DU Projectile Shot Straight Up, includes variation in atmospheric density with altitude

$mach := 1100\frac{ft}{s}$

$MJ := 10^6 \cdot J$

$Velocity := 4 \cdot mach$

$Mass := 66gm$

$Angle := 89.5deg$

Initial Constituent Velocity Calculations

$V_h := Velocity \cdot \cos(Angle)$ $\qquad V_h = 0.035 \cdot mach$

$V_v := Velocity \cdot \sin(Angle)$ $\qquad V_v = 4 \cdot mach$

Time to top of projectile arc, neglecting drag

$\Delta t := \frac{V_v}{g}$ $\qquad \Delta t = 136.751s$

$t_T := 2 \cdot \Delta t$ $\qquad t_T = 273.502s$

Estimated max altitude, neglecting drag (kinetic energy/potential energy balance)

$EstMaxAltitude := \frac{\frac{1}{2} \cdot V_v^{\,2}}{g}$ $\qquad EstMaxAltitude = 91.696km$

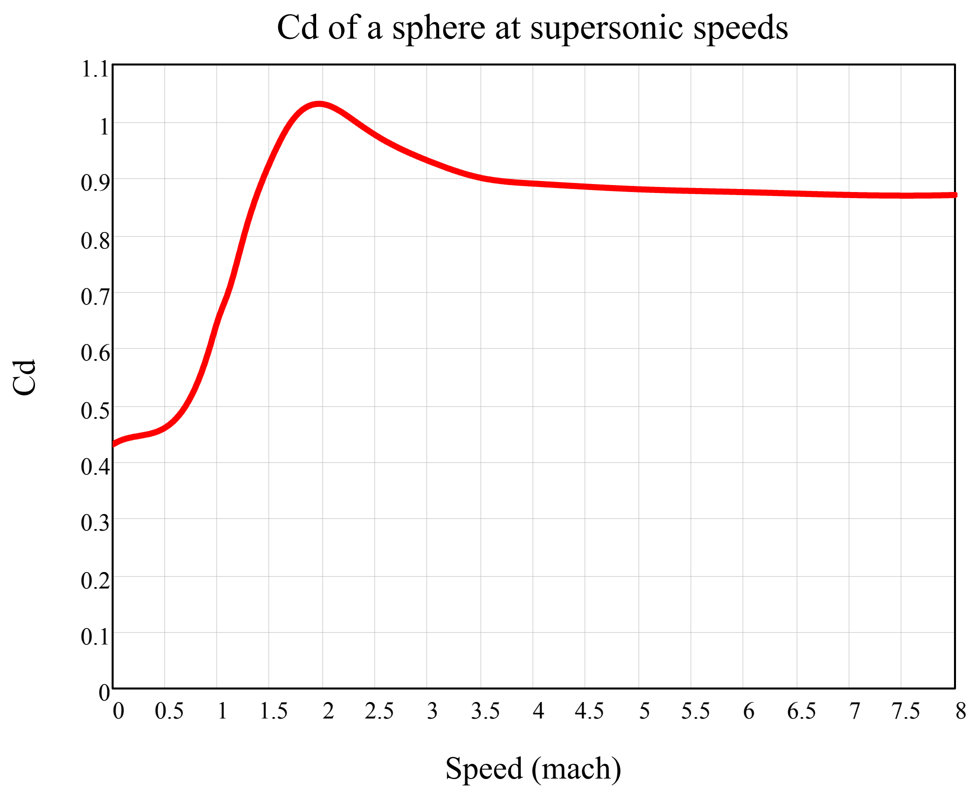
Gravity as a function of alititude

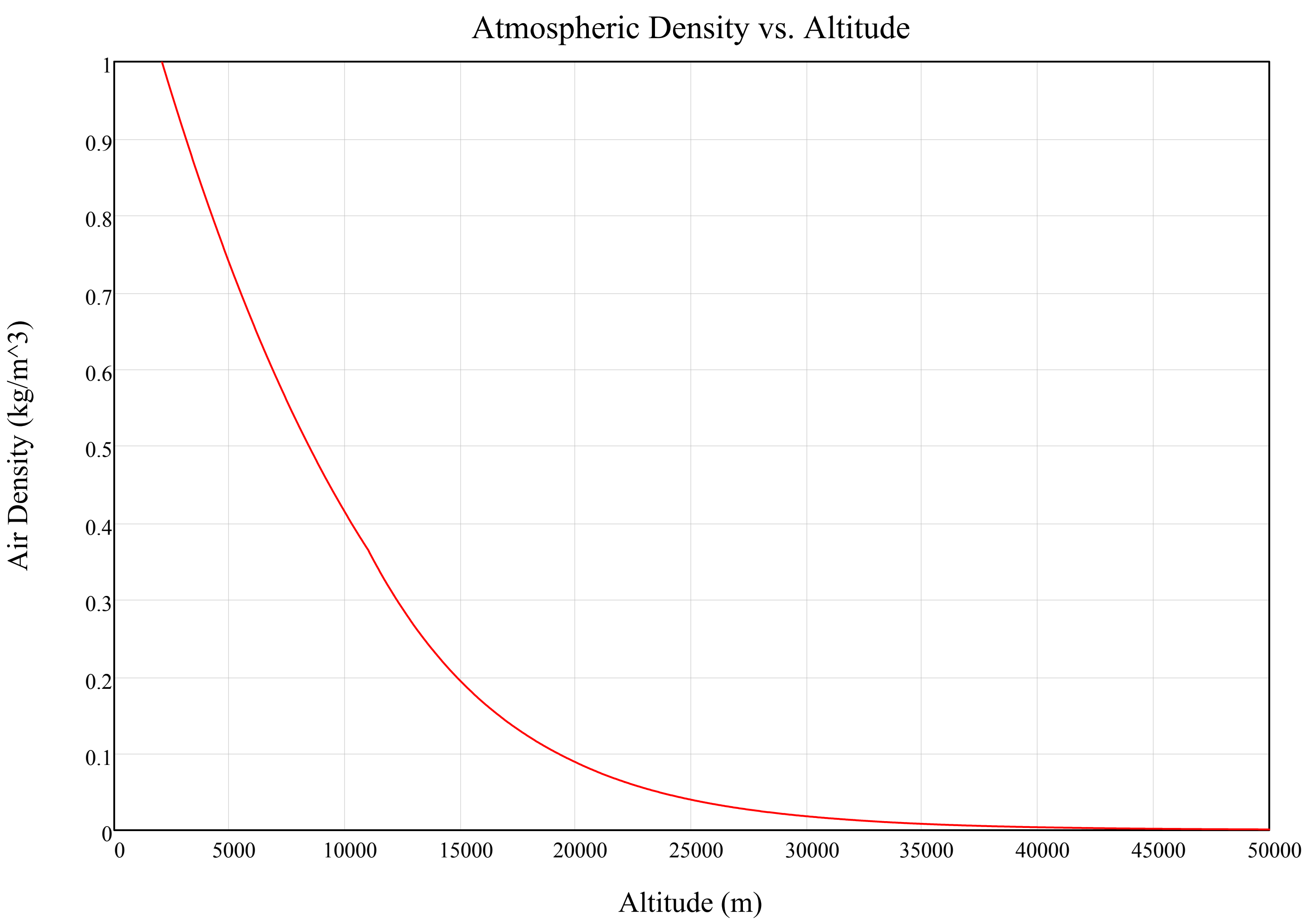
$M_{earth} := 5.9736 \cdot 10^{24}$

$r_{earth} := 6.37101 \cdot 10^{6}$

$G_{const} := 6.6742 \cdot 10^{-11}$

$$gravity(altitude) := G_{const} \cdot \frac{M_{earth}}{\left(r_{earth} + altitude\right)^2}$$


Cd of a sphere at supersonic speeds
Cd
Speed (mach)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1
0.5
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
6.5
7
7.5
8


Atmospheric Density vs. Altitude
