MISCELLANEOUS PROBLEMS

47. A number is called an *algebraic number* if it is a solution of a polynomial equation $a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0$ where $a_0, a_1, \ldots, a_n$ are integers. Prove that (*a*) $\sqrt{3} + \sqrt{2}$ and (*b*) $\sqrt[3]{4} - 2i$ are algebraic numbers.

(*a*) Let $z = \sqrt{3} + \sqrt{2}$ or $z - \sqrt{2} = \sqrt{3}$. Squaring, $z^2 - 2\sqrt{2}\,z + 2 = 3$ or $z^2 - 1 = 2\sqrt{2}\,z$. Squaring again, $z^4 - 2z^2 + 1 = 8z^2$ or $z^4 - 10z^2 + 1 = 0$, a polynomial equation with integer coefficients having $\sqrt{3} + \sqrt{2}$ as a root. Hence $\sqrt{3} + \sqrt{2}$ is an algebraic number.

(*b*) Let $z = \sqrt[3]{4} - 2i$ or $z + 2i = \sqrt[3]{4}$. Cubing, $z^3 + 3z^2(2i) + 3z(2i)^2 + (2i)^3 = 4$ or $z^3 - 12z - 4 = i(8 - 6z^2)$. Squaring, $z^6 + 12z^4 - 8z^3 + 48z^2 + 96z + 80 = 0$, a polynomial equation with integer coefficients having $\sqrt[3]{4} - 2i$ as a root. Hence $\sqrt[3]{4} - 2i$ is an algebraic number.