Chapter 1

Problem 1.1

Use Gauss' theorem [and (1.21) if necessary] to prove the following:

(a) Any excess charge placed on a conductor must lie entirely on its surface. (A conductor by definition contains charges capable of moving freely under the action of applied electric fields.)

(b) A closed, hollow conductor shields its interior from fields due to charges outside, but does not shield its exterior from the fields due to charges placed inside it.

(c) The electric field at the surface of a conductor is normal to the surface and has a magnitude σ/ϵ_0, where σ is the charge density per unit area on the surface.

Part a

Inside the conductor, we must have $\mathbf{E} = 0$, otherwise the charges inside would move. Application of Gauss' law

$$\oint_S \mathbf{E} \cdot \mathbf{n}\, da = \frac{1}{\epsilon_0} \int_V \rho(\mathbf{x})\, d^3x$$

with a surface S just inside the surface of V yields $\rho(\mathbf{x})$=0 so excess charge must lie entirely on the surface of a conductor.

Part b

With a charge outside the conductor, one can construct a surface S completely inside the conducting region and thus containing the hollow part. There is not electric flux through S and thus the inner surface always has zero surface charge density when no charge is contained within the hollow part. The effect of the external charge is to induce a non-zero surface charge density on the outer conductor surface in order to maintain $\mathbf{E}$=0 inside the conductor. Since no net charge resides on the inner surface, $\mathbf{E}$ is still zero in the hollow part. Thus the conductor shields its entire interior from the fields of the external charge.

If on the other hand a charge is placed within the hollow part, a corresponding charge of opposite sign is induced on the inner surface to keep $\mathbf{E}$=0 in the

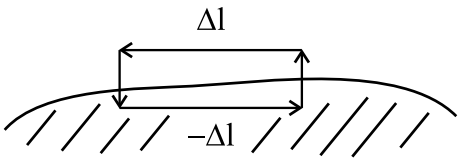

Figure 1.1: Integration path used in 1.1c

conducting region. The charge induced on the inner surface obviously has to come from the conductor itself, so a total charge equal to the charge in the hollow part is induced on the outer surface. The fields from charges inside the conductor are thus still present outside the conductor.

Part c

We prove the first part by the use of (1.21)

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0.$$

We use the integration path shown in Fig. 1.1. Inside the conductor $\mathbf{E}$=0 so

$$0 = \oint \mathbf{E} \cdot d\mathbf{l} = -\Delta\mathbf{l} \cdot 0 + \int_{\uparrow} +\mathbf{E} \cdot \Delta\mathbf{l} + \int_{\downarrow}$$

where the two integrals tend to zero as their path lengths are diminished. Any field $\mathbf{E}$ at the surface can be divided into components parallel and perpendicular to the surface: $\mathbf{E} = \mathbf{E}_\perp + \mathbf{E}_\parallel$ and since $\Delta\mathbf{l}$ is parallel to the surface, $\mathbf{E}_\perp \cdot \Delta\mathbf{l}$=0 and we are left with $E_\parallel$=0 and we have shown that the electric field is normal to the surface.

For the second part, introduce a Gauss box with on end inside the conductor and the other in vacuum. Again $\mathbf{E}$=0 inside the conductor and we just showed that the electric field at the surface is normal to the surface. If we let the thickness across the surface tend to zero, there is no contribution to the electric flux through the sides of the box and the only contribution is from the box surface just outside the conductor surface. Thus from Gauss' law

$$E_\perp da = \frac{\sigma}{\epsilon_0}\, da$$

and $\mathbf{E} = \sigma/\epsilon_0\, \mathbf{n}$.

Problem 1.2

The Dirac delta function in three dimensions can be taken as the improper limit as $\alpha \to 0$ of the Gaussian function

$$D(\alpha;\, x,\, y,\, z) = (2\pi)^{-3/2}\alpha^{-3} \exp\left[-\frac{1}{2\alpha^2}(x^2 + y^2 + z^2)\right]$$

Consider a general orthogonal coordinate system specified by the surfaces $u=$ constant, $v=$ constant, $w=$ constant, with length elements du/U, dv/V, dw/W in the three perpendicular directions. Show that

$$\delta(\mathbf{x} - \mathbf{x}') = \delta(u - u')\,\delta(v - v')\,\delta(w - w') \cdot UVW$$

by considering the limit of the Gaussian above. Note that as $\alpha \to 0$ only the infinitesimal length element need be used for the distance between the points in the exponent.

Solution

I'm sorry to say that this problem is the only problem in Chapter 1 I have not been able to solve. If any of you mathematician out there can present me a simple solution, I would be most grateful.

Problem 1.3

Using Dirac delta function in the appropriate coordinates, express the following charge distributions as three-dimensional charge densities $\rho(\mathbf{x})$

(a) In spherical coordinates, a charge Q uniformly distributed over a spherical shell of radius R.

(b) In cylindrical coordinates, a charge λ per unit length uniformly distributed over a cylindrical surface of radius b.

(c) In cylindrical coordinates, a charge Q spread uniformly over a flat circular disc of negligible thickness and radius R.

(d) The same as part (c), but using spherical coordinates.

Part a

We work in spherical coordinates (r, θ, ϕ) and volume element $d^3x = r^2 d(\cos\theta)\, d\phi\, dr$. We can write charge density as

$$\rho(\mathbf{x}) = f(\mathbf{x})\delta(r - R) = f(r)\delta(r - R) = f(R)\delta(r - R)$$

where the factor f is still to be determined. An integration over all space should give the total charge Q:

$$\begin{aligned} \int \rho(\mathbf{x})\, d^3x &= f(R) \int r^2 dr\, d\Omega\, \delta(r - R) = 4\pi f(R) R^2 = Q \;\Rightarrow \\ f(R) &= \frac{Q}{4\pi R^2} \;\Rightarrow \\ \rho(\mathbf{x}) &= \frac{Q}{4\pi R^2}\, \delta(r - R) \end{aligned}$$

Part b

Cylindrical coordinates (r, ϕ, z) and volume element $d^3x = \rho\, d\rho\, d\phi\, dz$, see Fig. 1.2. We consider an arbitrary length L along the z-axis.

$$\begin{aligned} \rho(\mathbf{x}) &= f(\mathbf{x})\delta(\rho - b) = f(b)\delta(\rho - b) \\ \int \rho(\mathbf{x})\, d^3x &= f(b) \int \rho\, d\rho\, d\phi\, dz\, \delta(\rho - b) = 2\pi b L f(b) = \lambda L \;\Rightarrow \\ f(b) &= \frac{\lambda}{2\pi b} \;\Rightarrow \\ \rho(\mathbf{x}) &= \frac{\lambda}{2\pi b}\, \delta(\rho - b) \end{aligned}$$

Part c

Cylindrical coordinates (r, ϕ, z) and volume element $d^3x = \rho\, d\rho\, d\phi\, dz$.

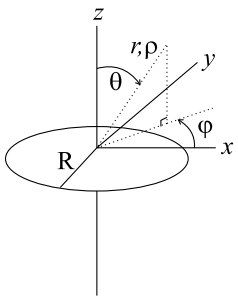

Figure 1.2: Definition of variables

$$\begin{aligned}
\rho(\mathbf{x}) &= f(R)\delta(z)\,\Theta(R-\rho) \\
\int \rho(\mathbf{x})\,d^3x &= f(R)\int \rho\,d\rho\,d\phi\,dz\,\delta(z)\,\Theta(R-\rho) = 2\pi f(R)\int \rho\,d\rho\,\Theta(R-\rho)\int dz\delta(z) \\
&= 2\pi f(R)\left[\frac{\rho^2}{2}\right]_0^R = 2\pi f(R)\frac{R^2}{2} = Q \;\Rightarrow \\
f(R) &= \frac{Q}{\pi R^2} \;\Rightarrow \\
\rho(\mathbf{x}) &= \frac{Q}{\pi R^2}\,\delta(z)\,\Theta(R-\rho)
\end{aligned}$$

Part d

(r,θ,ρ), $d^3x = r^2 dr\,d(\cos\theta)\,d\phi$. This is the only example where the factor f will depend on one of the independent variables, r.

$$\begin{aligned}
\rho(\mathbf{x}) &= f(\mathbf{x})\delta(\cos\theta)\,\Theta(R-r) = f(r)\delta(\cos\theta)\Theta(R-r) \\
\int \rho(\mathbf{x})\,d^3x &= \int f(r)\,\delta(\cos\theta)\,d(\cos\theta)\Theta(R-r)r^2 dr\,d\phi
\end{aligned}$$

The δ-function kills the $d(\cos\theta)$ and the step function defines the limits on the r-integration to $0 \le r \le R$.

$$\int \rho(\mathbf{x})\,d^3x = \int_0^R f(r)r^2\,dr\,d\phi = \int_0^R [f(r)r]\;r\,dr\,d\phi$$

Now, $r\,dr\,d\phi$ is an area element. Since the surface charge density is constant, the factor multiplying the area element must be a constant. Thus $f(r)r$ can be moved outside the integration.

$$\begin{aligned}
\int \rho(\mathbf{x})\,d^3x &= 2\pi f(r)r\int_0^R r\,dr = \pi R^2 f(r)r = Q \;\Rightarrow \\
f(r) &= \frac{Q}{\pi R^2}\frac{1}{r} \;\Rightarrow \\
\rho(\mathbf{x}) &= \frac{Q}{\pi R^2}\frac{1}{r}\,\delta(\cos\theta)\Theta(R-r)
\end{aligned}$$

Problem 1.4

Each of three charged spheres of radius a, one conducting, one having a uniform charge density within its volume, and one having a spherically symmetric charge density that varies radially as r^n $(n > -3)$, has a total charge Q. Use Gauss's theorem to obtain the electric fields both inside and outside each sphere. Sketch the behavior of the fields as a function of radius for the first two spheres, and for the third with $n = -2, +2$.

