The maximum likelihood estimations of p and q were $p = 0.074$ and $q = 0.101$. The expected frequency for a score of 8 or more was set equal to $\mathrm{P}\left(T_5 \geqslant 8\right)$ times 156. The resulting χ^2 tests were not significant at the 5%-level ($\chi^2 = 3.359$, df = 3, $p = 0.340$).

Hole 3: The frequencies f_n which belong to the various scores (n) of this hole were as follows:

n	3	4	5	6	7	8
f_n	1	2	31	64	40	18

Maximum likelihood estimation yielded the values $p = 0.074$ and $q = 0.101$. For the computation of χ^2 the frequencies belonging to scores 3 and 4 were taken together as well as the respective expected frequencies. The expected frequency for a score of 8 or more was set equal to $\mathrm{P}\left(T_6 \geqslant 8\right)$ times 156. The resulting χ^2 was not significant at the 5%-level ($\chi^2 = 0.829$, df = 2, $p = 0.661$).

In all these examples the value of χ^2 was not significant, which of course is in favour of the model. However, these favourable outcomes may be due to chance. Further empirical research is needed to explore the validity of the model.

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The Hardy distribution for golf hole scores

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In an article entitled ‘A Mathematical Theorem about Golf’ [1] G.H. Hardy introduced a simple model of golfing. He assumed, that, at one hole, a golfer has probability p of gaining a stroke with a single shot, and probability q that his shot costs him a stroke. Such strokes will be described as good (G) or bad (B), respectively, leaving probability $1 - p - q$ for an ordinary (O) stroke (see also [2]). For example, on a par four hole, successive strokes *OGO* will result in a birdie (a score which is one stroke less than par) and *BBGOO* in a bogey (a score which is one stroke more than par). In this paper the probability distribution $\mathrm{P}(T_k = n)$ will be derived for the number of strokes T a player may take on a hole of par k. The distribution will be derived separately for a par three hole $\mathrm{P}(T_3 = n)$, a par four hole $\mathrm{P}(T_4 = n)$ and a par five hole $\mathrm{P}(T_5 = n)$. A par is a term in the game of golf used to denote the predetermined number of strokes that a scratch golfer should require to complete a hole. A ‘scratch golfer’ is one whose handicap is 0 or lower; or, in common usage, a golfer who averages shooting par or better. Subsequently, a general formula will be given which holds for any par k, $k = 1, 2, \ldots$. In addition, some attention will be given to the matter of how to validate the obtained distribution using real data.

1. *Hardy's model as a random walk*

A possible approach to translate Hardy's idea into a mathematical form is to adapt the situation to a Markov chain. On a par three hole, for example, there are transition states 0, 1, 2 corresponding to the result of an initial bad, ordinary or good shot, respectively, and there are two absorption states 3 and 4 corresponding to holing out. More generally, on a par N hole the states of the system are $0, 1, \ldots, N - 1$, and the transitions between the states are governed by the following rule: once the player reaches state N or $N + 1$, no further transition into another state is possible; when the player is at state k, with $0 \leqslant k \leqslant N - 1$, then the next transition is either to the same state with probability q, or to $k + 1$ with probability $(1 - p - q)$, or to $k + 2$ with probability p. This type of system is called a *random walk* with absorbing barriers at states N and $N + 1$. In the case of a par three, one may consider the Markov chain* $X_0, X_1, X_2, \ldots, X_k, \ldots, X_n, \ldots$ on states 0, 1, 2, 3, 4 whose transition probability matrix is:

$$\begin{array}{cc} \text{States} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} q & 1-p-q & p & 0 & 0 \\ 0 & q & 1-p-q & p & 0 \\ 0 & 0 & q & 1-p-q & p \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

* We follow the notation of Taylor and Karlin [3].

observed data set obtained from amateur players, one should use only players with a handicap index of 18 or 36 (not both). Players should all have about the *same* handicap index in order to ensure that there are no systematic differences between players. In addition they should either have a handicap index of 18 or a handicap index of 36 in order to make it possible to correct the official par of the hole by adding a value of *one* for bogey golfers and a value of *two* for double bogey golfers.

