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Experimental Investigation of Faraday Waves of Maximum Height

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Abstract— The profiles of standing gravity waves of maximum height, parametrically excited on the free surface of a deep fluid in a vertically oscillating rectangular vessel (Faraday waves), are investigated experimentally. For a small modulation index of the excitation parameter, waves of three types are distinguished: regular, temporally periodic and symmetric about the vertical line passing through their crest; irregular but retaining the connectivity of the liquid volume; and breaking waves with drops separating from the free surface of the fluid. It is established that the profile of the maximum-height regular waves is smooth with a steepness of 0.255 and a limiting angle at the crest of less than 80°. Certain realizations of irregular and breaking waves, with profiles similar to those of regular waves but with much smaller steepnesses, 0288 and 0.429, respectively, are detected.

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The first solutions of the problem of the profile shape of a traveling stationary wave of maximum height on a free fluid surface were obtained [1] as early as by Stokes and Mitchell on the assumption that the crest of the limiting profile is pointed. Stokes analytically established that such a wave has a sharpness angle equal to 120° and Mitchell numerically obtained an estimate $\Gamma_m = 0.142$ for its maximum steepness $\Gamma = H/\lambda$, where H is the height of the crest of the wave over its bottom and λ is the wavelength. These results were confirmed by numerous theoretical and experimental investigations.

For standing waves the question of the limiting profile shape is still open [1] and investigations in this field have been and are being disputed in the scientific literature [2]. We will briefly mention the best known results. In the theoretical study [3], it was assumed that the limiting profile is pointed and it was found that the sharpness angle $\varphi_m = 90°$ and the steepness $\Gamma_m = 0.218$. In the subsequent theoretical studies the existence of sharpening and the estimate for the angle were doubted and the limiting steepness Γ_m was estimated on the range from 0.196 in [4] to 0.285 in [5]. The conclusions made in [3] were experimentally verified in the study by Taylor [6] where standing waves were excited by two wave generators mounted on the end walls of a rectangular vessel. An estimate $\Gamma_m = 0.228$ was proposed and reasons in support of the predicted φ_m value were formulated, although the wave profile observed was not pointed. In other experiments [7–11] the limiting steepness of standing waves was estimated on the range $0.108 < \Gamma_m < 0.266$.

The scatter of these experimental results may be attributed to a difference in the methods and conditions of wave excitation. The discussion of the shape of the limiting profile of standing waves is made more complicated by the lack of mathematically correct results for nonlinear standing waves in a fluid (in contrast to the theory of traveling waves). Taking this into account, it seems reasonable to continue the experimental investigations of high standing waves, using a single convenient method of wave excitation.

In this study, the parametric wave excitation discovered by Faraday is used. In this method, standing waves are formed due to destabilization of the horizontal free fluid surface in a vessel vertically oscillating under parametric resonance conditions. Since the Faraday waves are excited without using wave generators, their observation and measurement proves to be simpler and more accurate than with other methods of excitation; moreover, the wave height can easily be varied by varying the vessel oscillation frequency.