Solution

In all cases, the electric field is radially symmetric so $\mathbf{E} = E(r)\mathbf{e}_r$. Furthermore, since all charge is found at $r \leq a$, the electric field behavior for $r > a$ is the same for all cases, namely the field from a point charge at the origin.

(i) The sphere is conductive so Q resides on the surface only, see Sec. 1a, and $\sigma = Q/4\pi a^2$. According to Gauss's theorem, $\mathbf{E}$ makes a sudden jump at $r = a$ and drops off as if Q were centered at the origin. Thus

$$E(r) = \frac{Q}{4\pi\epsilon_0 r^2}\,\Theta(r-a)$$

(ii) Here Q is distributed evenly in the sphere so the constant space charge density is $\rho = Q/\frac{4}{3}\pi a^3$. Application of Gauss's theorem for $r \leq a$ yields

$$\begin{aligned} E(r)\cdot 4\pi r^2 &= \frac{Q}{\frac{4}{3}\pi a^3\epsilon_0}\cdot\frac{4}{3}\pi r^3 = \frac{Q}{\epsilon_0}\left(\frac{r}{a}\right)^3 \Rightarrow \\ E(r) &= \frac{Q}{4\pi\epsilon_0 a^3}\,r \end{aligned}$$

The behavior for $r > a$ has already been mentioned above.

(iii) Application of Gauss's theorem for $r \leq a$ yields

$$\begin{aligned} E(r)\cdot 4\pi r^2 &= \frac{4\pi}{\epsilon_0}\int_0^r r'^2\cdot Kr^n\,dr' = \frac{4\pi K}{\epsilon_0}\frac{r^{n+3}}{n+3} \Rightarrow \\ E(r) &= \frac{K}{\epsilon_0(n+3)}r^{n+1} \end{aligned}$$

The constant K can be determined easily since the volume integral over all space of the charge distribution is the total charge Q. Plugging in the expression for K gives

$$E(r) = \frac{Q}{4\pi\epsilon_0 r^2}\left(\frac{r}{a}\right)^{n+3}$$

Fig. 1.3 shows sketches of the electric fields.

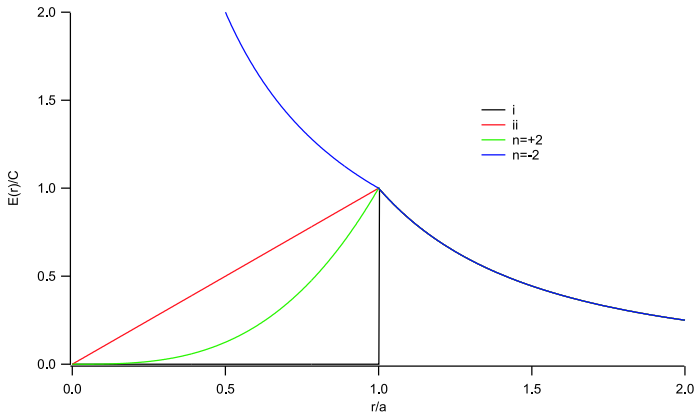

Figure 1.3: Sketch of the electric field behavior of the charge distribution in Problem 1.4. The constant C is defined as $C = \frac{Q}{4\pi\epsilon_0 a^2}$

Problem 1.5

The time-averaged potential of a neutral hydrogen atom is given by

$$\Phi = \frac{q}{4\pi\epsilon_0}\frac{e^{-\alpha r}}{r}\left(1+\frac{\alpha r}{2}\right)$$

where q is the magnitude of the electronic charge, and $\alpha^{-1} = a_0/2$, a_0 being the Bohr radius. Find the distribution of charge (both continuous and discrete) that will give this potential and interpret your result physically.

Solution

Obviously we have to employ Poisson's equation (1.28) here. Φ only depends on r so we need the first term in the Laplacian in spherical coordinates (look inside the back cover). Note that the radial Laplacian can be written in two ways:

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)$$

For some reason, only the first version automatically gives you the discrete charge (the nucleus). It comes about when you manage to make one term look like $\nabla^2(1/r)$, which equals $-4\pi\delta(\mathbf{x})$. The second version is much easier, but you have to add a δ-function after doing the derivative. Here goes:

$$\begin{aligned}\nabla^2\Phi &= \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(-\alpha\frac{e^{-\alpha r}}{r}+e^{-\alpha r}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)-\frac{\alpha^2}{2}e^{-\alpha r}\right)\right]\\ &= \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\left[\alpha^2 r e^{-\alpha r}-\alpha e^{-\alpha r}+\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right)e^{-\alpha r}\right.\\ &\quad\left.-\alpha e^{-\alpha r}r^2\left(-\frac{1}{r^2}\right)-\frac{\alpha^2}{2}2re^{-\alpha r}+\frac{\alpha^3}{2}r^2e^{-\alpha r}\right]\end{aligned}$$

The second term in the square bracket is the $-4\pi\delta(\mathbf{x})$ and since it only contributes for $r = 0$, the factor multiplying it is just unity.

$$\begin{aligned}
\nabla^2\Phi &= \frac{q}{4\pi\epsilon_0}\left[\frac{\alpha^3}{2}e^{-\alpha r} - 4\pi\delta(\mathbf{x})\right] \\
&= -\frac{1}{\epsilon_0}\left[q\delta(\mathbf{x}) - \frac{q\alpha^3}{8\pi}e^{-\alpha r}\right] \Rightarrow \\
\rho(\mathbf{x}) &= q\delta(\mathbf{x}) - \frac{q\alpha^3}{8\pi}e^{-\alpha r}
\end{aligned}$$

If you try to write up the wave function for a hydrogen s-state and form the squared norm, you will find the second term is indeed the corresponding charge density.

Problem 1.6

A simple capacitor is a device formed by two insulated conductors adjacent to each other. If equal and opposite charges are placed on the conductors, there will be a certain difference of potential between them. The ratio of the magnitude of the charge on one conductor to the magnitude of the potential difference is called the capacitance (in SI units it is measured in farads). Using Gauss's law, calculate the capacitance of

(a) two large, flat, conducting sheets of area A, separated by a small distance d;

(b) two concentric conducting spheres with radii a, b $(b > a)$;

(c) two concentric conducting cylinders of length L, large compared to their radii a, b $(b > a)$.

(d) What is the inner diameter of the outer conductor in an air-filled coaxial cable whose center conductor is a cylindrical wire of diameter 1 mm and whose capacitance is 3×10^{-11} F/m? 3×10^{-12} F/m?

Solution

The technique here is to first find the behavior of E as a function of Q, the magnitude of the charge on either conductor, and the geometric quantities describing the system from Gauss's law (1.11):

$$\oint_S \mathbf{E} \cdot \mathbf{n}\, da = \frac{1}{\epsilon_0} \int_V \rho(\mathbf{x})\, d^3x$$

From this, one can find the potential difference V between the two conductors (1.20):

$$\int_A^B \mathbf{E} \cdot d\mathbf{l} = -(\Phi_B - \Phi_A)$$

Part a

Use a Gauss box with one end inside the conductor where $\mathbf{E} = 0$ (this is the same as Problem 1.1c) and we get $E = Q/A\epsilon_0$. Then the potential difference is simply $V = Ed = Qd/A\epsilon_0$ and capacitance

$$C = \frac{Q}{V} = \frac{\epsilon_0 A}{d}$$

Part b

Here the electric field is only non-zero between the two shells

$$\begin{aligned} E(r) &= \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2} \\ V &= -\int_b^a E\, dr = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) \\ C &= \frac{Q}{V} = 4\pi\epsilon_0 \left(\frac{1}{a} - \frac{1}{b}\right)^{-1} \end{aligned}$$

Part c

$$\begin{aligned} E(r) &= \frac{Q}{2\pi\epsilon_0 L}\frac{1}{r} \\ V &= -\int_b^a E\,dr = \frac{Q}{2\pi\epsilon_0 L}\ln\frac{b}{a} \\ C &= \frac{Q}{V} = 2\pi\epsilon_0 L\left(\ln\frac{b}{a}\right)^{-1} \end{aligned}$$

Part d

With $2a = 1$ mm and specified values for C/L, we can find $2b$ from

$$2b = 2a\exp\left(\frac{2\pi\epsilon_0}{C/L}\right)$$

$C/L = 3 \times 10^{-11}$ F/m: $2b = 6.4$ mm
$C/L = 3 \times 10^{-12}$ F/m: $2b = 113$ km!
This final example is probably included to demonstrate the difficulties in changing the capacitance by geometric means only.