Air Density (kg/m^3)
Altitude (m)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
5000
10000
15000
20000
25000
30000
35000
40000
45000
50000

Numerically solving differential solution for rising edge

$F_d = -\frac{1}{2}\cdot\rho\cdot v\cdot A\cdot C_d$ $\quad r := 8\,\frac{mm}{m} = 8\times10^{-3}$ $\quad C_d(v) := C_{dspline}\left(\frac{v\cdot\frac{m}{s}}{mach}\right)$ $\quad M := \frac{Mass}{kg} = 0.066$

$Area := \pi\cdot(r)^2$ $\quad Area = 2.011\times10^{-4}$

$V_{init} := \frac{Velocity}{\frac{m}{s}}$

$LaunchPower := \frac{1}{2}\cdot(M\ kg)\cdot\left(V_{init}\,\frac{m}{s}\right)^2$ $\quad LaunchPower = 0.059\cdot MJ$

Given

$$hor_1''(t) = \frac{-\frac{1}{2}\cdot Density\left(ver_1(t)\right)\cdot hor_1'(t)\cdot Area\cdot C_d\left(\sqrt{ver_1'(t)^2 + hor_1'(t)^2}\right)}{M}$$

$$ver_1''(t) = \frac{-\frac{1}{2}\cdot Density\left(ver_1(t)\right)\cdot ver_1'(t)\cdot Area\cdot C_d\left(\sqrt{ver_1'(t)^2 + hor_1'(t)^2}\right)}{M} - gravity\left(ver_1(t)\right)$$

$hor_1(0) = 0$ $\quad ver_1(0) = 15$ $\quad hor_1'(0) = V_{init}\cos(Angle)$ $\quad ver_1'(0) = V_{init}\sin(Angle)$

$\begin{pmatrix} hor_1 \\ ver_1 \end{pmatrix} := Odesolve\left[\begin{pmatrix} hor_1 \\ ver_1 \end{pmatrix}, t, 250\right]$ $\quad mach = 335.28\,\frac{m}{s}$