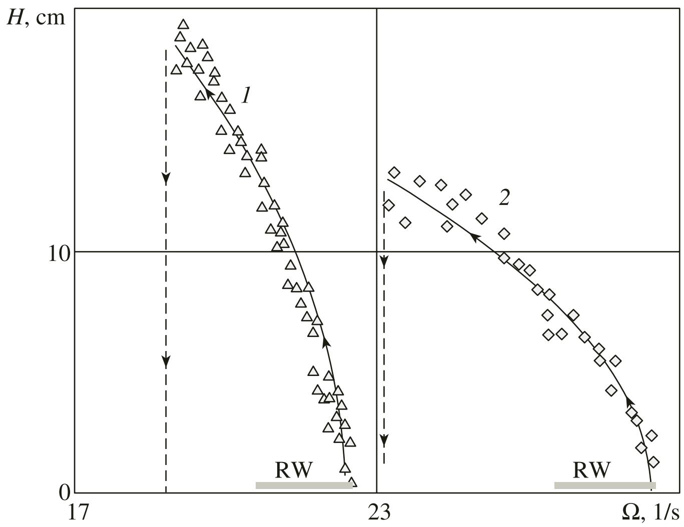

Fig. 1. Dependence of the height H of the Faraday waves on the oscillation frequency Ω of a vessel measuring $50 \times 4 \times 40$ cm for $h = 15$ cm and $s = 0.25$ cm: (*1*) second mode, $n = 2$, $\omega_2 = 10.85\ \mathrm{s}^{-1}$; (*2*) third mode, $n = 3$, $\omega_3 = 13.55\ \mathrm{s}^{-1}$, RW frequency range of regular wave excitation.

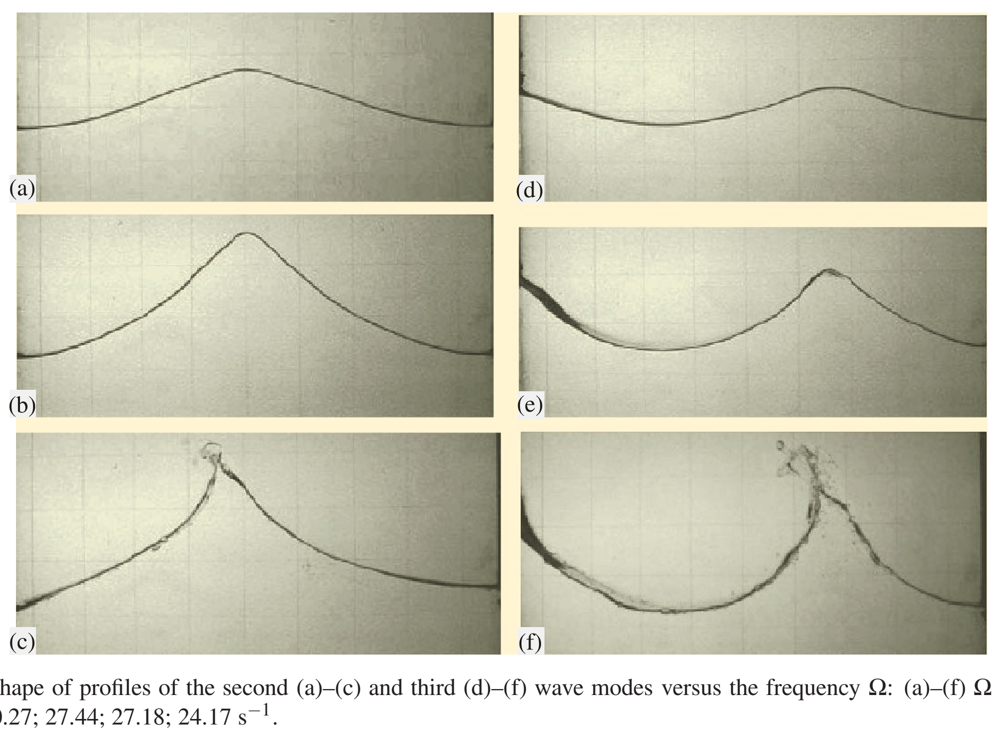

Fig. 2. Shape of profiles of the second (a)–(c) and third (d)–(f) wave modes versus the frequency Ω: (a)–(f) $\Omega = 21.74$; 20.74; 20.27; 27.44; 27.18; 24.17 s^{-1}.

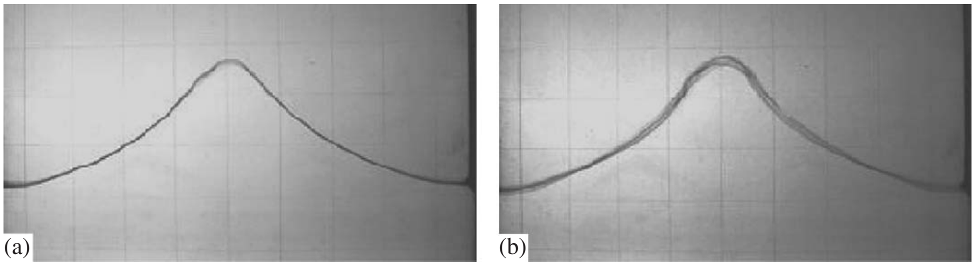

Fig. 3. Superposition of ten successive profiles corresponding to maximum development of the second mode for $h = 15$ cm, $s = 0.25$ cm, and $\omega_2 = 10.85\ \mathrm{s}^{-1}$: $p = 0.949$, $\Gamma = 0.255$ (a); 0.946, 0.259 (b).