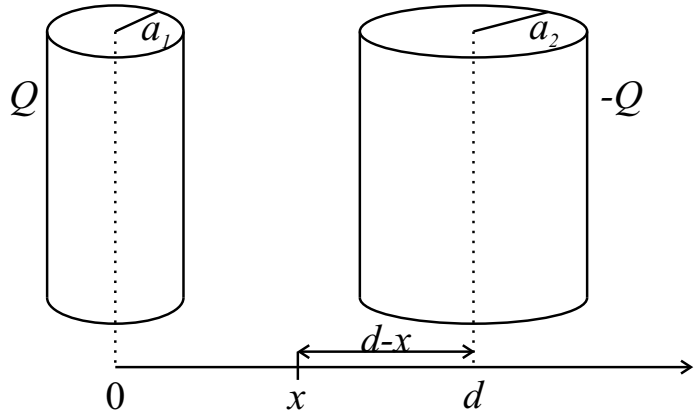

Figure 1.4: The situation in Problem 1.7. Notice that $d \gg a_1, a_2$.

Problem 1.7

Two long, cylindrical conductors of radii a_1 and a_2 are parallel and separated by a distance d, which is large compared with either radius. Show that the capacitance per unit length is given approximately by

$$C \backsimeq \pi\epsilon_0 \left(\ln \frac{d}{a} \right)^{-1}$$

where a is the geometrical mean of the two radii.

Approximately what gauge wire (state diameter in millimeters) would be necessary to make a two-wire transmission line with a capacitance of 1.2×10^{-11} F/m if the separation of the wires was 0.5 cm? 1.5 cm? 5.0 cm?

Solution

The plan of attack is similar to that of the preceding solution. First, the geometrical mean G is defined as $G = \sqrt[n]{a_1 a_2 \ldots a_n}$ so here we have $a = (a_1 a_2)^{1/2}$. The configuration of conductors i seen in Fig. 1.4 The electric field in all space in easily found from Gauss's law:

$$E(x) = \frac{1}{2\pi\epsilon_0 L} \left(\frac{Q}{x} - \frac{-Q}{d-x} \right)$$

We take $E(x)$ to be positive in the direction of the axis. Next we find the potential V:

$$\begin{aligned} V &= -\int_{d-a_2}^{a_1} \frac{Q}{2\pi\epsilon_0 L} \left(\frac{1}{x} + \frac{1}{d-x} \right) dx = \frac{Q}{2\pi\epsilon_0 L} \ln \frac{d-a_1}{a_1} \frac{d-a_2}{a_2} \\ &\overset{d \gg a_1, a_2}{=} \frac{Q}{\pi\epsilon_0 L} \ln \frac{d}{a} \end{aligned}$$

We then find the capacitance per unit length:

$$\frac{C}{L} = \frac{Q}{VL} = \pi\epsilon_0 \left(\ln \frac{d}{a} \right)^{-1}$$

To determine the gauge wire diameter $2a$ corresponding to a given C/L=1.2 $\times$ 10^{-11} F/m and separation d, isolate a:

$$2a = 2d \exp\left(-\frac{\pi\epsilon_0}{C/L}\right)$$

d = 5 mm: 2a= 1 mm
d = 15 mm: 2a= 3 mm
d = 50 mm: 2a= 10 mm
What to say about these number? Well, $2a$ scales linearly in d but exponentially in C/L.

Problem 1.8

(a) For the three capacitor geometries in Problem 1.6 calculate the total electrostatic energy and express it alternatively in terms of the equal and opposite charges Q and $-Q$ placed on the conductors *and* the potential difference between them.

(b) Sketch the energy density of the electrostatic field in each case as a function of the appropriate linear coordinate.

Solution

At this point, the progression in the problems deviates from that of the text sections in that we bypass the sections on Green functions and move on to Section 1.11 on electrostatic energy. Three expressions for the electrostatic energy W are given; one emphasizing the charge distribution (1.53), one emphasizing the electric field (1.54), and one involving the capacitance C_{ij} and the potentials of a system of conductors (1.62):

$$W = \frac{1}{2}\int \rho(\mathbf{x})\Phi(\mathbf{x})\, d^3x = \frac{\epsilon_0}{2}\int |\mathbf{E}|^2\, d^3x = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}V_iV_j$$

We already have expressions for $\mathbf{E}$ involving Q and equations connecting Q and V from Problem 1.6 so the second form (1.54) is easily exploited. Even easier is the application of the third form, which reduces to $W = \frac{1}{2}CV^2$ when only two conductors are present.

Part a

For the parallel-plate capacitor, $C = \epsilon_0 A/d$ and $V = Qd/A\epsilon_0$ so

$$W = \frac{1}{2}\frac{\epsilon_0 A}{d}V^2 = \frac{1}{2}\frac{d}{\epsilon_0 A}Q^2$$

For the spherical capacitor,

$$\begin{aligned} V &= \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right) \\ C &= \frac{4\pi\epsilon_0}{1/a-1/b} \\ W &= \frac{1}{2}\frac{4\pi\epsilon_0}{1/a-1/b}V^2 = \frac{2\pi\epsilon_0}{1/a-1/b}V^2 = \frac{1/a-1/b}{8\pi\epsilon_0}Q^2 \end{aligned}$$

For the cylindrical capacitor,

$$\begin{aligned} V &= \frac{Q}{2\pi\epsilon_0 L}\ln\frac{b}{a} \\ C &= \frac{2\pi\epsilon_0 L}{\ln b/a} \\ W &= \frac{\pi\epsilon_0 L}{\ln b/a}V^2 = \frac{\ln b/a}{4\pi\epsilon_0 L}Q^2 \end{aligned}$$

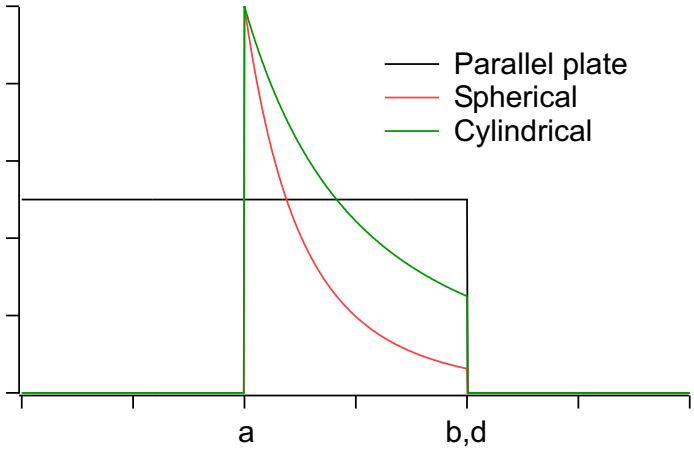

Figure 1.5: Sketches of the energy densities w in the three capacitor geometries.

Part b

The energy density w is defined in (1.55) as

$$w = \frac{\epsilon_0}{2}|\mathbf{E}|^2$$

We already found $\mathbf{E}$ in Problem 1.6. The three expressions are easily found:

$$\begin{aligned} w &= \frac{Q^2}{2\epsilon_0 A^2} = \frac{\sigma^2}{2\epsilon_0},\ 0 < r < d \\ w &= \frac{Q^2}{32\pi^2\epsilon_0}\frac{1}{r^4},\ a < r < b \\ w &= \frac{Q^2}{8\pi^2\epsilon_0 L^2}\frac{1}{r^2} = \frac{\lambda^2}{8\pi^2\epsilon_0}\frac{1}{r^2},\ a < r < b \end{aligned}$$

It is difficult to directly compare the energy densities for the three cases since different geometric quantities are involved in the expressions for w. Sketches are shown in Fig. 1.5. All we can say is that for the parallel-plate capacitor, w is constant and for the spherical capacitor, the energy is more strongly concentrated close to the inner conductor than in the case of a parallel-cylinder capacitor.

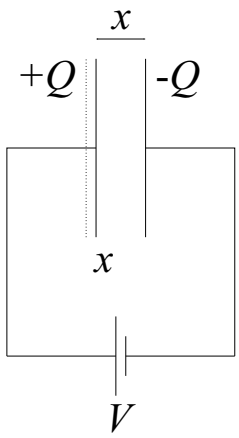

Figure 1.6: Sketch showing the concept of a small virtual displacement, Δx.

Problem 1.9

Calculate the attractive force between conductors in the parallel plate capacitor (Problem 1.6a) and the parallel cylinder capacitor (Problem 1.7) for

(a) fixed charges on each conductor;

(b) fixed potential difference between conductors.

Solution

The technique for solving this problem is described in Section 1.11, starting below the middle of p. 42. We shall make sure to follow Jackson's advice: "Care must be taken to exhibit the energy in a form showing clearly the factors that vary with a change in configuration and those that are kept constant." The change in total electrostatic energy (due to the work we have to do to change the system) has two contributions when a virtual displacement is performed. We can write up an expression taking both effects into account:

- A change in electrostatic energy ΔW as a result of the change in geometry when the capacitor is subject to a small virtual displacement Δx. This term is illustrated in the example beginning p. 42 and on Fig. 1.6.

- A change due to the work necessary to move the charge added during the displacement against a potential difference, $V\Delta Q$.

$$\begin{aligned} F\Delta x &= \Delta W + V\Delta Q \;\Rightarrow \\ F &= \frac{\partial W}{\partial x} + \left|\frac{\partial Q}{\partial x}\right| V \end{aligned}$$

Part a

Here Q is maintained constant, i.e. the conductors are electrically isolated from each other, so the second term above is zero. Following Jackson's warning, we thus aim for expressions involving Q and not V.