The following examples may clarify the approach in more detail. The data* were obtained from a sample of 156 male amateur bogey golfers (handicap 18-24). They played the so-called Nijmegen Course of the golf course Het Rijk van Nijmegen in Groesbeek (the Netherlands). The data were obtained in 1997. It would have been better to have at one's disposal handicap 18 players only. However, the players participated in games in which only players with handicap 24 or lower were allowed. Players with handicap index less then 18 were removed from the sample in order to obtain a homogeneous sample. The correlation between the handicap index and the total score was not 1-tailed significant at the 5%-level ($r = 0.074$, $N = 156$, $p = 0.178$). Therefore it is justifiable to conclude that there are no systematic differences between the players. All players can be considered as equivalent. A goodness-of-fit test was performed for a par three hole, a par four hole and a par five hole. The analysis was performed on holes which were as close as possible to the average stroke index†, which is equal to 9.5. The actual hole scores were obtained from hole 5 (par 3, stroke 11), hole 8 (par 4, stroke 9) and hole 13 (par 5, stroke 4). The other two par five holes had strokes 2 and 3 respectively. For each hole the parameter m was taken equal to the par of the hole plus one.

Hole 5: The frequencies f_n which belong to the various scores (n) of this hole were as follows:

n	2	3	4	5	6	7
f_n	1	34	74	35	9	3

Maximum likelihood estimation yielded the values $p = 0.104$ and $q = 0.119$. For the computation of χ^2 the expected frequency for a score of 7 or more was set equal to $\mathrm{P}(T_4 \geqslant 7)$ times 156. The resulting χ^2 was not significant at the 5%-level ($\chi^2 = 1.222$, df = 3, $p = 0.748$).

Hole 8: The frequencies f_n which belong to the various scores (n) of these holes were as follows:

n	3	4	5	6	7	8
f_n	1	31	71	40	12	1

* I thank the golf course Het Rijk van Nijmegen for allowing me to use these data.

† Stroke index is where the holes on a golf course are ranked in order of difficulty, stroke 1 being the hardest and stroke 18 being the easiest.

This was done separately for the case where $m = 4$ and for the case where $m = 5$. For each of these cases also a chi-square goodness-of-fit test was performed.

The case $m = 4$: The estimated values for p and q were $p = 0.014$ and $q = 0.144$. For the computation of χ^2 the observed frequencies for $n = 2$ and $n = 3$ were taken together as well as the expected frequencies for $n = 2$ and $n = 3$. In addition the observed frequencies for $n = 7$ and $n = 8$ were combined. The expected frequency was set equal to $\mathrm{P}(T_4 \geqslant 7)$ times 155. The resulting χ^2 was significant ($\chi^2 = 13.356$, df $= 2$, $p = 0.001$).

The case $m = 5$: The estimated values for p and q were $p = 0.159$ and $q = 0.044$. Note the switch of values. For the computation of χ^2 the observed frequencies for $n = 7$ and $n = 8$ were combined again. The expected frequency was again set equal to the right tail probability $\mathrm{P}(T_5 \geqslant 7)$ times 155. The resulting χ^2 was now not significant ($\chi^2 = 5.352$, df $= 2$, $p = 0.069$).