The profiles of maximum-height Faraday waves were investigated experimentally in [7, 8] for different wave excitation regimes and vessel dimensions. Our experimental data are compared with [7, 8] below.

In our experiments we used a regime known in the theory of oscillations as the regime with a small modulation index of the excitation parameter. In this regime, the vessel oscillation accelerations are small as compared with the gravity acceleration g, which made the shape of the Faraday wave profile similar to that of free standing waves and, hence, made it possible to compare the measurement results with the conclusions of standing wave theory.

1. ORGANIZATION OF THE EXPERIMENT

The Faraday waves were generated on a free fluid surface using the setup described in [12, 13]. The second and third wave modes were investigated in a rectangular vessel under fundamental Faraday resonance conditions when the frequency of the excited waves is equal to half the vessel oscillation frequency. In most experiments the vessel measured $50 \times 4 \times 40$ cm, which ensured the two-dimensionality of the wave modes and made the waves similar to pure gravity waves. The fluid depth h was equal to 15 cm, which ensured that the wave motion was independent of of h. As the working fluid we used water. Only in a few experiments was the water depth varied or kerosene used, as noted below.

Vertical oscillations with amplitude s and frequency Ω were imparted to the vessel. Waves were excited on the free fluid surface when s exceeded a certain threshold value. As mentioned above, $\Omega \sim 2\omega_n$, where ω_n is the natural frequency of the wave n-mode. At a fixed value $s = 0.25$ cm, varying Ω led to a change in the steepness of the Faraday waves excited. On going over from one value of the frequency Ω to another, the measurements were begun only after an interval of the order of 100–200 vessel oscillation periods, necessary for the new wave regime to become established in the fluid.

The wave profiles were digitally video-recorded (24–30 fr/s). The accuracy in measuring the height of the wave profiles from the record was equal to 0.1 cm, its mean value H in each regime being calculated from eight successive values of the difference between the crest and hollow heights measured at the moments of maximum wave development.

2. RESULTS AND DISCUSSION

First of all, the Faraday waves observed were divided up into three groups. Since under the conditions of our experiment the profiles of the Faraday waves differed only slightly from the profiles of the free waves, we took into account the main properties of free waves known from standing nonlinear wave theory [14]. By definition, in a free standing wave the fluid surface oscillates periodically with time and displays a typical spatial distribution of its nodes and antinodes. The antinodes of a standing wave are at rest, while its nodes perform horizontal oscillations. In the two-dimensional waves in question, the distance between the antinodes is equal to $\lambda_n/2$ and the trajectories of the fluid particles are directed along vertical planes passing through the wave antinodes. The wave profile is symmetrical about the vertical planes passing through the antinodes.

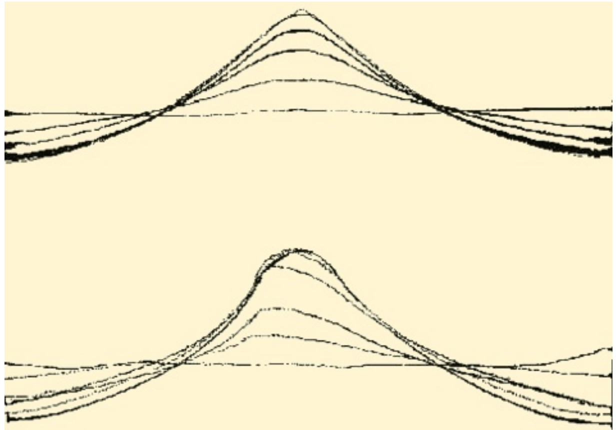

Fig. 4. Variation of the wave profile on the interval $T/4$: (a) regular wave, time step $\Delta t = T/18$, $T = 0.600$ s, $p = 0.965$; irregular wave (b), $\Delta t = T/19$, $T = 0.612$ s, $p = 0.946$.