(i) For the parallel plate capacitor, we already found the expression for W in Problem 1.8a. The variable quantity is d, the distance between the plates:

$$F = \frac{\partial}{\partial x}\left(\frac{Q^2}{2\epsilon_0 A}x\right) = \frac{Q^2}{2\epsilon_0 A}$$

(ii) For the parallel cylinders, we found the capacitance in Problem 1.7. From (1.61) and (1.62) we get an expression involving Q and C:

$$W = \frac{1}{2}QV = \frac{Q^2}{2C} = \frac{Q^2}{2\pi\epsilon_0 L}\ln\frac{d}{a}$$

The variable quantity here is again $d = x$, the distance between the conductors:

$$F = \frac{\partial}{\partial x}\left(\frac{Q^2}{2\pi\epsilon_0 L}\ln\frac{x}{a}\right) = \frac{Q^2}{2\pi\epsilon_0 x L}$$

Part b

Here the potential difference V is maintained constant so Q varies with the conductor distance. We thus aim for expressions involving V, not Q.

(i) In Problem 1.8a we found the necessary expressions:

$$\begin{aligned} F &= \frac{\partial}{\partial x}\left(\frac{\epsilon_0 A}{2x}V^2\right) + \left|\frac{\partial}{\partial x}\left(\frac{\epsilon_0 A V}{x}\right)\right| \cdot V \\ &= \frac{\epsilon_0 A}{2x^2}V^2 = \frac{Q^2}{2\epsilon_0 A} \end{aligned}$$

The same expression as in part a!

(ii) Here we need $W = \frac{1}{2}CV^2$ and the connection between Q and V from Problem 1.7.

$$\begin{aligned} W &= \frac{\pi\epsilon_0 L}{2\ln x/a}V^2 \\ Q &= \frac{\pi\epsilon_0 L}{\ln x/a}V \\ F &= \frac{\partial W}{\partial x} + \left|\frac{\partial Q}{\partial x}\right| V = \frac{\pi\epsilon_0 L V^2}{2x\ln^2 d/a} = \frac{Q^2}{2\pi\epsilon_0 x L} \end{aligned}$$

Again the same expression as i part a!

The reemergence of the force expressions in part b is not surprising since for a given V, the charge on the conductors is always $Q = CV$. Thus part b could have been solved much easier simply by inserting $Q = CV$ in the expressions from part a. I used another method to demonstrate that the force in part b has two contributions (and also because of Jackson's remarks on p. 42).

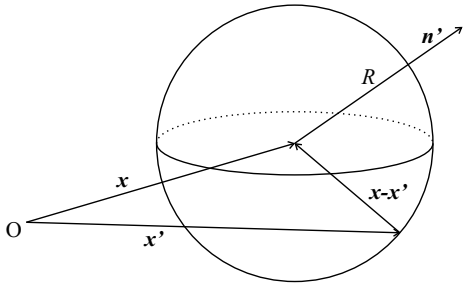

Figure 1.7: The geometry in Problem 1.10.

Problem 1.10

Prove the *mean value theorem*: For charge-free space the value of the electrostatic potential at any point is equal to the average of the potential over the surface of *any* sphere centered on that point.

Solution

There are many ways to do this, e.g. by direct integration, but here I only give one method, which involves the integral form of the Poisson equation (1.36). The quantities involved are shown if Fig. 1.7. Note that $1/R \equiv 1/|\mathbf{x} - \mathbf{x}'|$, that R is constant, and that $\mathbf{x} - \mathbf{x}'$ is parallel to $\mathbf{n}'$. $\mathbf{x}$ is the vector ending in the point of interest and $\mathbf{x}'$ is the integration variable running over the spherical surface S.

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{x}')}{R}d^3x' + \frac{1}{4\pi}\oint_S \left[\frac{1}{R}\frac{\partial\Phi}{\partial n'} - \Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right] da'$$

We work in charge-free space, so the first term above is zero.

$$\begin{aligned}
\Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\oint_S \nabla\Phi\cdot\mathbf{n}'\,da' - \frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\left(-\frac{1}{R^2}\right)da' \\
&= \frac{1}{4\pi R}\oint_S (-\mathbf{E})\cdot\mathbf{n}'\,da' + \frac{1}{4\pi R^2}\oint_S \Phi(\mathbf{x}')\,da' \\
&= \frac{1}{4\pi R^2}\oint_S \Phi(\mathbf{x}')\,da'
\end{aligned}$$

In the last step, I used Gauss's theorem; the first term is zero since S enclosed no charge. This equation says that the potential in $\mathbf{x}$ is the *mean value* of the potential on a spherical surface S centered on $\mathbf{x}$.

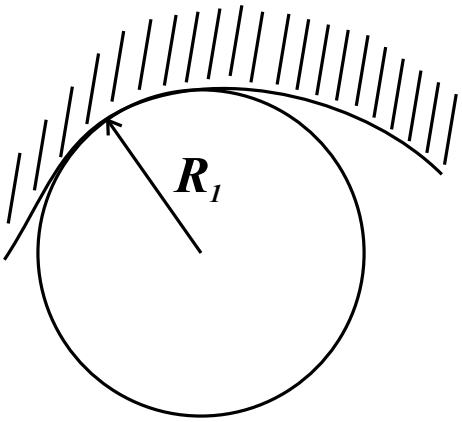

Figure 1.8: The geometry in Problem 1.11.

Problem 1.11

Use Gauss's theorem to prove that at the surface of a curved charged conductor, the normal derivative of the electric field is given by

$$\frac{1}{E}\frac{\partial E}{\partial n} = -\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

where R_1 and R_2 are the principal radii of curvature of the surface.

Solution

The situation is shown in Fig. 1.8. Imagine a second circle perpendicular to the paper plane with radius of curvature R_2. As directed, we use Gauss's theorem, although I prefer to do it without the integrals, so we consider a infinitesimal area da of the conductor containing a charge dq. At this particular area, the radii of curvature are R_1 and R_2. Thus the area da can be written as $da = R_1 d\theta_1 R_2 d\theta_2$. Furthermore, since we need the normal derivative at da, we write the normal unit vector $\mathbf{n}$ at da as

$$\mathbf{n} = \frac{\mathbf{R_1}}{R_1} + \frac{\mathbf{R_2}}{R_2}$$

The electric field just outside the conductor (remember $\mathbf{E}$ is perpendicular to the conductor surface) due to the area da is then

$$E = \frac{1}{\epsilon_0}\frac{dq}{da} = \frac{1}{\epsilon_0}\frac{dq}{R_1 R_2 d\theta_1 d\theta_2}$$

We now need the normal derivative $\nabla\mathbf{E}\cdot\mathbf{n}$ where you project the gradient onto the normal vector. Since the direction of $\mathbf{n}$ is given from the combined directions

of R_1 and R_2, the derivative contains two terms as follows

$$\begin{aligned}
\frac{\partial E}{\partial n} &= \frac{\partial E}{\partial R_1} + \frac{\partial E}{\partial R_2} \\
&= \frac{1}{\epsilon_0} \frac{dq}{d\theta_1 d\theta_2} \left(-\frac{1}{R_1^2 R_2} - \frac{1}{R_1 R_2^2} \right) \\
&= -E \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \Rightarrow \\
\frac{1}{E} \frac{\partial E}{\partial n} &= -\left(\frac{1}{R_1} - \frac{1}{R_2} \right)
\end{aligned}$$

Problem 1.12

Prove *Green's reciprocation theorem*: If Φ is the potential due to a volume-charge density ρ within a volume V and a surface-charge density σ on the conducting surface S bounding the volume V, while Φ' is the potential due to another charge distribution ρ' and σ', then

$$\int_V \rho\Phi'\, d^3x + \int_S \sigma\Phi'\, da = \int_V \rho'\Phi\, d^3x + \int_S \sigma'\Phi\, da$$

Solution

This theorem is most easily proven by invoking Green's theorem (1.35) with $\phi = \Phi$ and $\psi = \Phi'$. This enables a quick introduction of the space and surface charge densities through the Poisson equation (1.28) $\nabla^2\Phi = -\rho/\epsilon_0$ and $\sigma = \epsilon_0(\partial\Phi/\partial n)$.

$$\int_V (\psi\nabla^2\phi - \phi\nabla^2\psi)\, d^3x = \oint_S \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right] da$$
$$\int_V \left(-\frac{\rho'}{\epsilon_0}\Phi + \frac{\rho}{\epsilon_0}\Phi'\right)\, d^3x = \oint_S \left[\frac{\sigma'}{\epsilon_0}\Phi - \frac{\sigma}{\epsilon_0}\Phi'\right] da$$

Multiplying out ϵ_0 gives the desired theorem

$$\int_V \rho\Phi'\, d^3x + \int_S \sigma\Phi'\, da = \int_V \rho'\Phi\, d^3x + \int_S \sigma'\Phi\, da$$

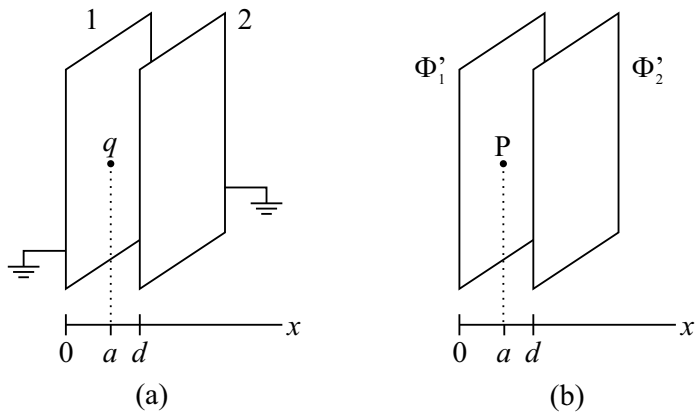

Figure 1.9: Problem 1.13. (a) The "real" system. (b) The mirror setup.