The assumption of $m = 5$ yields an acceptable value for χ^2 where the assumption of $m = 4$ did not. This makes it clear that for scratch players the value of m is not always equal to the official par of the hole. It also makes it clear that the parameter m, in principle, cannot be considered as an *a priori* given parameter. Like the parameters p and q it must be estimated from the data. However, in the case of scratch golfers, in most cases, the official par of the hole is a good guess for m.*

5. *Validation using amateur players*

For professional players and scratch golfers (handicaps around 0) the value of m usually corresponds with the official par of the hole. For bogey golfers (handicaps around 18) the value of m usually corresponds with the official par of the hole *plus one*. Similarly, for double bogey golfers (handicaps around 36) the value of m usually corresponds with the official par of the hole *plus two*. For bogey players, playing a par four hole, a correct guess for the value of the parameter m would, therefore, generally be a value of 5. Now one could argue that, since the hole is officially a par four hole, it is still possible for bogey golfers to make a score of two (a so-called ‘albatross’), which according to Hardy's model is impossible for the case where $m = 5$. However, a similar situation occurs in the case where $m = 3$. According to Hardy's model for $m = 3$ a score of 1 (a so-called ‘hole in one’) is impossible. Therefore, in Hardy's model a 'hole in one' is treated as a birdie. Similarly, if, in the case of an official par four, the value of m is taken equal to 5, then an albatross should be treated as an eagle (a score of two under par). If the model is tested for goodness-of-fit against an

* The results of actual goodness-of-fit tests, in which data are used from the above mentioned Majors, will be published in a more empirically oriented paper.

where $0 \leqslant p + q \leqslant 1$. The Markov chain starts at time zero in state $X_0 = 0$. If the Markov chain begins on state 0, it oscillates in states 0, 1 and 2 for a random duration and then proceeds either to state 3 or to state 4, where it is trapped or absorbed.

Generally, in the case of a par k, $k = 1, 2, 3, \ldots$ the Markov chain $X_0, X_1, X_2, \ldots$, on states $0, 1, 2, 3, \ldots, k + 1$ has a similar transition probability matrix with q on the diagonal cells, $(1 - p - q)$ on the first upper off-diagonal cells and p, on the second upper off-diagonal cells except for the submatrix:

$$\begin{array}{cc} \text{States} & \begin{array}{cc} k & k+1 \end{array} \\ \begin{array}{c} k \\ k+1 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

which has unity diagonal entries and zero off-diagonal entries. The Markov chain starts at time zero in state $X_0 = 0$. If the Markov chain begins on state 0, it oscillates in state $0, 1, \ldots, k - 1$ for a random duration and then proceeds either to state k or to state $k + 1$, where it is trapped or absorbed.

2. *Derivation of the Hardy distributions*

The Hardy distribution is derived for a par three, a par four and a par five hole separately. The general formula will be given in the next section. This formula holds for a par k with $k = 1, 2, 3, \ldots$.

2.1 *The Hardy distribution for a par three hole*

Let

$$T_3 = \min(0 \leqslant n;\, X_n = 3 \text{ or } X_n = 4)$$

be the time of absorption of the process for a par *three*. The subscript 3 refers to a par three! Note that T_3 simply represents the score on a par *three* hole. Consider the case $T_3 = 2$. One has the following sequences of strokes with transitions (i, j), where $(i, j) = (X_{k+1} = j \mid X_k = i)$:

$$\begin{array}{lll} OG & \text{with} & (0,\ 1) \text{ and } (1,\ 3) \\ GO & \text{with} & (0,\ 2) \text{ and } (2,\ 3) \\ GG & \text{with} & (0,\ 2) \text{ and } (2,\ 4). \end{array}$$

Therefore for $T_3 = 2$ one obtains:

$$\mathrm{P}(T_3 = 2) = 2p(1 - p - q) + p^2.$$

Consider the case $T_3 = 3$. One has the following sequences of strokes with transitions (i, j):

$$\begin{array}{lll} BGO & \text{with} & (0,0) \text{ and } (0,2) \text{ and } (2,3) \\ GBO & \text{with} & (0,2) \text{ and } (2,2) \text{ and } (2,4) \\ BOG & \text{with} & (0,0) \text{ and } (0,1) \text{ and } (1,3) \end{array}$$

$$\begin{array}{lll} OBG & \text{with} & (0,1) \text{ and } (1,1) \text{ and } (1,4) \\ BGG & \text{with} & (0,0) \text{ and } (0,2) \text{ and } (2,4) \\ GBG & \text{with} & (0,2) \text{ and } (2,2) \text{ and } (2,4) \\ OOO & \text{with} & (0,1) \text{ and } (1,2) \text{ and } (2,3) \\ OOG & \text{with} & (0,1) \text{ and } (1,2) \text{ and } (2,4). \end{array}$$

