**Problem A.28 2x2 Hermitian matrix**

Quantum Mechanics – by Griffiths & Schroeter Problem A.28

# Question

Let

$$\mathrm{T} = \begin{pmatrix} 1 & 1-i \\ 1+i & 0 \end{pmatrix} \tag{1}$$

(a) Verify that T is hermitian.
(b) Find its eigenvalues (note that they are real).
(c) Find and normalize the eigenvectors (note that they are orthogonal).
(d) Construct the unitary diagonalizing matrix S, and check explicitly that it diagonalizes T.
(e) Check that det(T) and Tr(T) are the same for T as they are for its diagonalized form.

# Answers

By George Keeling Created 2 Oct 2023.

## (a) Verify that T is hermitian.

$$\tilde{\mathrm{T}} = \begin{pmatrix} 1 & 1+i \\ 1-i & 0 \end{pmatrix} \tag{2}$$

$$\mathrm{T}^\dagger = \tilde{\mathrm{T}}^* = \begin{pmatrix} 1 & 1-i \\ 1+i & 0 \end{pmatrix} = \mathrm{T} \tag{3}$$

Therefore T is hermitian.

## (b) Find its eigenvalues (note that they are real).

The eigenvector equation A.70 is

$$\mathrm{Ta} = \lambda \mathrm{a} \qquad \mathrm{a} \neq 0 \tag{4}$$

which gives the determinant equation

$$\begin{vmatrix} 1-\lambda & 1-i \\ 1+i & 0-\lambda \end{vmatrix} = 0 \tag{5}$$

$$\Rightarrow \lambda^2 - \lambda - 2 = 0 \tag{6}$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = 2 \text{ or } -1 \tag{7}$$

The eigenvalues $\lambda$ are indeed real.

## (c) Find and normalize the eigenvectors

The eigenvector $\mathrm{a}^{(1)}$ for $\lambda_1 = 2$ satisfies

$$\mathrm{Ta}_1 = \lambda_1 \mathrm{a}^{(1)} \rightarrow \begin{pmatrix} 1 & 1-i \\ 1+i & 0 \end{pmatrix} \begin{pmatrix} \mathrm{a}_1^{(1)} \\ \mathrm{a}_2^{(1)} \end{pmatrix} = 2 \begin{pmatrix} \mathrm{a}_1^{(1)} \\ \mathrm{a}_2^{(1)} \end{pmatrix} \tag{8}$$

which gives two equations

$$\mathrm{a}_1^{(1)} + \mathrm{a}_2^{(1)} - i\mathrm{a}_2^{(1)} = 2\mathrm{a}_1^{(1)} \tag{9}$$

$$a_1^{(1)} + i a_1^{(1)} = 2a_2^{(1)} \tag{10}$$

substitute for $a_2^{(1)}$ in (9) and we get

$$a_1^{(1)} + \frac{1}{2}a_1^{(1)} \cancel{+\frac{1}{2}ia_1^{(1)} - \frac{1}{2}ia_1^{(1)}} + \frac{1}{2}a_1^{(1)} = 2a_1^{(1)} \tag{11}$$

so $a_1^{(1)}$ can be anything, let's say 2 then (10) gives us $a_2^{(1)} = 1 + i$. That works in (9) as well. Say we said $a_1^{(1)} = 1 - i$ then we would get $a_2^{(1)} = 1$ which agree with wolfram.

Similarly for $\lambda_2 = -1$

$$\begin{pmatrix} 1 & 1-i \\ 1+i & 0 \end{pmatrix}\begin{pmatrix} a_1^{(2)} \\ a_2^{(2)} \end{pmatrix} = -1\begin{pmatrix} a_1^{(2)} \\ a_2^{(2)} \end{pmatrix} \tag{12}$$

which gives

$$a_1^{(2)} + a_2^{(2)} - ia_2^{(2)} = -a_1^{(2)} \tag{13}$$

$$-a_1^{(2)} - ia_1^{(2)} = a_2^{(2)} \tag{14}$$

(14) into (13) gives

$$a_1^{(2)} - a_1^{(2)} - ia_1^{(2)} + ia_1^{(2)} - a_1^{(2)} = -a_1^{(2)} \tag{15}$$

Once again $a_1^{(2)}$ can be anything let's pick 1, then $a_2^{(2)} = -1 - i$

Collecting our results we have

$$\boxed{\lambda_1 = 2, a^{(1)} = \begin{pmatrix} 2 \\ 1+i \end{pmatrix}; \qquad \lambda_2 = -1, a^{(2)} = \begin{pmatrix} 1 \\ -1-i \end{pmatrix}} \tag{18}$$

These agree with wolfram if we multiply $a^{(1)}$ by $(1 + i)$ and $a^{(2)}$ by $(-1 + i)$, which is always possible with eigenvectors.