Problem 1.13

Two infinite grounded parallel conducting planes are separated by a distance d. A point charge q is placed between the planes. Use the reciprocation theorem of Green to prove that the total induced charge on one of the planes is equal to $(-q)$ times the fractional perpendicular distance of the point charge from the other plane. (*Hint*: As your comparison electrostatic problem with the same surfaces choose one whose charge densities and potential are known and simple.)

Solution

As our comparison problem, we use a "mirror" setup with no charge in V but the two planes maintained at potentials Φ'_1 and Φ'_2, see Fig. 1.9. For this system, the potential is known and simple (a linear function).

First, several simplifications can be made in Green's reciprocation theorem:

$$\int_V \rho\Phi'\, d^3x + \int_S \sigma\Phi'\, da = \int_V \rho'\Phi\, d^3x + \int_S \sigma'\Phi\, da$$

- The whole RHS is zero since $\Phi = 0$. ρ' is of course zero but now we don't have to bother with the induced charges σ' for the primed case.

- All induced surface charges in the unprimed case are multiplied by the same potential and can be collected, i.e.

$$\int_{S_1} \sigma\Phi'\, da = \Phi' \int_{S_1} \sigma\, da = q_1\Phi'$$

  for the left plane.

- The first term on the LHS is also simple since we can write

$$\int_V \rho\Phi'\, d^3x = \int_V q\delta(x-a)\delta(y)\delta(z)\Phi'_P\, d^3x = q\Phi'_P$$

- We also know that $q_1+q_2 = -q$ since the induced charges on the two planes must shield the point charge to maintain $\mathbf{E} = 0$ inside the conductors.

We thus end up with

$$\begin{aligned} q\Phi'_P + \int_{S_1} \sigma\Phi'\, da + \int_{S_2} \sigma\Phi'\, da &= 0 \;\Leftrightarrow \\ q\Phi'_P + q_1\Phi'_1 + q_2\Phi'_2 &= 0 \\ q_1 = -q\frac{\Phi'_P - \Phi'_2}{\Phi'_1 - \Phi'_2} \quad &\wedge \quad q_2 = -q\frac{\Phi'_1 - \Phi'_P}{\Phi'_1 - \Phi'_2} \end{aligned}$$

Here only Φ'_P is unknown but can be found from known quantities since

$$\Phi' = \Phi'_1 + \frac{x}{d}(\Phi'_2 - \Phi'_1)$$

Inserting $x = a$ yields

$$\Phi'_P = \left(1 - \frac{a}{d}\right)\Phi'_1 + \frac{a}{d}\Phi'_2$$

Finally, the two arbitrary potentials Φ'_1 and Φ'_2 can be eliminated by inserting the latter expression into the expressions for q_1 and q_2

$$q_1 = -q\frac{d-a}{d} \;\wedge\; q_2 = -q\frac{a}{d}$$

This is what we wanted to prove since the factors multiplying $-q$ are the fractional perpendicular distances of the point charge from the other plane.

Problem 1.14[1]

Consider the electrostatic Green functions of Section 1.10 for Dirichlet and Neumann boundary conditions on the surface S bounding the volume V. Apply Green's theorem (1.35) with integration variable $\mathbf{y}$ and $\phi = G(\mathbf{x},\mathbf{y}), \psi = G(\mathbf{x}',\mathbf{y})$, with $\nabla_y^2 G(\mathbf{z},\mathbf{y}) = -4\pi\delta(\mathbf{y}-\mathbf{z})$. Find an expression for the difference $[G(\mathbf{x},\mathbf{x}') - G(\mathbf{x}',\mathbf{x})]$ in terms of an integral over the boundary surface S.

(a) For Dirichlet boundary conditions on the potential and the associated boundary condition on the Green function, show that $G_D(\mathbf{x},\mathbf{x}')$ must be symmetric in $\mathbf{x}$ and $\mathbf{x}'$.

(b) For Neumann boundary conditions, use the boundary condition (1.45) for $G_N(\mathbf{x},\mathbf{x}')$ to show that $G_N(\mathbf{x},\mathbf{x}')$ is not symmetric in general, but that $G_N(\mathbf{x},\mathbf{x}') - F(\mathbf{x})$ is symmetric in $\mathbf{x}$ and $\mathbf{x}'$, where

$$F(\mathbf{x}) = \frac{1}{S}\oint_S G_N(\mathbf{x},\mathbf{y})\, da_y$$

(c) Show that the addition of $F(\mathbf{x})$ to the Green function does not affect the potential $\Phi(\mathbf{x})$.

See Problem 3.26 for an example of the Neumann Green function.

Solution

We first apply Green's theorem (1.35) with integration variable $\mathbf{y}$ and $\phi = G(\mathbf{x},\mathbf{y}), \psi = G(\mathbf{x}',\mathbf{y})$, with $\nabla_y^2 G(\mathbf{z},\mathbf{y}) = -4\pi\delta(\mathbf{y}-\mathbf{z})$ (this works for both Dirichlet and Neumann boundary conditions):

$$\begin{aligned}
\int_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) d^3y &= \oint_S \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right] da_y \\
\int_V \left[G(\mathbf{x},\mathbf{y})\nabla_y^2 G(\mathbf{x}',\mathbf{y}) - G(\mathbf{x}',\mathbf{y})\nabla_y^2 G(\mathbf{x},\mathbf{y})\right] d^3y &= \oint_S \left[G(\mathbf{x},\mathbf{y})\frac{\partial G(\mathbf{x}',\mathbf{y})}{\partial n} - G(\mathbf{x}',\mathbf{y})\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n}\right] da_y \\
G(\mathbf{x},\mathbf{x}') - G(\mathbf{x}',\mathbf{x}) &= -\frac{1}{4\pi}\oint_S \left[G(\mathbf{x},\mathbf{y})\frac{\partial G(\mathbf{x}',\mathbf{y})}{\partial n} - G(\mathbf{x}',\mathbf{y})\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n}\right] da_y
\end{aligned}$$

This is the integral over S required before getting started with the next parts.

Part a

For Dirichlet boundary conditions we demand (1.43):

$$G_D(\mathbf{x},\mathbf{x}') = 0 \quad \text{for } \mathbf{x}' \text{ on } S$$

[1] I solve the new problem from Jackson's Errata. The last two sentences of the first paragraph constitute a separate introductory part of the problem and should maybe have been marked as part a of the problem. The progression of this problem is identical to that of Jackson's paper cited bottom p. 40. By the way, Problem 3.26 is an example from the same paper. Also notice that in the Errata, there is a small mistake in the last sentence of the first paragraph...

In the integral over S above, our integration variable is $\mathbf{y}$ so substitute $\mathbf{y}$ for $\mathbf{x}'$ in the boundary condition and we find

$$G_D(\mathbf{x},\mathbf{x}') - G_D(\mathbf{x}',\mathbf{x}) = 0$$

$G_D(\mathbf{x},\mathbf{x}')$ is clearly symmetric in $\mathbf{x}$ and $\mathbf{x}'$.

Part b

The simplest allowable boundary condition on $G_N(\mathbf{x},\mathbf{x}')$ is (1.45):

$$\frac{\partial G_N}{\partial n'}(\mathbf{x},\mathbf{x}') = -\frac{4\pi}{S} \quad \text{for } \mathbf{x}' \text{ on } S$$

Inserting this in the difference $[G(\mathbf{x},\mathbf{x}') - G(\mathbf{x}',\mathbf{x})]$ above yields

$$\begin{aligned} G(\mathbf{x},\mathbf{x}') - G(\mathbf{x}',\mathbf{x}) &= \frac{1}{S}\oint_S G_N(\mathbf{x},\mathbf{y})\,da_y - \frac{1}{S}\oint_S G_N(\mathbf{x}',\mathbf{y})\,da_y \\ G(\mathbf{x},\mathbf{x}') - G(\mathbf{x}',\mathbf{x}) &= F(\mathbf{x}) - F(\mathbf{x}') \end{aligned}$$

with the definition of $F(\mathbf{x})$ given in the problem text. $[G(\mathbf{x},\mathbf{x}') - G(\mathbf{x}',\mathbf{x})]$ is thus not symmetric in general, but upon rearrangement of the latter equation

$$G(\mathbf{x},\mathbf{x}') - F(\mathbf{x}) = G(\mathbf{x}',\mathbf{x}) - F(\mathbf{x}')$$

we clearly see that the combination $[G(\mathbf{x},\mathbf{x}') - F(\mathbf{x})]$ is symmetric in $\mathbf{x}$ and $\mathbf{x}'$.

