mathcal{J}}^{(j)}_{\lambda'\lambda} = \vec{\mathbb{J}}^0{}_\nu \epsilon^\nu(\vec{0},\lambda) = 0$$

$\Longrightarrow \epsilon^0(\vec{0},\lambda) = \epsilon^0(\vec{0},0) \equiv \epsilon^0(\vec{0})$ describes particles with $j=0$.

$$2\epsilon^i(\vec{0},\lambda) = 2\delta^i_k \epsilon^k(\vec{0},\lambda) = \vec{\mathbb{J}}^i{}_j \vec{\mathbb{J}}^j{}_k \epsilon^k(\vec{0},\lambda) = \sum_{\lambda'\lambda''} \epsilon^i(\vec{0},\lambda')\vec{\mathcal{J}}^{(j)}_{\lambda'\lambda''}\vec{\mathcal{J}}^{(j)}_{\lambda''\lambda}$$

$$= \sum_{\lambda'} \epsilon^0(\vec{0},\lambda')j(j+1)\delta_{\lambda'\lambda} = j(j+1)\epsilon^0(\vec{0},\lambda) \Longrightarrow \epsilon^i(\vec{0},\lambda) \to j=1$$

$$(\mathbb{J}_3)^i{}_j \epsilon^j(\vec{0},\lambda) = \sum_{\lambda'} \epsilon^i(\vec{0},\lambda')(\mathcal{J}_3)^{(1)}_{\lambda'\lambda} = \sum_{\lambda'} \epsilon^i(\vec{0},\lambda')\lambda\delta_{\lambda'\lambda} = \lambda\epsilon^i(\vec{0},\lambda)$$

$\Longrightarrow \epsilon^\mu(\vec{0},0) \propto (0,0,0,1)^T$. We normalize so that $\epsilon^\mu(\vec{0},0) = (0,0,0,1)^T$.

$$(\mathbb{J}_1 \pm i\mathbb{J}_2)^i{}_j \epsilon^j(\vec{0},0) = \begin{pmatrix} 0 & 0 & \mp 1 \\ 0 & 0 & -i \\ \pm 1 & +i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \mp 1 \\ -i \\ 0 \end{pmatrix}$$

$$= \sum_{\lambda'} \epsilon^i(\vec{0},\lambda')(\mathcal{J}_1 \pm i\mathcal{J}_2)^{(1)}_{\lambda' 0} = \sum_{\lambda'} \epsilon^i(\vec{0},\lambda')\sqrt{2}\delta_{\lambda',\pm 1} = \sqrt{2}\epsilon^i(\vec{0},\pm 1)$$

$$\Longrightarrow \epsilon^\mu(\vec{0},+1) \equiv \epsilon^\mu_R(\vec{0}) = -\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix} \qquad \epsilon^\mu(\vec{0},-1) \equiv \epsilon^\mu_L(\vec{0}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}$$

where ϵ^μ_R appears with an extra overall sign relative to usual conventions.

We can write the standard boost as $L(\vec{p}) = R_3(\phi)R_2(\theta)B_3(\xi)$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cosh\xi & 0 & 0 & \sinh\xi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\xi & 0 & 0 & \cosh\xi \end{pmatrix}$$