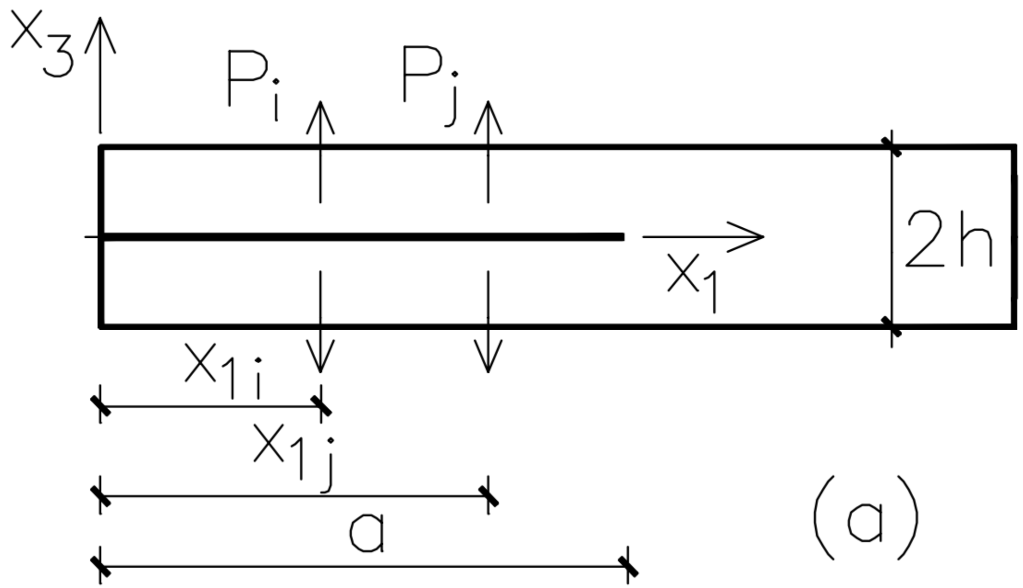
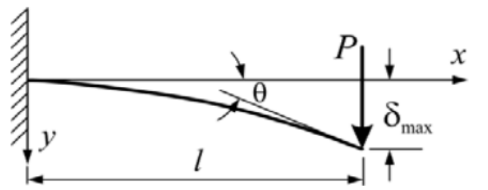
The scope of this note is to assess the localized compliance of a double cantilever beam specimen determined by stress intensity factors and Castigliano's theorem. It will be compared to the compliance obtained from simple beam theory and thus, the stress intensity factors will also be from simple beam theory.

The local compliance is denoted λij which translates to the deflection of P_i at x_{1i} due to a force couple, P_j, acting at x_{1j}. Castigliano's theorem states:

$$\lambda_{ij} = \frac{2}{E} \cdot \int_0^a \frac{K_I(x_{1i}, P_i) \cdot K_I(x_{1j}, Pj)}{P_i \cdot P_j} \, da$$



Consider a single cantilever beam with a concentrated load, P, at distance l from the base

1. Cantilever Beam – Concentrated load *P* at the free end		
	$\theta = \dfrac{Pl^2}{2EI}$	$y = \dfrac{Px^2}{6EI}(3l - x)$
2. Cantilever Beam – Concentrated load *P* at any point		

The deflection at distance x from base is:

$$y(x, P, E, I, l) := \frac{P \cdot x^2}{6 \cdot E \cdot I} \cdot (3l - x)$$

With I=(h^3)/12 for a beam of unit depth and the height h with δ=2y being the normal opening displacement of the double cantilever we get:

$$\delta(x, P, E, h, l) := 2\,y\left(x, P, E, \frac{h^3}{12}, l\right) \text{ factor } \rightarrow \frac{4 \cdot P \cdot x^2 \cdot (3 \cdot l - x)}{E \cdot h^3}$$

We now switch to a coordinate system where x_1 originates at the free end of the beams and is positive toward the crack tip. Thus, we need to replace x and l appropriately with respect to the new reference:

l was the distance from the load P_j to the crack tip: $l(x_{1j}, a) := a - x_{1j}$

And x the distance from the crack tip: $x(x_{1i}, a) := a - x_{1i}$

The displacement at x_{1i} due to a force, P_j, acting at X_{1j} is then:

$$\delta(x_{1i}, x_{1j}, a, P, E, h) := \delta(a - x_{1i}, P, E, h, a - x_{1j}) \text{ factor } \rightarrow \frac{4 \cdot P \cdot (a - x_{1i})^2 \cdot (2 \cdot a + x_{1i} - 3 \cdot x_{1j})}{E \cdot h^3}$$

And the the compliance at x_{1i} for a force couple acting at x_{1j} is thus:

$$\lambda_{ij}(x_{1i}, x_{1j}, a, E, h) := \frac{\delta(x_{1i}, x_{1j}, a, P, E, h)}{P} \rightarrow \frac{4 \cdot (a - x_{1i})^2 \cdot (2 \cdot a + x_{1i} - 3 \cdot x_{1j})}{E \cdot h^3}$$

To determine the local complaince using stress intensity factors we use:

$$K_I(x_1, a, h, P) := \frac{\sqrt{12}\,P}{h^{\frac{3}{2}}} \cdot (a - x_1)$$

,where a-x_1 is the distance from the force couple P to the crack tip.

For P_i and P_j respectively we get:

$$K_I(x_{1i}, a, h, P_i) := \frac{\sqrt{12}\,P_i}{h^{\frac{3}{2}}} \cdot (a - x_{1i})$$

$$K_I(x_{1j}, a, h, P_j) := \frac{\sqrt{12}\,P_j}{h^{\frac{3}{2}}} \cdot (a - x_{1j})$$

According to the Castigliano theorem, the localized compliance can be determined by:

$$\lambda_{ij}(x_{1i}, x_{1j}, a, h, E) := \frac{2}{E} \cdot \int_0^a \frac{K_I(x_{1i}, a, h, P_i) \cdot K_I(x_{1j}, a, h, P_j)}{P_i \cdot P_j} \, da \text{ factor } \rightarrow \frac{4 \cdot a \cdot \left(2 \cdot a^2 - 3 \cdot a \cdot x_{1i} - 3 \cdot a \cdot x_{1j} + 6 \cdot x_{1i} \cdot x_{1j}\right)}{E \cdot h^3}$$

but this is different from the local compliance obtained directly from simple beam theory. However, we get the same result if we change the limits of the integral from 0 to a in to from x_{1i} to a!:

$$\lambda_{ij}(x_{1i}, x_{1j}, a, h, E) := \frac{2}{E} \cdot \int_{x_{1i}}^a \frac{K_I(x_{1i}, a, h, P_i) \cdot K_I(x_{1j}, a, h, P_j)}{P_i \cdot P_j} \, da \text{ factor } \rightarrow \frac{4 \cdot (a - x_{1i})^2 \cdot (2 \cdot a + x_{1i} - 3 \cdot x_{1j})}{E \cdot h^3}$$