Part c

The solution for the potential $\Phi(\mathbf{x})$ corresponding to a Neumann Green function is (1.46). The added contribution to the potential from $F(\mathbf{x})$ is

$$\begin{aligned} \Delta\Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}')F(\mathbf{x})\,d^3x' + \frac{1}{4\pi}\oint_S \frac{\partial\Phi}{\partial n'}F(\mathbf{x})\,da' \\ &= \frac{F(\mathbf{x})}{4\pi}\left(\frac{1}{\epsilon_0}\int_V \rho(\mathbf{x}')\,d^3x' + \oint_S \frac{\partial\Phi}{\partial n'}\,da'\right) \\ &= \frac{F(\mathbf{x})}{4\pi}\left(\frac{1}{\epsilon_0}\int_V \rho(\mathbf{x}')\,d^3x' + \oint_S \nabla\Phi\cdot\mathbf{n}'\,da'\right) \\ &= \frac{F(\mathbf{x})}{4\pi}\left(\frac{1}{\epsilon_0}\int_V \rho(\mathbf{x}')\,d^3x' - \oint_S \mathbf{E}\cdot\mathbf{n}'\,da'\right) \\ &= 0 \end{aligned}$$

In the last step, I used Gauss's law. We have thus shown that the addition of $F(\mathbf{x})$ does not affect the potential $\Phi(\mathbf{x})$.

Problem 1.15

Prove *Thomson's theorem*: If a number of surfaces are fixed in position and a given total charge is placed on each surface, then the electrostatic energy in the region bounded by the surfaces is an absolute minimum when the charges are placed so that every surface is an equipotential, as happens when they are conductors.

Solution

For the electrostatic situation, the following equations are valid:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0},\ \nabla \times \mathbf{E} = 0,\ \mathbf{E} = -\nabla\Phi$$

We consider n conductors bounded by surfaces S_i with total charges q_i ($i = 1, 2, \ldots, n$). Now let $\mathbf{E}$ be created by a distribution of charge such that

$$\Phi_i = constant,\ \int_{S_i} \mathbf{E} \cdot \mathbf{n}\, da = \frac{q_i}{\epsilon_0}$$

$\Phi', \mathbf{E}'$ is another possible electrostatic field satisfying

$$\nabla \cdot \mathbf{E}' = \frac{\rho}{\epsilon_0},\ \nabla \times \mathbf{E}' = 0,\ \mathbf{E}' = -\nabla\Phi',\ \int_{S_i} \mathbf{E}' \cdot \mathbf{n}\, da = \frac{q_i}{\epsilon_0}$$

but not the condition that every surface be an equipotential. We immediately find two relations, which will prove useful later:

$$\nabla \cdot (\mathbf{E}' - \mathbf{E}) = 0,\ \int_{S_i} (\mathbf{E}' - \mathbf{E}) \cdot \mathbf{n}\, da = 0$$

As the last quantities, we introduce W and W' to be the electrostatic energies corresponding to $\mathbf{E}$ and $\mathbf{E}'$ and here we clearly aim to calculate the difference between W and W' and show that W' is always greater than W:

$$W' - W = \frac{\epsilon_0}{2} \int_V E'^2\, d^3x - \frac{\epsilon_0}{2} \int_V E^2\, d^3x = \frac{\epsilon_0}{2} \int_V \left(E'^2 - E^2\right) d^3x$$

The integrand involves the difference between the two squared fields, which inhibits the use of the two relations above. Instead we should aim for an integrand which involves the difference between the fields themselves. We thus rewrite the integrand in the following manner:

$$E'^2 - E^2 = (\mathbf{E}' - \mathbf{E})^2 - 2\mathbf{E} \cdot (\mathbf{E}' - \mathbf{E})$$

The integral of the first term is obviously positive so we press on with the second term (forgetting the factor of 2):

$$\mathbf{E} \cdot (\mathbf{E}' - \mathbf{E}) = -\nabla\Phi \cdot (\mathbf{E}' - \mathbf{E}) = -\left(\nabla[\Phi(\mathbf{E}' - \mathbf{E})] - \Phi\nabla \cdot (\mathbf{E}' - \mathbf{E})\right)$$

Here, by the relation above, the second term is zero and the integral can be written

$$\int_V \mathbf{E} \cdot (\mathbf{E}' - \mathbf{E})\, d^3x = -\int_V \nabla\left[\Phi(\mathbf{E}' - \mathbf{E})\right] d^3x = -\int_S \Phi(\mathbf{E}' - \mathbf{E}) \cdot \mathbf{n}\, da$$

In the last step, I used the divergence theorem. So, now we are getting close, because the last integral can be split up into n integrals over each surface separately. We also know that Φ_i is a constant on each of these surfaces and Φ_i can thus be moved outside the integral for each surface. In short, we can write

$$\int_V \mathbf{E} \cdot (\mathbf{E}' - \mathbf{E})\, d^3x = -\sum_{i=1}^{n} \Phi_i \left(\int_{S_i} (\mathbf{E}' - \mathbf{E}) \cdot \mathbf{n}\, da \right)$$

The integral in the large curved parentheses is the other relation from above and is zero (it simply says that the total charges q_i are the same in the two cases). Then we have shown

$$\begin{aligned} W' - W &= \frac{\epsilon_0}{2} \int_V (\mathbf{E}' - \mathbf{E})^2\, d^3x > 0 \;\Rightarrow \\ W' &> W \end{aligned}$$

for any electrostatic field $\mathbf{E}'$ different from the one due to equipotential surfaces.

Problem 1.16

Prove the following theorem: If a number of conducting surfaces are fixed in position with a given total charge on each, the introduction of an uncharged, insulated conductor into the region bounded by the surfaces lowers the electrostatic energy.

Solution

This problem is very similar to the previous one, the only difference being the necessity to consider different integration volumes for the two electrostatic energies.

Consider n conductors with bounding surfaces S_i and total charges q_i ($i = 1, 2, \ldots, n$). The electric field is in this case $\mathbf{E}$, the electrostatic energy W, and the volume exterior to S_i we designate V. Now introduce another conductor with bounding surface S_0 and volume V_0. We designate the volume exterior to both S_i and S_0 $V_1 \equiv V - V_0$. The electric field is now $\mathbf{E}'$ and the electrostatic energy W'. The electrostatic-energy difference can now be written (notice the different volumes being considered)

$$\begin{aligned} W - W' &= \frac{\epsilon_0}{2}\int_V E^2\, d^3x - \frac{\epsilon_0}{2}\int_{V_1} E'^2\, d^3x \\ &= \frac{\epsilon_0}{2}\int_{V_0} E^2\, d^3x + \frac{\epsilon_0}{2}\int_{V_1} \left(E^2 - E'^2\right) d^3x \end{aligned}$$

The second term is almost identical to the one considered in Problem 1.15; primed and unprimed quantities are simply exchanged. We can then again split the integral in two

$$W - W' = \frac{\epsilon_0}{2}\int_{V_0} E^2\, d^3x + \left[\frac{\epsilon_0}{2}\int_{V_1} \left(\mathbf{E} - \mathbf{E}'\right)^2 d^3x + \epsilon_0 \int_{V_1} \mathbf{E}' \cdot \left(\mathbf{E} - \mathbf{E}'\right) d^3x\right]$$

We are dealing with equipotential surfaces so by analogy with Problem 1.15, the last term is zero. We end up with

$$W - W' = \frac{\epsilon_0}{2}\int_{V_0} E^2\, d^3x + \frac{\epsilon_0}{2}\int_{V_1} \left(\mathbf{E} - \mathbf{E}'\right)^2 d^3x > 0$$

and we have shown that $W' < W$, i.e. introducing V_0 lowers the electrostatic energy.

Problem 1.17

A volume V in vacuum is bounded by a surface S consisting of several separate conducting surfaces S_i. One conductor is held at *unit* potential and all the other conductors at zero potential.

(a) Show that the capacitance of the one conductor is given by

$$C = \epsilon_0 \int_V |\nabla\Phi|^2 \, d^3x$$

where $\Phi(\mathbf{x})$ is the solution for the potential.

(b) Show that the true capacitance C is always less than or equal to the quantity

$$C[\Psi] = \epsilon_0 \int_V |\nabla\Psi|^2 \, d^3x$$

where Ψ is any trial function satisfying the boundary condition on the conductors. This is a variational principle for the capacitance that yields an *upper bound*.

Solution

This problem relates to the discussion at the end of Section 1.11 and Section 1.12 in that we shall use energy-like functionals to obtain an upper bound on the true capacitance.

Part a

In order to be concrete we let S_1 be at potential $V_1 = 1$ and all the others at potential zero. We can use both (1.54) and (1.62) to express the electrostatic energy of the system.

$$W = \frac{\epsilon_0}{2} \int_V |\nabla\Phi|^2 \, d^3x = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} V_i V_j$$

Only S_1 is at non-zero potential so the double sum is simply $\frac{1}{2} C_{11} V_1^2 = \frac{1}{2} C$. Thus

$$C = \epsilon_0 \int_V |\nabla\Phi|^2 \, d^3x$$

Part b

The trial function Ψ satisfies the Laplace equation $\nabla^2\Psi = 0$ in V and is given on S and thus fulfills $\delta\Psi = 0$ there. Additionally, since $C[\Psi]$ is effectively an energy-like functional through $W = \frac{1}{2} C V^2$, we can use the variational approach from Section 1.12. We can split any deviation from the true potential $\Phi(\mathbf{x})$ into infinitesimal changes like $\Phi \to \Phi + \delta\Phi$. The first-order change in the difference $C[\Phi] = C[\Phi + \delta\Phi] - C[\Phi]$ then vanishes since Ψ satisfies the Laplace equation and $C[\Phi]$, the true capacitance C, is a stationary minimum. We have thus shown that the true capacitance C is always less than or equal to $C[\Psi]$:

$$C \leq C[\Psi] = \epsilon_0 \int_V |\nabla\Psi|^2 \, d^3x$$

Problem 1.18

Consider the configuration of conductors of Problem 1.17, with all conductors except S_1 at zero potential.