Therefore for $T_3 = 3$ one obtains

$$P(T_3 = 3) = 4pq(1-p-q) + 2p^2q + (1-p-q)^3 + p(1-p-q)^2.$$

Consider the case $T_3 = 4$. A score of four can be obtained by the case $T_3 = 2$ in combination with two bad strokes. This results in $\binom{3}{2} \times 3 = 3 \times 3 = 9$ sequences of strokes. However, a score of four can also be obtained with each of the sequences OOO and OOG in combination with one bad stroke. This results in $\binom{3}{1} \times 2 = 3 \times 2 = 6$ sequences of scores. Therefore for $T_3 = 4$ one obtains

$$P(X_3 = 4) = 3q^2\left(p^2 + 2(1-p-q)p\right) + 3q\left((1-p-q)^3 + (1-p-q)^2p\right).$$

More generally, for $2 \leqslant n$ one obtains

$$P(T_3 = n) = \binom{n-1}{n-2} q^{n-2}\left(p^2 + 2p(1-p-q)\right) + \binom{n-1}{n-3} q^{n-3}\left(p(1-p-q)^2 + (1-p-q)^3\right).$$

Note that this formula also applies for $1 \leqslant n$.

2.2 *The Hardy distribution for a par four hole*

In the case of a par four one has the Markov chain $X_0, X_1, X_2, \ldots$, on states 0, 1, 2, 3, 4, 5. The Markov chain starts at time zero in state $X_0 = 0$. If the Markov chain begins on state 0, it oscillates in states 0, 1, 2 and 3 for a random duration and then proceeds either to state 4 or to state 5, where it is trapped or absorbed. Let

$$T_4 = \min\left(0 \leqslant n;\, X_n = 4 \text{ or } X_n = 5\right)$$

be the time of absorption of the process for a par *four*. The subscript 4 refers to a par four! Note that T_4 simply represents the score on a par *four* hole. An eagle (a score two less than par) can be obtained by shooting GG with transitions (0,2) and (2,4). Therefore for $T_4 = 2$ one obtains

$$P(T_4 = 2) = p^2.$$

Consider the case $T_4 = 3$. The possible sequences of strokes with no bad strokes are (see also [2]):

$$\begin{array}{lll} GOO & \text{with} & (0,2) \text{ and } (2,3) \text{ and } (3,4) \\ OGO & \text{with} & (0,1) \text{ and } (1,3) \text{ and } (3,4) \end{array}$$

The correlation between the total score of rounds 1 and 2 was significant for the British Open of 2011 and almost significant for the US Open of 2011. Therefore one might argue that these data cannot be used for a goodness-of-fit test, because there is at least some evidence against the assumption of equivalence of the players. One way to get a homogeneous set of players would be to remove the amateur players from the data set. If the amateur players ($N = 12$) were removed from the US Open data set, the resulting correlation between rounds 1 and 2 was *not* significant any more ($r = 0.030$, $p = 0.362$, $N = 143$). However, if the amateur players ($N = 39$) were removed from the British Open data set, the resulting correlation between round 1 and 2 was still significant ($r = 0.231$, $p = 0.006$, $N = 116$).

