A simple model for negative feedback

We consider a very simple type of feedback:

1) $\frac{dx}{dt} = H(x) = F\left(\frac{x}{K}\right) - \gamma x$
2) $F(x) = \begin{cases} \beta_H \, for \, x < 1 \\ \beta_L \, for \, x \geq 1 \end{cases}$

Steady state solution:

The steady-state depends on the value of γ:

For $\gamma > \beta_H / K$ the steady-state is $x_{st} = \beta_H / \gamma$

For $\gamma < \beta_L / K$ the steady-state is $x_{st} = \beta_L / \gamma$

In between there is no solution, but if we assume a rapid slope near x=K, then the steady state is

$$x_{st} \cong K$$

What about the dynamics? If we start from $x = 0$, then if $\gamma > \beta_H / K$ the system never reaches a high enough level to activate the feedback and the approach to steady-state is:

$$x = \frac{\beta_H}{\gamma}(1 - e^{-\gamma t})$$

If $\gamma < \beta_L / K$ the system the system has few dynamical phases:

Initial – no feedback: $x < K$: $x = \frac{\beta_H}{\gamma}(1 - e^{-\gamma t})$ for $t \leq t_0 = -\gamma^{-1} \log\left(1 - \frac{K\gamma}{\beta_H}\right)$ – fast rise to a value K

Later – with feedback: $x = \frac{\beta_L}{\gamma} + \left(K - \frac{\beta_L}{\gamma}\right) e^{-\gamma(t - t_0)}$

What about a time delay? We consider only the case $\gamma < \beta_L / K$ for which we find a strong negative feedback?

1) $\frac{dx}{dt} = F\left(\frac{x(t-\tau)}{K}\right) - \gamma x$
2) $F(x) = \begin{cases} \beta_H \, for \, x < 1 \\ \beta_L \, for \, x \geq 1 \end{cases}$

One way by which we can solve such equation is by solving separately for different intervals, $0 < t < \tau; \tau < t < 2\tau; \ldots; n\tau < t < (n+1)\tau$. We assume that $x = 0$ for $t < 0$ and therefore, for the first interval we can solve:

$$\frac{dx}{dt} = F(0) - \gamma x \to x(t; 0 < t < \tau) = g_1(x)$$

For the next interval we use this solution to solve:

$$\frac{dx}{dt} = F\big(g_1(t - \tau)\big) - \gamma x \to x(t; \tau < t < 2\tau) = g_2(t)$$

And generally:

$$\frac{dx}{dt} = F\big(g_n(t-\tau)\big) - \gamma x \rightarrow x(t; n\tau < t < (n+1)\tau) = g_{n+1}(x)$$

For the step negative feedback function, we can consider two extreme cases:

$\tau \ll t_0$ – small delay – in this case, the function would remain $F\big(x(t)\big) = \beta_H$ for many intervals and the function would simply behave as the non-delayed function:

$$x = \frac{\beta_H}{\gamma}(1 - e^{-\gamma t}), t < (n+1)\tau; n\tau < t_0 < (n+1)\tau; n = \frac{t_0}{\tau} \gg 1$$

Then the system will switch to almost the same solution as with the no delay case. Not exactly the same, as this will occur with a delay τ which will allow x to rise a little bit above K, but not significantly.

$\tau \gg t_0$ – long delay.

In this case, at the first interval, the system will behave as it is in open loop:

$$x = \frac{\beta_H}{\gamma}(1 - e^{-\gamma t})$$

Only at time $t = \tau + t_0 \gg t_0$ the delayed value of $x(t-\tau)$ would reach a value of K and the feedback will kick in, setting the production at β_L. By this time, the value of $x(\tau + t_0) = \frac{\beta_H}{\gamma}\big(1 - e^{-\gamma(t+t_0)}\big) \gg K$. The equation at later stages will therefore be:

$$x(t > \tau + t_0) = x(\tau + t_0) + \left(\frac{\beta_L}{\gamma} - x(\tau + t_0)\right)e^{-\gamma(t-\tau-t_0)}$$