(a) Show that the potential $\Phi(\mathbf{x})$ in the volume V *and* on any of the surfaces S_i can be written

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \oint_{S_1} \sigma_1(\mathbf{x}')G(\mathbf{x},\mathbf{x}')\, da'$$

where $\sigma_1(\mathbf{x}')$ is the surface charge density on S_1 and $G(\mathbf{x},\mathbf{x}')$ is the Green function potential for a point charge in the presence of all the surfaces that are held at zero potential (but with S_1 absent). Show also that the electrostatic energy is

$$W = \frac{1}{8\pi\epsilon_0} \oint_{S_1} da \oint_{S_1} da'\, \sigma_1(\mathbf{x})G(\mathbf{x},\mathbf{x}')\sigma_1(\mathbf{x}')$$

where the integrals are only over the surface S_1.

(b) Show that the variational expression

$$C^{-1}[\sigma] = \frac{\oint_{S_1} da \oint_{S_1} da'\, \sigma(\mathbf{x})G(\mathbf{x},\mathbf{x}')\sigma(\mathbf{x}')}{4\pi\epsilon_0 \left[\oint_{S_1} \sigma(\mathbf{x})\, da\right]^2}$$

with an arbitrary integrable function $\sigma(\mathbf{x})$ defined on S_1, is stationary for small variations of σ away from σ_1. Use Thomson's theorem to prove that the reciprocal of $C^{-1}[\sigma]$ gives a *lower bound* to the true capacitance of the conductor S_1.

Solution

Here we shall exploit some formal aspects of the Green function from Section 1.10 as well as the formulas for the electrostatic energy in Section 1.11.

Part a

Consider the configuration of conductors held at zero potential, i.e. without S_1 present. $\Phi(\mathbf{x})$ is specified to be zero on the surface $S_i,\ i \neq 1$. With Dirichlet boundary conditions on the Green function, the expression for the potential $\Phi(\mathbf{x})$ (1.44) becomes

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{x}')G_D(\mathbf{x},\mathbf{x}')\, d^3x'$$

The surface charge distribution on S_1 can be introduced stepwise as a distribution of unit charges. The contribution to the potential from a single unit charge at $\mathbf{x}'$ is

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_V 4\pi\epsilon_0\delta(\mathbf{x}')G(\mathbf{x},\mathbf{x}')\, d^3x' = G(\mathbf{x},\mathbf{x}')$$

and as mentioned in the problem text, $G(\mathbf{x},\mathbf{x}')$ is the Green function for a point charge in the presence of all the surfaces that are held at zero potential (but

with S_1 absent). Since the charges on S_1 are all at its surface, constituting the surface charge density $\sigma_1(\mathbf{x}')$, we can write the potential anywhere in the volume V as wanted

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \oint_{S_1} \sigma_1(\mathbf{x}') G(\mathbf{x}, \mathbf{x}')\, da'$$

We get the desired expression for the electrostatic energy W most easily by inserting the expression for $\Phi(\mathbf{x})$ into (1.53)

$$\begin{aligned} W &= \frac{1}{2} \int_V \rho(\mathbf{x}) \Phi(\mathbf{x})\, d^3x \\ &= \frac{1}{8\pi\epsilon_0} \int_V \rho(\mathbf{x})\, d^3x \oint_{S_1} \sigma_1(\mathbf{x}') G(\mathbf{x}, \mathbf{x}')\, da' \\ &= \frac{1}{8\pi\epsilon_0} \oint_{S_1} \sigma_1(\mathbf{x})\, da \oint_{S_1} \sigma_1(\mathbf{x}') G(\mathbf{x}, \mathbf{x}')\, da' \\ &= \frac{1}{8\pi\epsilon_0} \oint_{S_1} da \oint_{S_1} da'\, \sigma_1(\mathbf{x}) G(\mathbf{x}, \mathbf{x}') \sigma_1(\mathbf{x}') \end{aligned}$$

since the only charge is located on S_1.

Part b

The total charge Q_1 on S_1 can be written

$$Q_1 = \oint_{S_1} \sigma_1(\mathbf{x})\, da$$

and the electrostatic energy W can be expressed as

$$W = \frac{1}{2} Q_1 \Phi_1 = \frac{1}{2} \frac{Q_1^2}{C}$$

Thus

$$C^{-1} = 2W/Q_1^2 = \frac{\oint_{S_1} da \oint_{S_1} da'\, \sigma_1(\mathbf{x}) G(\mathbf{x}, \mathbf{x}') \sigma_1(\mathbf{x}')}{4\pi\epsilon_0 \left[\oint_{S_1} \sigma_1(\mathbf{x})\, da \right]^2}$$

Next, we need to show that the variational expression

$$C^{-1}[\sigma] = \frac{\oint_{S_1} da \oint_{S_1} da'\, \sigma(\mathbf{x}) G(\mathbf{x}, \mathbf{x}') \sigma(\mathbf{x}')}{4\pi\epsilon_0 \left[\oint_{S_1} \sigma(\mathbf{x})\, da \right]^2}$$

is stationary for small variations σ away from σ_1. This can be done formally by varying σ:

$$4\pi\epsilon_0 \left[\oint_{S_1} \sigma(\mathbf{x})\, da \right]^2 C^{-1} = \oint_{S_1} da \oint_{S_1} da'\, \sigma(\mathbf{x}) G(\mathbf{x}, \mathbf{x}') \sigma(\mathbf{x}')$$

$$4\pi\epsilon_0 \left\{ Q^2 \delta(C^{-1})) + 2C^{-1} Q \oint_{S_1} \delta\sigma\, da \right\} = 2 \oint_{S_1} da \oint_{S_1} da'\, G(\mathbf{x}, \mathbf{x}') \sigma(\mathbf{x}') \delta\sigma(\mathbf{x})$$

On the RHS, I used the symmetry property of $G(\mathbf{x}, \mathbf{x}') = G(\mathbf{x}', \mathbf{x})$. Rearranging,

$$4\pi\epsilon_0 Q^2 \delta(C^{-1}) = 2\left\{ \oint_{S_1} \delta\sigma(\mathbf{x}) \left[-\frac{Q}{C} + \oint_{S_1} da'\, G(\mathbf{x}, \mathbf{x}')\sigma(\mathbf{x}') \right] da \right\}$$

The square parentheses contains two ways of expressing the equipotential Φ on S_1, with opposite signs. Thus the variation in C^{-1} vanishes.

Last thing to prove is that $C^{-1}[\sigma]$ gives an upper bound on C^{-1}. The denominator of the variational expression is obviously positive and the numerator constitutes an expression for the potential energy. Thomson's theorem states that the potential-energy content of the volume V bounded by the surfaces S_i is a minimum, since all surfaces are conducting and thus equipotentials. Any other potential configuration is thus of higher electrostatic energy and $C^{-1}[\sigma]$ gives an upper bound on C^{-1} or equivalently a *lower bound* to the true capacitance C of the conductor S_1.

Problem 1.19

For the cylindrical capacitor of Problem 1.6c, evaluate the variational upper bound of Problem 1.17b with the naive trial function, $\Psi_1(\rho) = (b-\rho)/(b-a)$. Compare the variational result with the exact result for $b/a = 1.5, 2, 3$. Explain the trend of your results in terms of the functional form of Ψ_1. An improved trial function is treated by *Collin* (pp. 275–277).

Solution

The functional

$$C[\Psi_1] = \epsilon_0 \int_V |\nabla \Psi_1|^2 \, d^3x$$

gives a variational upper bound on the true C.

$$\begin{aligned} C[\Psi_1] &= \epsilon_0 \int_V \left| \frac{\partial \Psi_1}{\partial \rho} \right|^2 d^3x \\ &= \epsilon_0 \int_V \frac{1}{(b-a)^2} \, d^3x \\ &= \frac{2\pi\epsilon_0 L}{(b-a)^2} \int_a^b \rho \, d\rho \\ &= 2\pi\epsilon_0 L \frac{b^2-a^2}{2(b-a)^2} \\ &= 2\pi\epsilon_0 L \frac{1}{2} \frac{b+a}{b-a} \\ &= 2\pi\epsilon_0 L \frac{1}{2} \frac{b/a+1}{b/a-1} \end{aligned}$$

The exact expression for C is

$$C_{\text{exact}} = 2\pi\epsilon_0 L \frac{1}{\ln b/a}$$

Below we compare the values from the variational expression and exact expression for C:

b/a	$2\pi\epsilon_0 L C[\Psi_1]$	$2\pi\epsilon_0 L C_{\text{exact}}$	Deviation
3/2	5/2	2.466	1%
2	3/2	1.443	4%
3	1	0.910	10%

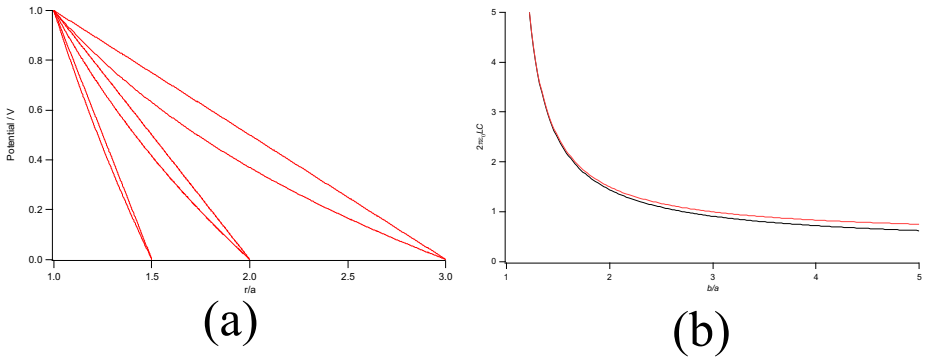

Figure 1.10: Problem 1.19. (a) Variational (straight lines) and exact potential $C_{exact} = \frac{\ln b/r}{\ln b/a}$ for selected values of b/a. (b) Variational (red) and exact (black) values for the capacitance.