The waves with profiles closest to those of the theoretically predicted free standing waves are called regular. More precisely, **regular** Faraday waves are those waves whose profile possesses two properties: it must be (1) temporally periodic and (2) symmetrical about the vertical planes passing through the wave antinodes.

Waves for which at least one of these properties is violated but the oscillating fluid volume retains its connectivity are called **irregular** Faraday waves.

Finally, waves from whose free surface fluid drops or jets break away are called **breaking** Faraday waves.

In Fig. 1, for the second and third wave modes the dependences of the height H of the wave excited on the frequency Ω of the vessel oscillations (resonance dependences) are presented. It can be seen that with decrease in the frequency Ω the wave height monotonically increases until at a certain vessel oscillation frequency the waves suddenly disappear (downward arrows in Fig. 1). On the Ω axis the symbols RW denote the frequency intervals $20.58 < \Omega_{n=2} < 22.52$ s^{-1} and $26.46 < \Omega_{n=3} < 28.58$ s^{-1} on which regular waves of the second and third wave modes, respectively, were detected.

In Fig. 2, the two columns show photos of the second (on the left) and third (on the right) wave modes at their moments of maximum development for the corresponding Ω values. From the photos it can be seen that at the highest frequencies $\Omega = 21.74$ and 27.44 s^{-1} the shape of the wave profiles (a), (d) is almost sinusoidal. At lower Ω values the crests of the wave profiles become narrower and their feet are flattened (b), (e), whereas the height of the waves increases. These changes in the profiles can be predicted from the linear theory. At even lower frequencies $\Omega = 20.27$ and 24.17 s^{-1} the waves become still higher and the separation of drops from the crest, that is, breakdown of the waves, can be observed (c), (f).

The difference in profile between the regular and irregular waves at the moments of their maximum development, when during the oscillations the waves are highest, is illustrated by Figs. 3a and 3b. In each figure, for the second wave mode the superposition of ten successive profiles at given moments in the wave period is presented. The profiles in Figs. 3a and 3b were obtained for the dimensionless vessel oscillation frequencies $p = 0.949$ and 0.946, respectively, where $p = \Omega/(2\omega_2)$. For these frequencies the modulation index of the excitation parameter (overload) $\varepsilon = (s\Omega^2)/g$ is equal to 0.108 and 0.107, respectively. The thin curve in Fig. 3a testifies to the exact repeatability of the profile shape over the wave period. Since the profile is symmetrical about the vertical line passing through the wave crest, this wave observed at $p = 0.949$ is regular. In Fig. 3b the "smeared curve" indicates that the profile shape is not repeatable over the wave period. Therefore, the wave observed at $p = 0.946$ is not regular. Since in Fig. 3b the curve is only moderately smeared, we may conclude that the connectivity of the fluid volume is conserved; therefore, this wave is irregular (but not breaking).

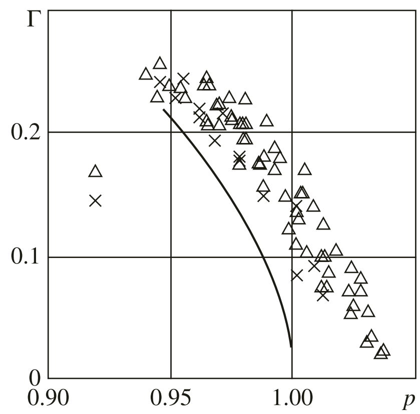

Fig. 5. Steepness Γ of regular Faraday waves vs. dimensionless oscillation frequency p of a vessel measuring $50 \times 4 \times 40$ cm for $h = 15$ cm and $s = 0.25$ cm: (*1*), (*2*) $n = 2$, 3.