The inner product of the eigenvectors is

$$\langle a^{(1)} | a^{(2)} \rangle = (2 \quad 1-i)\begin{pmatrix} 1 \\ -1-i \end{pmatrix} = 2 - 2 = 0 \tag{19}$$

so the are orthogonal. The norms of the vectors are

$$\|a^{(1)}\| = \sqrt{\langle a^{(1)} | a^{(1)} \rangle} = \sqrt{(2 \quad 1-i)\begin{pmatrix} 2 \\ 1+i \end{pmatrix}} = \sqrt{4+2} = \sqrt{6} \tag{20}$$

$$\|a^{(2)}\| = \sqrt{\langle a^{(2)} | a^{2} \rangle} = \sqrt{(1 \quad -1+i)\begin{pmatrix} 1 \\ -1-i \end{pmatrix}} = \sqrt{1+2} = \sqrt{3} \tag{21}$$

wolfram agrees. So the normalised vectors are

$$a^{(1)} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ 1+i \end{pmatrix}; \qquad a^{(2)} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1-i \end{pmatrix} \tag{22}$$

## (d) Construct the unitary diagonalizing matrix S

Just before A.81 we are told that the similarity matrix S that effects the diagonalisation can be constructed by using the eigenvectors (in the old basis) as the columns of $S^{-1}$:

$$(S^{-1})_{ij} = \left(a^{(j)}\right)_i \tag{23}$$

In our case that is

$$S^{-1} = \begin{pmatrix} 2 & 1 \\ 1+i & -1-i \end{pmatrix} \tag{24}$$

and

$$S = \frac{1}{\det S^{-1}} \tilde{C} \tag{25}$$

and we have

$$\det S^{-1} = -2 - 2i - 1 - i = -3 - 3i \tag{26}$$

$$C = \begin{pmatrix} -1-i & -1-i \\ -1 & 2 \end{pmatrix} \tag{27}$$

so

$$S = \frac{1}{-3-3i} \begin{pmatrix} -1-i & -1 \\ -1-i & 2 \end{pmatrix} = -\frac{(1-i)}{3(1+i)(1-i)} \begin{pmatrix} -1-i & -1 \\ -1-i & 2 \end{pmatrix} \tag{29}$$

$$= \boxed{\frac{1}{6} \begin{pmatrix} 2 & 1-i \\ 2 & -2+2i \end{pmatrix}} \tag{30}$$

which agrees with wolfram. Even do, we check that

$$SS^{-1} = \frac{1}{6} \begin{pmatrix} 2 & 1-i \\ 2 & -2+2i \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1+i & -1-i \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 4+2 & 2-2 \\ 4-4 & 2+4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark \tag{31}$$

What happens if we apply S to T or the eigenvectors. Last first:

$$a'^{(1)} = Sa^{(1)} = \frac{1}{6} \begin{pmatrix} 2 & 1-i \\ 2 & -2+2i \end{pmatrix} \begin{pmatrix} 2 \\ 1+i \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 4+2 \\ 4-4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \checkmark \tag{33}$$

$$a'^{(2)} = Sa^{(2)} = \frac{1}{6} \begin{pmatrix} 2 & 1-i \\ 2 & -2+2i \end{pmatrix} \begin{pmatrix} 1 \\ -1-i \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 2-2 \\ 2+4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \checkmark \tag{34}$$

$$T' = STS^{-1} = \frac{1}{6} \begin{pmatrix} 2 & 1-i \\ 2 & -2+2i \end{pmatrix} \begin{pmatrix} 1 & 1-i \\ 1+i & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1+i & -1-i \end{pmatrix} \tag{35}$$

$$= \frac{1}{6} \begin{pmatrix} 2+2 & 2-2i \\ 2-4 & 2-2i \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1+i & -1-i \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & 1-i \\ -1 & 1-i \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1+i & -1-i \end{pmatrix} \tag{36}$$

$$= \frac{1}{3} \begin{pmatrix} 4+2 & 2-2 \\ -2+2 & -1-2 \end{pmatrix} = \boxed{\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}} \checkmark \tag{37}$$

which also agrees with wolfram. So we have successfully found a diagonalizing matrix S. **However it is not unitary**. If it were its inverse would be the same as its hermitian conjugate: $U^\dagger = U^{-1}$. We have

$$S^\dagger = \frac{1}{6} \begin{pmatrix} 2 & 2 \\ 1+i & -2-2i \end{pmatrix} \tag{38}$$

## (e) Check that det(T) and Tr(T) are the same for T and T'

$$\det T = -2 \tag{39}$$

$$\det \mathrm{T}' = -2 \tag{40}$$

$$\mathrm{Tr}\,\mathrm{T} = 1 \tag{41}$$

$$\mathrm{Tr}\,\mathrm{T}' = 1 \tag{42}$$

Hurrah!