$I_1 := 1000$ $\quad i_1 := 0..I_1$

$Range1_{i_1} := 0.0 + i_1\cdot.1$

$VerValues1_{i_1} := ver_1\left(Range1_{i_1}\right)$

$HorValues1_{i_1} := hor_1\left(Range1_{i_1}\right)$

$ProjectileFunction_1 := cspline\left(HorValues_1, VerValues_1\right)$

$v_{p1}(x) := interp\left(ProjectileFunction_1, HorValues_1, VerValues_1, x\right)$

$TimeFunction_{v1} := cspline\left(Range_1, VerValues_1\right)$

$v_{t1}(t) := interp\left(TimeFunction_{v1}, Range_1, VerValues_1, t\right)$

$TimeFunction_{h1} := cspline\left(Range_1, HorValues_1\right)$

$h_{t1}(t) := interp\left(TimeFunction_{h1}, Range_1, HorValues_1, t\right)$

$$Velocity_{scalar1}(t) := \left[\left(\frac{d}{dt}v_{t1}(t)\right)^2 + \left(\frac{d}{dt}h_{t1}(t)\right)^2\right]^{\frac{1}{2}}$$

$Velocity_{horiz1}(t) := \frac{d}{dt}h_{t1}(t)$ $\quad Velocity_{vert1}(t) := \frac{d}{dt}v_{t1}(t)$ $\quad Slope_{proj1}(t) := \frac{d}{dt}v_{p1}(t)$

$Accel_{horiz1}(t) := \frac{d^2}{dt^2}h_{t1}(t)$ $\quad Accel_{vert1}(t) := \frac{d^2}{dt^2}v_{t1}(t)$

$t_{guess1} := \frac{V_{init}\cdot\sin(Angle)}{9.8}$

$t_{guess1} = 136.844$

$Time_{peak} := root\left(Velocity_{vert1}\left(t_{guess1}\right), t_{guess1}\right)$

$Time_{peak} = 137.711$

$t_i := 0, .1 .. Time_{peak}$

$x_{guess1} := Time_{peak} \cdot V_{init} \cdot \cos(Angle)$ $\qquad x_{guess1} = 1.612 \times 10^{3}$

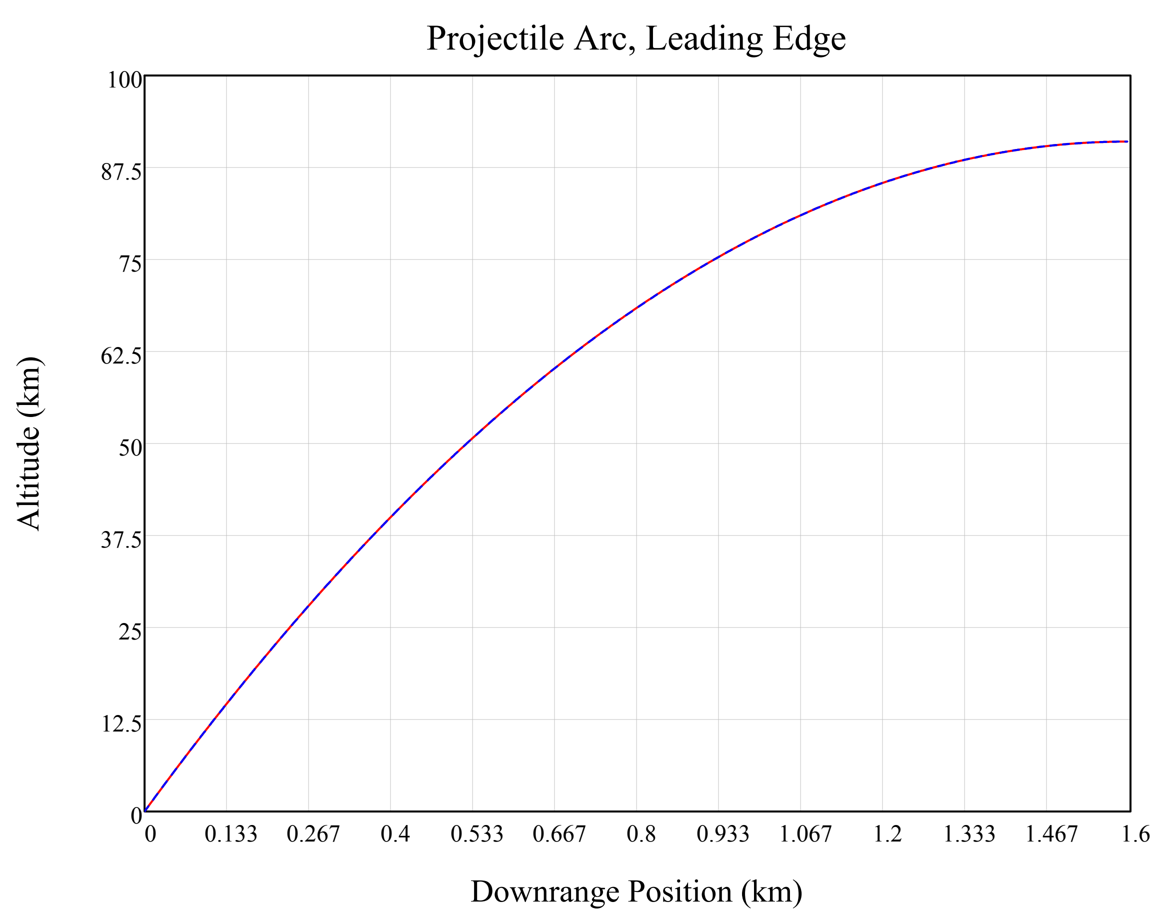
$Range_{peak} := root\left(Slope_{proj1}\left(x_{guess1}\right), x_{guess1}\right)$ $\qquad Range_{peak} = 1.595 \times 10^{3}$