Another way to circumvent the problem of a too high correlation between round 1 and round 2 scores would be to take into consideration that the goodness-of-fit tests to be used are performed on hole scores instead of on round scores. Therefore, instead of taking the correlation between round scores, one could take the correlation between hole scores and this for each hole separately. For each hole for which this correlation is not significant one may conclude that the scores are obtained from an imaginary 'single' player. In the case of the US Open Championship 2011 only for one hole a 1-tailed significant correlation was found at the level of 5%. This was hole 7 ($r = 0.171$, $p = 0.016$, $N = 155$). In the case of the British Open Championship 2011 also a 1-tailed significant correlation was found at the level of 5% for only one hole. This was hole 16 ($r = 0.162$, $p = 0.022$, $N = 155$). Goodness-of-fit tests could be performed on the remaining 17 holes.

On some occasions the score frequencies of a hole are not according to what would be expected for the par of the hole. For example, the score frequencies of a hole, which was originally designated as a par four hole, could be more in agreement with a par five than with a par four hole. This could be the case if the mean is closer to 5 than to 4. In these cases the Hardy distribution should be tested for the more appropriate par. An example of such a situation is given by hole four on the second day of the British Open 2011. The frequencies f_n which belong to the various scores (n) were as follows:

n	3	4	5	6	7	8
f_n	5	69	66	12	2	1

Maximum likelihood estimation (MLE) for p and q was performed by maximising the log-likelihood $\log L$:

$$\log L = \sum_{n=3}^{8} f_n \ln\left(P\left(T_m = n\right)\right).$$

Each of the partial derivatives of $\log L$ with respect to p and to q was set equal to zero and the resulting two equations were solved for p and q. Solving was done numerically using the function 'fsolve' from *Maple* 12.

The latter is a necessary condition for the application of a goodness-of-fit test, which is based on a comparison of the observed and expected frequency distributions. A possible way to circumvent this problem would be to use different players, each player playing the hole one time. However, this would only make sense if there was sufficient evidence that the players (competitors) would all perform at the same proficiency level.

In analogy with the concept of *replicate measurements* in test psychology (see [4] chap. 1.12, p. 46) one may consider the first two rounds of a tournament as two replicate tests. This would also be the case for the second two rounds. However, in the case of these rounds the number of players is considerably reduced, which make these rounds less suitable for a goodness-of-fit test. According to the *classical test theory model* (see [4[, part 2, p. 55) the correlation between replicate tests is a measure for the reliability of the test. The reliability of the observed test (or round) score X, which is denoted as $\rho_{XX'}$, is defined as the ratio of the true score variance σ_X^2 to the observed score variance:

$$\rho_{XX'} = \frac{\sigma_T^2}{\sigma_X^2} = \frac{\sigma_T^2}{\sigma_T^2 + \sigma_E^2}$$

where X and X' are replicate tests (round 1 and round 2). The observed score X equals the true score T plus some error E, i.e. $X = T + E$. Therefore, if the correlation between round 1 and round 2 is equal to zero (or not significantly different from zero), one may validly conclude that the systematic differences between players may be neglected, and the round scores may be considered as being produced by an imaginary 'single' player. What has been said here for round scores naturally also holds for hole scores.

For any goodness-of-fit test the sample size has to be sufficiently large. Therefore it is recommended only to use the results of round 1 and 2 for a goodness-of-fit test and to perform the test for day 1 and 2 separately as a kind of double check. In most of the professional golf tournaments the correlation between round 1 and round 2 significantly differs from zero, although at the same time these correlations are generally very low. For example, in the US Open Championship of 2011 this correlation was almost 1-tailed significant at the 5%-level ($r = 0.124$, $p = 0.063$, $N = 155$). Note, that $N = 155$ and not $N = 156$. The total number of players during the first and second round was equal to 156 (52 flights of three players each). However, during the second round one player was 11 over par with one hole to play when darkness suspended his round. Rather than return for one hole only to miss the cut anyway, this player withdrew. This player was removed from the data set resulting in a total of 155 players. In the British Open Championship of 2011 the correlation between the total score of rounds 1 and 2 was really 1-tailed significant ($r = 0.331$, $p = 0.000$, $N = 155$). One of the players was forced to withdraw due to injury during the second round. This player was also removed from the data set resulting in a total of 155 players.