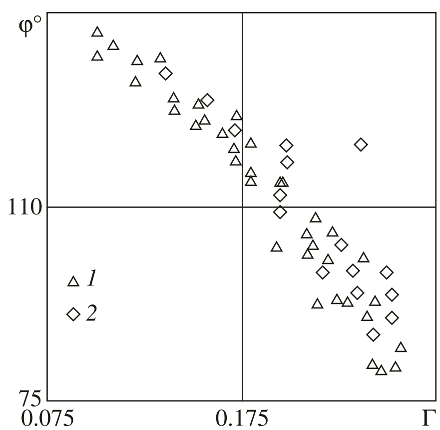

Fig. 6. Angle between the tangents at the wave crest inflection points for a vessel measuring $50 \times 4 \times 40$ cm and $s = 0.25$: (*1*), (*2*) $n = 2$ (water), 3 (water).

We note that the profile depicted in Fig. 3a is the profile of the wave of maximum height observed in the experiments. We repeat that this regular wave of maximum height for the second wave mode was excited at the vessel oscillation frequency $p = 0.949$ and its steepness $\Gamma_m = 0.255$ for $\varepsilon = 0.108$.

For the third mode the steepness of the profile of the maximum-height regular Faraday wave was determined as $\Gamma_m = 0.236$ for $p = 0.954$ and $\varepsilon = 0.167$.

For the second mode the maximum steepness obtained $\Gamma_m = 0.255$ proved to be greater than the values $\Gamma_m = 0.216$ ($\varepsilon = 0.163$) and 0.154–0.224 ($\varepsilon = 0.054$–0.079) obtained in [7] and [8], respectively.

Figures 4a and 4b show the change in the wave profiles on the interval $T/4$, where $T = 4\pi/\Omega$ is the wave period. Each figure was obtained by superposing successive profiles at time intervals $\Delta t = T/18$ and $T/19$, respectively. The moment $t = 0$ corresponds to a wave with an almost free horizontal surface, and $t = T/4$ to the most developed wave, that is, to the wave with the highest crest. For regular waves (Fig. 4a), the spatial symmetry of the profile about a vertical line through the wave antinode is typical. Moreover, an analysis of the experimental data showed that for these waves the periodic profile deformation with time is characterized by symmetry about the moment $t = T/2$ within each period. For the irregular waves (Fig. 4b), the violation of both spatial symmetry and periodicity is typical.

In Fig. 5, the resonance curves for the regular waves are presented in the dimensionless variables (p, Γ). From these curves it can be seen that regular waves of the second and third wave modes are observed on the interval $0.944 \le p \le 1.037$ of dimensionless vessel oscillation frequencies and the maximum wave steepness $\Gamma_m = 0.255$ corresponds to $p_m = 0.944$.

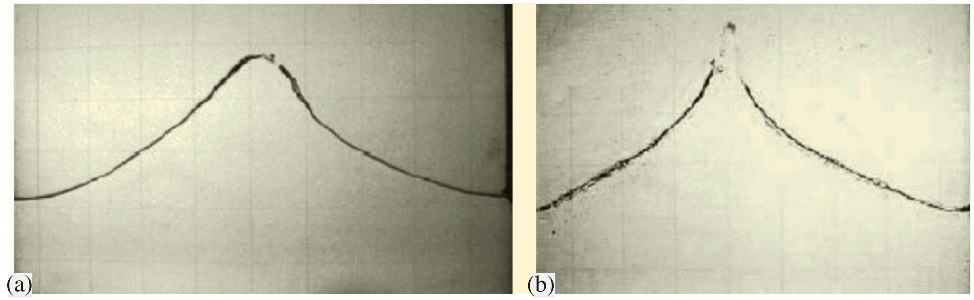

Fig. 7. Highest profiles similar in shape to regular wave profiles: (a) irregular wave, $p = 0.946$, $\Gamma = 0.288$; (b) breaking wave, $p = 0.877$, $\Gamma = 0.429$.