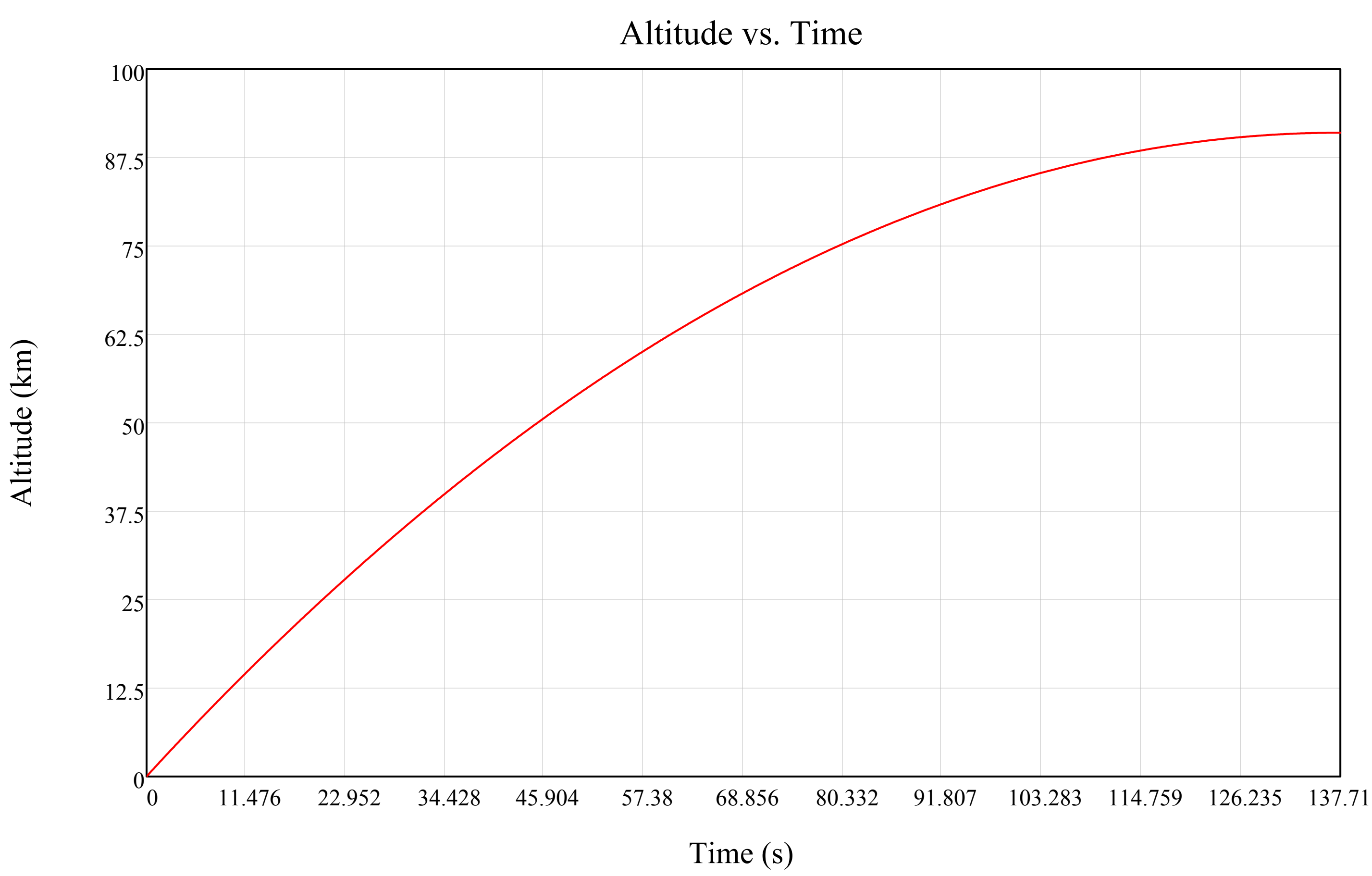
$$TotalEnergy_{peak} := \frac{1}{2} \cdot M \cdot kg \cdot \left(Velocity_{scalar1}\left(Time_{peak}\right) \cdot \frac{m}{s}\right)^2 + M \cdot kg \cdot v_{t1}\left(Time_{peak}\right) \cdot m \cdot gravity\left(v_{t1}\left(Time_{peak}\right)\right) \cdot \frac{m}{s^2} = 0.057\ MJ$$