OOG with (0,1) and (1,2) and (2,4)
OGG with (0,1) and (1,3) and (3,5)
GOG with (0,2) and (2,3) and (3,5).

The possible sequences of strokes with one bad stroke are (see also [2]):

GBG with (0,2) and (2,2) and (2,4)
BGG with (0,0) and (0,2) and (2,4).

Therefore for $T_4 = 3$ one obtains

$$P(T_4 = 3) = 3p(1 - p - q)^2 + 2p^2(1 - p - q) + 2p^2q.$$

Consider the case $T_4 = 4$. For the sequence *GG* in combination with two bad strokes one has $\binom{3}{2} \times 1 = 3 \times 1 = 3$ sequences of strokes. For each of the possible sequences of three strokes with no bad stroke in combination with one bad stroke one has $\binom{3}{1} \times 5 = 3 \times 5 = 15$ sequences of strokes. Finally one has the sequences of strokes *OOOO* and *OOOG*. Therefore for $T_4 = 4$ one obtains

$$P(X_4 = 4) = 3q^2p^2 + 3q\left(2p^2(1 - p - q) + 3p(1 - p - q)^3\right) + \\ + p(1 - p - q)^3 + (1 - p - q)^4.$$

More generally, one may obtain the probability distribution $P(T_4 = n)$ for $2 \leqslant n$:

$$P(T_4 = n) = \binom{n-1}{n-2}q^{n-2}p^2 + \\ + \binom{n-1}{n-3}q^{n-3}\left(2p^2(1 - p - q) + 3p(1 - p - q)^2\right) + \\ + \binom{n-1}{n-4}q^{n-4}\left(p(1 - p - q)^3 + (1 - p - q)^4\right).$$

Note the resemblance with $P(T_3 = n)$.

2.3 *The Hardy distribution for a par five hole*

In the case of a par five, one has the Markov chain $X_0, X_1, X_2, \ldots$, on states 0, 1, 2, 3, 4, 5, 6. The Markov chain starts at time zero in state $X_0 = 0$. If the Markov chain begins on state 0, it oscillates in states 0, 1, 2, 3 and 4 for a random duration and then proceeds either to state 5 or to state 6, where it is trapped or absorbed. Let

$$T_5 = \min\left(0 \leqslant n; X_n = 5 \text{ or } X_n = 6\right)$$

be the time of absorption of the process for a par *five*. The subscript 5 refers to a par five! Note that T_5 simply represents the score on a par *five* hole. An eagle can be obtained by the following sequences of strokes with transitions (i, j):

OGG with (0,1) and (2,4) and (4,6)
GOG with (0,2) and (2,3) and (3,5)
GGO with (0,2) and (2,4) and (4,5)
GGG with (0,2) and (2,4) and (4,6).

Therefore for $T_5 = 3$ one obtains

$$\mathrm{P}(T_5 = 3) = 3p^2(1 - p - q) + p^3.$$

Consider the case $T_5 = 4$ (a birdie). For each of the above triples *OGG*, *GOG*, *GGO* and *GGG* in addition to one bad shot (B) one obtains 12 possibilities to score a birdie. For each of the quadruples *OOGG*, *OGOG* and *GOOG* with two good shots one obtains 3 possibilities to score a birdie and for each of the quadruples *OOOG*, *OOGO*, *OGOO* and *GOOO* with one good shot 4 possibilities. This yields a total of 19 ways to score a birdie. Therefore the probability for a birdie is