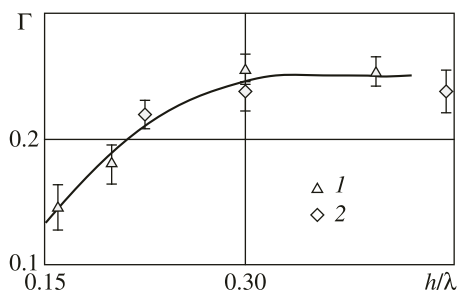

Fig. 8. Dependence of the limiting steepness Γ of regular Faraday waves on the dimensionless fluid depth h/λ for a vessel measuring $50 \times 4 \times 40$ cm and $s = 0.25$ cm: (*1*), (*2*) $n = 2$ (water), 3 (water).

On the basis of the data presented, we may conclude that in the case of the fundamental resonance in a vessel measuring $50 \times 4 \times 40$ cm for $h/\lambda = 0.3$ the maximum steepness Γ_m of the second and third Faraday wave modes can be estimated as 0.255 and 0.236, respectively.

We will now consider the problem of the limiting angle φ_m, which in [3, 4] was predicted and estimated as $\varphi_m = 90°$. Since all the profiles observed in the experiment were smooth, by analogy with [3, 6] the following passage to the limit was performed. We considered a sequence of profiles of maximum wave development whose steepness Γ increased with decrease in p to the value $\Gamma_m = 0.255$ corresponding to the maximum steepness of the regular waves. For each Γ we determined the angle $\varphi(\Gamma)$ between the tangents at the crest inflection points positioned symmetrically about a vertical line through the wave antinode. Figure 6 shows the dependence $\varphi = \varphi(\Gamma)$, from which it can be seen that the limiting steepness $\Gamma_m = 0.255$ corresponds to an angle $\varphi_m \approx 80°$ that decreases with further increase in Γ. Thus, the measured value $\varphi_m \approx 80°$ proved to be much smaller than the value given in [3, 6].

Generally, the profile shape of the irregular and breaking waves is stochastic. However, an analysis of a large number of photos of the various maximum-height profiles of such waves on the frequency range $p < 0.944$ showed that among them we can distinguish certain realizations in which the wave profiles are similar to the profiles of the regular waves: they are spatially symmetric and there is no fluid separation from the wave crest. Figures 7a and 7b show such maximum-height profiles chosen from the observed profiles of irregular and breaking waves, respectively. Their steepnesses are equal to $\Gamma_m = 0.288$ for the irregular and 0.429 for the breaking waves.

All the above results were obtained for Faraday waves on a free water surface at a depth $h = 15$ cm. Since $h/\lambda = 0.3$, this case is similar to the case of an infinitely deep fluid. A decrease in depth has a considerable effect on the maximum steepness of the regular waves Γ_m. This can be seen from Fig. 8 which shows the dependence of Γ_m on h/λ for the second and third wave modes.

In order to estimate the possible effect of the fluid surface tension on the maximum steepness of the Faraday waves in question, a series of experiments in which water was replaced by kerosene (with a surface tension of 30 dyn/cm) was carried out. For the second mode ($n = 2$ and $\lambda = 50$ cm), at $h = 15$ cm and

$s = 0.25$ cm the limiting steepness proved to be equal to $\Gamma_m = 0.242$ for $p_m = 0.946$. Correct to 4%, this value coincides with the value obtained for water (71 dyn/cm). Thus, for sufficiently long waves (as in this experiment) the maximum steepness is only slightly sensitive to variations of the surface tension.

Summary. Using Faraday parametric resonance to generate standing gravity waves on a free liquid surface made it possible to divide up the Faraday waves observed into three groups: regular, irregular, and breaking. It is established that on the surface of a deep fluid the maximum steepness of the regular waves is equal to 0.255, which is greater than the value obtained in other studies. The limiting crest angle is estimated as 80°.

Among the set of, in general, random profiles of irregular and breaking waves of maximum height there are certain realizations similar to the profiles of regular waves. The maximum steepnesses of these profiles are equal to 0.288 and 0.429, respectively.

It is shown that for relatively small fluid depths ($h/\lambda < 0.3$) the limiting steepness of the regular Faraday waves substantially depends on the depth.

It is preliminarily concluded that for sufficiently long waves (as in our experiment) the maximum steepness is only slightly sensitive to variations of the surface tension.

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