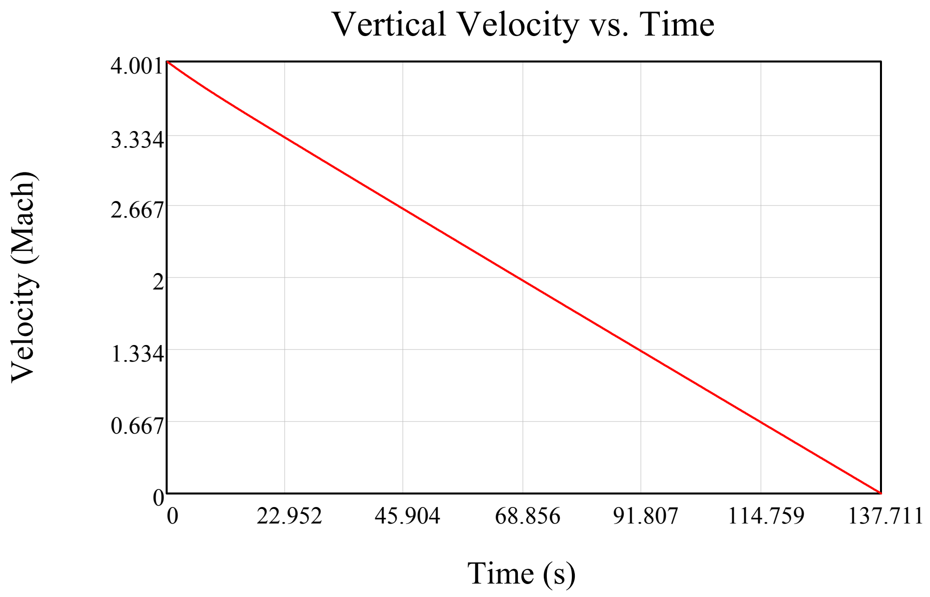
$$PercentTotalEnergy_{peak} := \frac{TotalEnergy_{peak}}{LaunchPower} \cdot 100 = 96.644$$

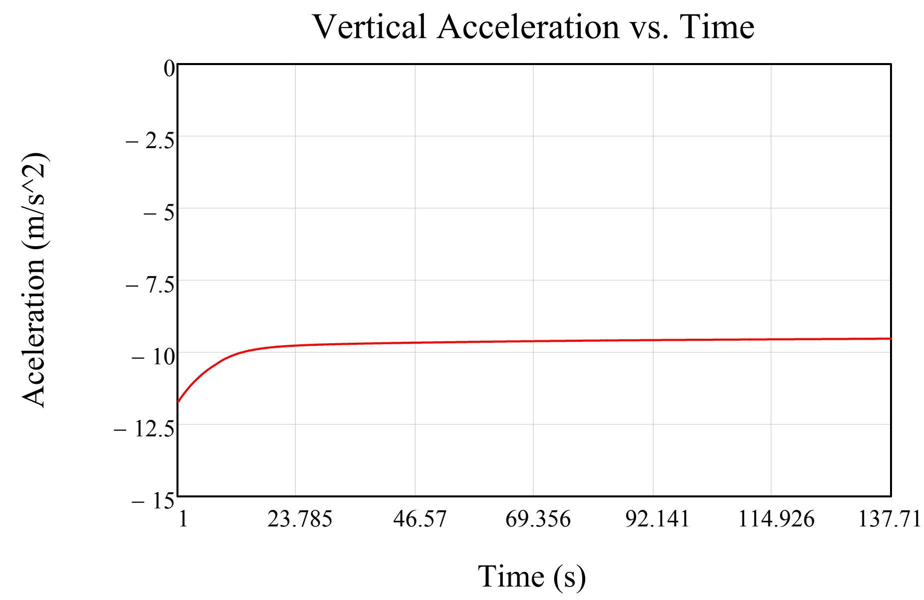
$$KineticEnergy_{peak} := \frac{1}{2} \cdot M \cdot kg \cdot \left(Velocity_{scalar1}\left(Time_{peak}\right) \cdot \frac{m}{s}\right)^2 = 4.421\ J$$