$$\mathrm{P}(T_5 = 4) = 9p^2(1-p-q) + 3p^3 + 3p^2(1-p-q)^2 + 4p(1-p-q)^3.$$

Consider the case $T_5 = 5$ (a par). For each of the above triples *OGG*, *GOG*, *GGO* and *GGG* in addition to two bad shots (B) one obtains $\binom{4}{2} \times 4 = 6 \times 4 = 24$ possibilities to score a par. For each of the quadruples *OOGG*, *OGOG* and *GOOG* in addition to one bad shot one obtains $\binom{4}{1} \times 3 = 4 \times 3 = 12$ possibilities to score a par and for each of the quadruples *OOOG*, *OOGO*, *OGOO* and *GOOO* with one bad shot $\binom{4}{1} \times 4 = 4 \times 4 = 16$ possibilities. Finally, one may obtain a par with the sequences *OOOOO* and *OOOOG*. This yields a total of 54 ways to score a par. Therefore the probability for a par is

$$\begin{aligned}\mathrm{P}(T_5 = 5) = 6q^2\left(p^3 + 3p^2(1 - p - q)\right) &+ \\ + 4q\left(3p^2(1 - p - q)^2 + 4p(1 - p - q)^3\right) &+ \\ + p(1 - p - q)^4 + (1 - p - q)^5&.\end{aligned}$$

More generally, one may obtain the probability distribution $\mathrm{P}(T_5 = n)$ for $3 \leqslant n$:

$$\begin{aligned}\mathrm{P}(T_5 = n) &= \binom{n-1}{n-3}q^{n-3}\left(p^3 + 3p^2(1 - p - q)\right) + \\ &+ \binom{n-1}{n-4}q^{n-4}\left(3p^2(1 - p - q)^2 + 4p(1 - p - q)^3\right) + \\ &+ \binom{n-1}{n-5}q^{n-5}\left(p(1 - p - q)^4 + (1 - p - q)^5\right).\end{aligned}$$

Note the resemblance to the formulas for $\mathrm{P}(T_3 = n)$ and $\mathrm{P}(T_4 = n)$.

3. *General formula for the Hardy distribution*

The probability distribution $\mathrm{P}(T_m = n)$ for $m = 1, 2, 3, \ldots$, where m denotes the par of a hole and $\lfloor y \rfloor = \max\{j \in \mathbb{Z} \mid j \leqslant y\}$, is as follows:

$$\mathrm{P}(T_m = n) = \sum_{j=\lfloor k \rfloor}^{m} \binom{n-1}{n-j} q^{n-j}(A_{jm} + B_{jm}) \text{ with } k = \frac{m+1}{2}$$

where

$$A_{jm} = \binom{j-1}{2j-m-1} p^{m+1-j}(1 - p - q)^{2j-m-1}$$

and

$$B_{jm} = \binom{j}{2j-m} p^{m-j}(1 - p - q)^{2j-m}.$$

The moment generating function $M_{T\,|\,m}(t) = \mathrm{E}\left(e^{tT_m}\right)$ is as follows:

$$M_{T\,|\,m}(t) = \sum_{j=\lfloor k \rfloor}^{m} \frac{(X_{jm} + Y_{jm})\,e^{jt}}{(1 - e^t q)^j}$$

where

$$X_{jm} = \binom{j-1}{2j-m-1} p^{m+1-j}(1 - p - q)^{2j-m-1}$$

and

$$Y_{jm} = \binom{j}{2j-m} p^{m-j}(1 - p - q)^{2j-m}.$$

The mean μ_m of T_m is as follows:

$$\mu_m = \sum_{j=1}^{m} -\frac{(m + 1 - j)\,p^{j-1}}{(q - 1)^j} \qquad m = 1, 2, 3, \ldots$$

The simplicity of this formula makes it attractive.

4. *Validation using professional players*

To inquire the validity of a certain probability distribution it is common practice to compare the observed frequency distribution of scores with the expected frequency distribution using a goodness-of-fit test such as, for example, the Kolmogorov-Smirnov Z test or Pearson's chi-square test. To do so, however, one must have at one's disposal many scores. It is very difficult though to find a player who is prepared to play a hole many times in order to obtain enough scores. Moreover, due to practice, the player might become more familiar with the hole in the course of playing. This would mean that subsequent hole scores would be subject to some learning effect and could not, therefore, be considered as a collection of pure replications.