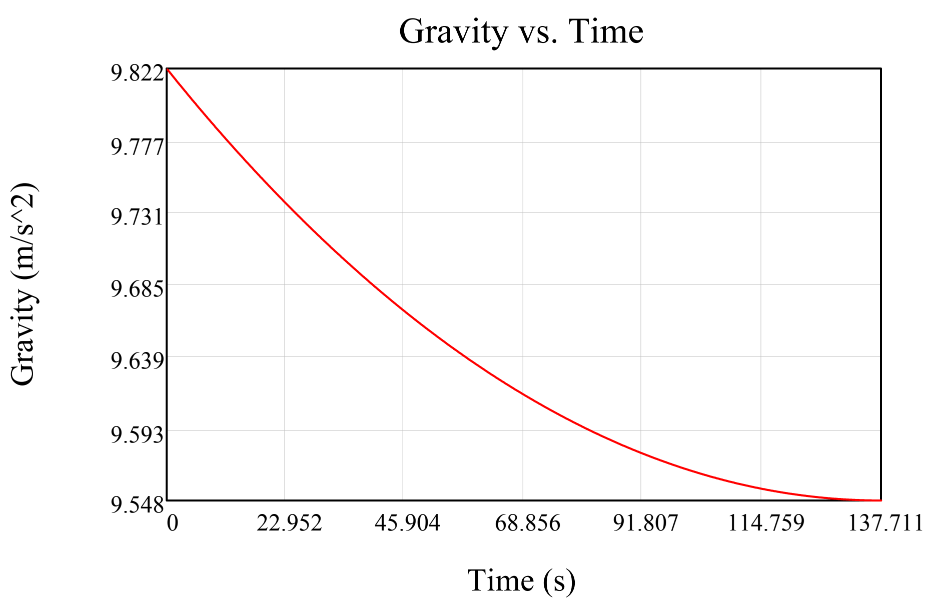
$$PercentKineticEnergy_{peak} := \frac{KineticEnergy_{peak}}{LaunchPower} \cdot 100 = 7.449 \times 10^{-3}$$

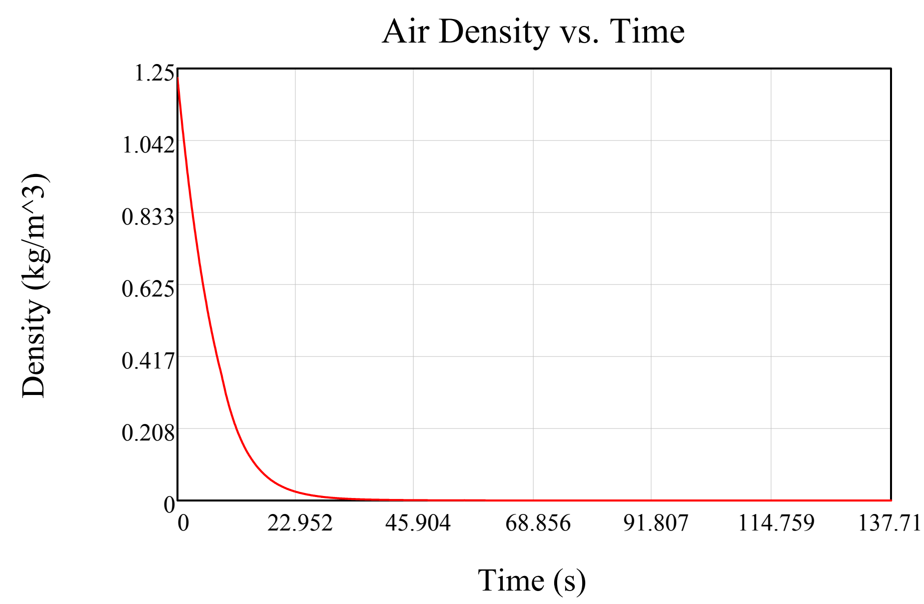


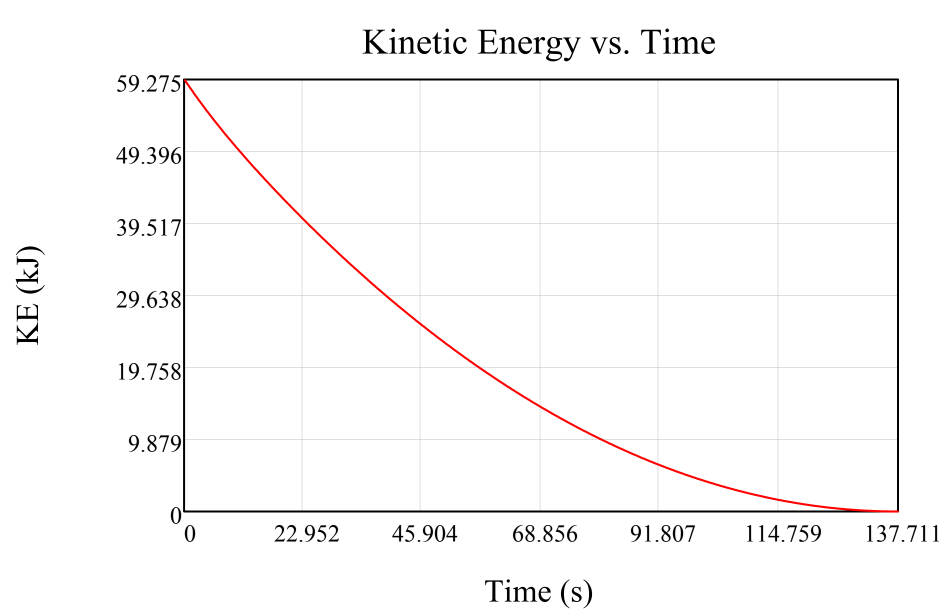



Vertical Velocity vs. Time
Velocity (Mach)
4.001
3.334
2.667
2
1.334
0.667
0
0
22.952
45.904
68.856
91.807
114.759
137.711
Time (s)


Vertical Acceleration vs. Time
Aceleration (m/s^2)
0
− 2.5
− 5
− 7.5
− 10
− 12.5
− 15
1
23.785
46.57
69.356
92.141
114.926
137.711
Time (s)


Gravity vs. Time
Gravity (m/s^2)
9.822
9.777
9.731
9.685
9.639
9.593
9.548
0
22.952
45.904
68.856
91.807
114.759
137.711
Time (s)

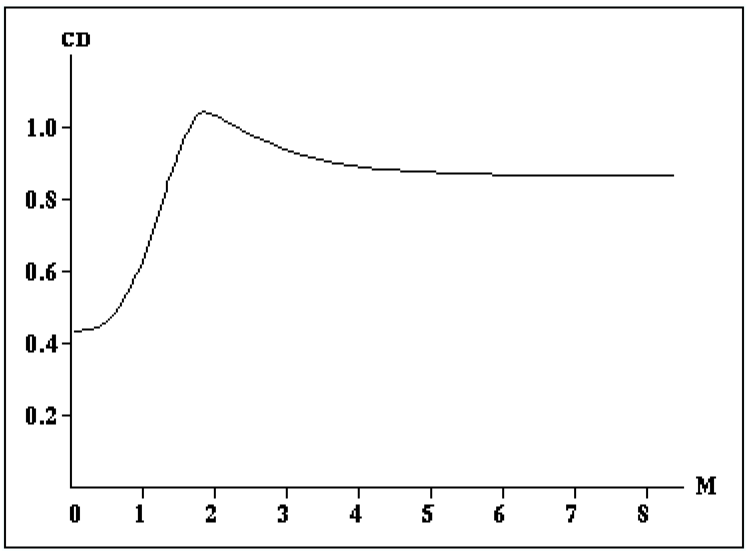

Air Density vs. Time
Density (kg/m^3)
1.25
1.042
0.833
0.625
0.417
0.208
0
0
22.952
45.904
68.856
91.807
114.759
137.711
Time (s)


Kinetic Energy vs. Time
KE (kJ)
59.275
49.396
39.517
29.638
19.758
9.879
0
0
22.952
45.904
68.856
91.807
114.759
137.711
Time (s)

CD of a sphere at supersonic speeds

http://www.aerodyn.org/Drag/speed-drag.html

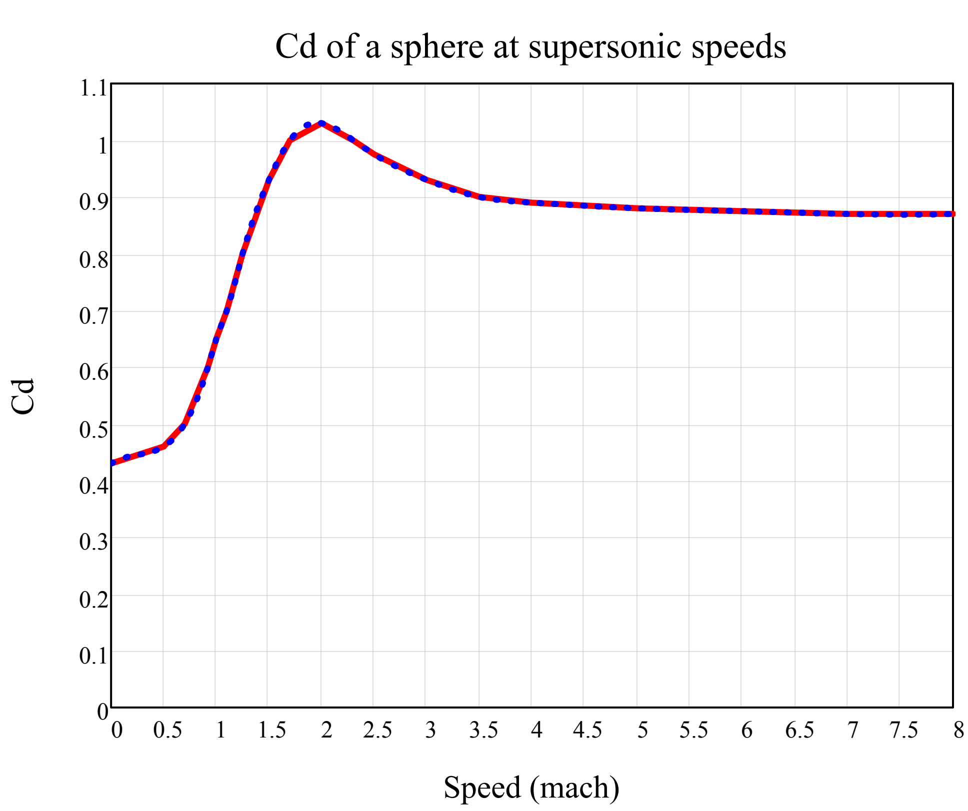
Table of values are estimated from chart



$$\text{ValuesTable} := \begin{pmatrix} 0 & 0.43 \\ 0.5 & 0.46 \\ 0.7 & 0.5 \\ 0.92 & 0.6 \\ 1.0 & 0.65 \\ 1.10 & 0.7 \\ 1.25 & 0.8 \\ 1.5 & 0.93 \\ 1.7 & 1.0 \\ 2.0 & 1.03 \\ 2.3 & 1.0 \\ 2.5 & 0.975 \\ 3.0 & 0.93 \\ 3.5 & 0.9 \\ 4.0 & 0.89 \\ 5.0 & 0.88 \\ 6.0 & 0.875 \\ 7.0 & 0.87 \\ 8.0 & 0.87 \end{pmatrix}$$

$$v_s := \text{cspline}\left(\text{ValuesTable}^{\langle 0\rangle}, \text{ValuesTable}^{\langle 1\rangle}\right)$$

$$C_{dspline}(\text{mach}) := \text{interp}\left(v_s, \text{ValuesTable}^{\langle 0\rangle}, \text{ValuesTable}^{\langle 1\rangle}, \text{mach}\right)$$



$$\text{Density(altitude)} := \left|\begin{array}{l}
M \leftarrow 28.9644 \\
R_{const} \leftarrow 8.31432 \cdot 10^3 \\
g_c \leftarrow 9.8 \\
\text{if altitude} > 86000 \\
\quad \left|\begin{array}{l} \rho \leftarrow 0 \\ \text{return } \rho \end{array}\right. \\
\text{if altitude} > 71000 \\
\quad \left|\begin{array}{l} \rho_b \leftarrow .000064 \\ T_b \leftarrow 214.65 \\ L_b \leftarrow -0.002 \\ h_b \leftarrow 71000 \\ \rho \leftarrow \rho_b \cdot \left[\dfrac{T_b}{T_b + L_b \cdot (\text{altitude} - h_b)}\right]^{\left(\frac{g_c \cdot M}{R_{const} \cdot L_b}\right)+1} \\ \text{return } \rho \end{array}\right. \\
\text{if altitude} > 51000 \\
\quad \left|\begin{array}{l} \rho_b \leftarrow .00086 \\ T_b \leftarrow 270.65 \\ L_b \leftarrow -0.0028 \\ h_b \leftarrow 51000 \\ \rho \leftarrow \rho_b \cdot \left[\dfrac{T_b}{T_b + L_b \cdot (\text{altitude} - h_b)}\right]^{\left(\frac{g_c \cdot M}{R_{const} \cdot L_b}\right)+1} \\ \text{return } \rho \end{array}\right. \\
\text{if altitude} > 47000 \\
\quad \left|\begin{array}{l} \rho_b \leftarrow .00143 \\ T_b \leftarrow 270.65 \\ L_b \leftarrow 0.0 \\ h_b \leftarrow 47000 \\ \rho \leftarrow \rho_b \cdot e^{\left[\frac{-g_c \cdot M \cdot (\text{altitude} - h_b)}{R_{const} \cdot T_b}\right]} \\ \text{return } \rho \end{array}\right. \\
\text{if altitude} > 32000 \\
\quad \left|\begin{array}{l} \rho_b \leftarrow .01322 \\ T_b \leftarrow 228.65 \\ L_b \leftarrow 0.0028 \\ h_b \leftarrow 32000 \\ \rho \leftarrow \rho_b \cdot \left[\dfrac{T_b}{T_b + L_b \cdot (\text{altitude} - h_b)}\right]^{\left(\frac{g_c \cdot M}{R_{const} \cdot L_b}\right)+1} \\ \text{return } \rho \end{array}\right. \\
\text{if altitude} > 20000 \\
\quad \left|\begin{array}{l} \rho_b \leftarrow .08803 \\ T_b \leftarrow 216.65 \\ L_b \leftarrow 0.001 \\ h_b \leftarrow 20000 \\ \rho \leftarrow \rho_b \cdot \left[\dfrac{T_b}{T_b + L_b \cdot (\text{altitude} - h_b)}\right]^{\left(\frac{g_c \cdot M}{R_{const} \cdot L_b}\right)+1} \\ \text{return } \rho \end{array}\right. \\
\text{if altitude} > 11000 \\
\quad \left|\begin{array}{l} \rho_b \leftarrow .36391 \\ T_b \leftarrow 216.65 \\ L_b \leftarrow 0.0 \\ h_b \leftarrow 11000 \\ \rho \leftarrow \rho_b \cdot e^{\left[\frac{-g_c \cdot M \cdot (\text{altitude} - h_b)}{R_{const} \cdot T_b}\right]} \\ \text{return } \rho \end{array}\right. \\
\rho_b \leftarrow 1.2250
\end{array}\right.$$

$$
\begin{array}{|l}
T_b \leftarrow 288.15 \\
L_b \leftarrow -.0065 \\
h_b \leftarrow 0 \\
\rho \leftarrow \rho_b \cdot \left[\dfrac{T_b}{T_b + L_b \cdot \left(\text{altitude} - h_b\right)}\right]^{\left(\frac{g_c \cdot M}{R_{const} \cdot L_b}\right)+1} \\
\text{return } \rho
\end{array}
$$