The variational expression is the first order Taylor expansion in b/a of C_{exact}. We thus expect the variational value for $C[\Psi_1]$ to be better for small b/a. This is indeed the case. Illustrative plot for the potentials and capacitances are shown in Fig. 1.10. In Fig. 1.10a we see how the variational potential trails the exact solution better for small b/a.

Problem 1.20

In estimating the capacitance of a given configuration of conductors, comparison with known capacitances is often helpful. Consider two configurations of n conductors in which the $(n-1)$ conductors held at zero potential are the same, but the one conductor whose capacitance we wish to know is different. In particular, let the conductor in one configuration have a closed surface S_1 and in the other configuration have surface S_1', with S_1' totally inside S_1.

(a) Use the extremum principle of Section 1.12 and the variational principle of Problem 1.17 to prove that the capacitance C' of the conductor with surface S_1' is less than or equal to the capacitance C of the conductor with surface S_1 that encloses S_1'.

(b) Set upper and lower limits for the capacitance of a conducting cube of side a. Compare your limits and also their average with the numerical value, $C \backsimeq 0.655(4\pi\epsilon_0 a)$.

(c) By how much do you estimate the capacitance per unit length of the two-wire system of Problem 1.7 will change (larger? smaller?) if *one* of the wires is replaced by a wire of square cross section whose side is equal to its diameter?

Solution

We shall consider two configurations. Both contain n conductors; $(n-1)$ are at zero potential.

(i) S_1 with capacitance C present. The correct expression for the potential in this configuration is $\Phi_1(\mathbf{x})$. The volume outside S_1 is termed V_1.

(ii) S_1 *not* present but instead the smaller S_1' with capacitance C'. The correct potential is now $\Phi_1'(\mathbf{x})$. The volume outside S_1 is termed V_1'.

Part a

The extremum principle of Section 1.12 states that C and C' have a stationary minimum and the variational principle of Problem 1.17 states that

$$C[\Psi] = \epsilon_0 \int_V |\nabla\Psi|^2 \, d^3x \geq C = \epsilon_0 \int_V |\nabla\Phi|^2 \, d^3x$$

where $\Phi(\mathbf{x})$ is the correct expression for the potential in the given configuration and $\Psi(\mathbf{x})$ is any potential (meeting the correct boundary conditions) different from $\Phi(\mathbf{x})$. We now construct an approximation $\Phi_a'(\mathbf{x})$ to $\Phi_1'(\mathbf{x})$, which is given by $\Phi(\mathbf{x})$ outside S_1, i.e. in V_1, and is zero between S_1' and S_1, i.e. in $V_1' - V_1$.

The correct capacitance of S_1 can be written

$$C = \epsilon_0 \int_{V_1} |\nabla\Phi_1|^2 \, d^3x$$

while for the capacitance of S_1' we can write

$$C' = \epsilon_0 \int_{V_1'} |\nabla\Phi_1'|^2 \, d^3x \leq \epsilon_0 \int_{V_1'} |\nabla\Phi_a'|^2 \, d^3x = \epsilon_0 \int_{V_1} |\nabla\Phi_1|^2 \, d^3x = C$$

We have thus shown that $C' \leq C$.

Part b

Here we use two spherical conductors: one with surface S_{out}, which completely encloses the cube with side a and another with surface S_{in} completely enclosed within the cube. The radii of S_{out} and S_{in} are $(\sqrt{3}/2)a$ and $a/2$, respectively. From part a we know that $C_{in} \leq C \leq C_{out}$. The capacitance of a sphere with radius R is $4\pi\epsilon_0 R$ (see Problem 1.6b). We thus find lower and upper bounds on the capacitance C of the cube:

$$\frac{1}{2}(4\pi\epsilon_0 a) \leq C \leq \frac{\sqrt{3}}{2}(4\pi\epsilon_0 a) \backsimeq 0.866(4\pi\epsilon_0 a)$$

The average of the lower and upper bounds is $C_{av} = \frac{1}{2}(C_{in}+C_{out}) \backsimeq 0.683(4\pi\epsilon_0 a)$. This corresponds very well to the numerical value of $C = 0.655(4\pi\epsilon_0 a)$.

Part c

Since the square conductor completely encloses the circular one, the capacitance per unit length of the circular/square configuration is certainly larger than the circular/circular configuration. But by how much then? Inspired by part b we approximate the capacitance per unit length of the circular/square system C_{cs} by the average capacitance per unit length of two circular/circular systems C_{out} and C_{in} with obvious dimensions. Considering the capacitance as a function of the geometrical mean $a = \sqrt{a_1 a_2}$ of the two conductor radii we can write approximately

$$C_{cs} \backsimeq \frac{1}{2}(C_{in} + C_{out}) = \frac{1}{2}\left[C(\,(a_1 a_2)^{1/2}\,)+C(\,(a_1 2^{1/2} a_2)^{1/2}\,)\right]$$

The first term is just the capacitance per unit length of the original system but the second term is slightly modified. We thus consider this term in more detail:

$$C_{out}/\pi\epsilon_0 = \left(\ln\frac{d}{2^{1/4}a}\right)^{-1} = \left(\ln d/a - \frac{1}{4}\ln 2\right)^{-1} = (\ln d/a)^{-1}\left(1 - \frac{1}{4}\frac{\ln 2}{\ln d/a}\right)^{-1}$$

The last term is <1 so indeed $C_{out} > C$. We can then write C_{cs} as

$$C_{cs} \backsimeq \frac{C}{2}\left[1 + \left(1 - \frac{1}{4}\frac{\ln 2}{\ln d/a}\right)^{-1}\right]$$

For example, let's use the set of parameters in Problem 1.7 ($d/a = 10$) with $C = 1.2 \times 10^{-11}$ F/m. This gives $C_{cs} = 1.04C$, i.e. a change of 4%.

Problem 1.21

A two-dimensional potential problem consists of a unit square area ($0 \le x \le 1$, $0 \le y \le 1$) bounded by "surfaces" held at zero potential. Over the entire square there is a uniform charge density of unit strength (per unit length in z).

(a) Apply the variational principle (1.63) for the Poisson equation with the "variational" trial function $\Psi(x,y) = A \cdot x(1-x) \cdot y(1-y)$ to determine the best value of the constant A. [I use quotation marks around variational because there are no parameters to vary except the overall scale.]

(b) The exact (albeit series) solution for this problem is [See Problems 2.15 and 2.16]

$$4\pi\epsilon_0\Phi(x,y) = \frac{16}{\pi^2}\sum_{m=0}^{\infty}\frac{\sin[(2m+1)\pi x]}{(2m+1)^3}\left\{1 - \frac{\cosh[(2m+1)\pi(y-\frac{1}{2})]}{\cosh[(2m+1)\pi/2]}\right\}$$

For $y = 0.25$ and $y = 0.5$, plot and compare the simple variational solution of part a with the exact solution as functions of x.

Part a

We need to minimize the functional in (1.63) wrt. A

$$I[\Psi] = \frac{1}{2}\int_V \nabla\Psi \cdot \nabla\Psi \, d^3x - \int_V g\Psi \, d^3x$$

The source term g is equal $1/\epsilon_0$ and the gradient operator is simple is Cartesian coordinates so we easily find

$$\begin{aligned} I[\Psi] &= \frac{A^2}{2}\int_0^1 dx \int_0^1 dy \left\{y^2(1-y)^2(1-2x)^2 + x^2(1-x)^2(1-2y)^2\right\} \\ &\quad -\frac{1}{\epsilon_0}\int_0^1 dx \int_0^1 dy\, A \cdot x(1-x) \cdot y(1-y) \\ &= \frac{A^2}{90} - \frac{A}{36\epsilon_0} \end{aligned}$$

Minimization wrt. A yields $A = 5/4\epsilon_0$. To facilitate comparison to the exact solution, let's write the result like this:

$$4\pi\epsilon_0\Psi(x,y) = 5\pi \cdot x(1-x) \cdot y(1-y)$$

Part b

The desired plots are shown in Fig. 1.11. The correspondence is better for smaller y. I also show the potential configuration in a 3D plot in Fig. 1.12. We notice that the variational expression has remarkable likeness to the exact potential. It just seems to be a bit too "pointy".

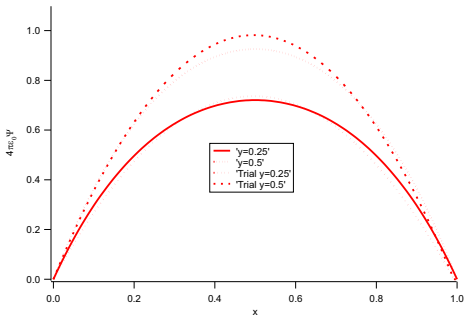

Figure 1.11: Problem 1.21. Variational and exact potential for selected values of y.

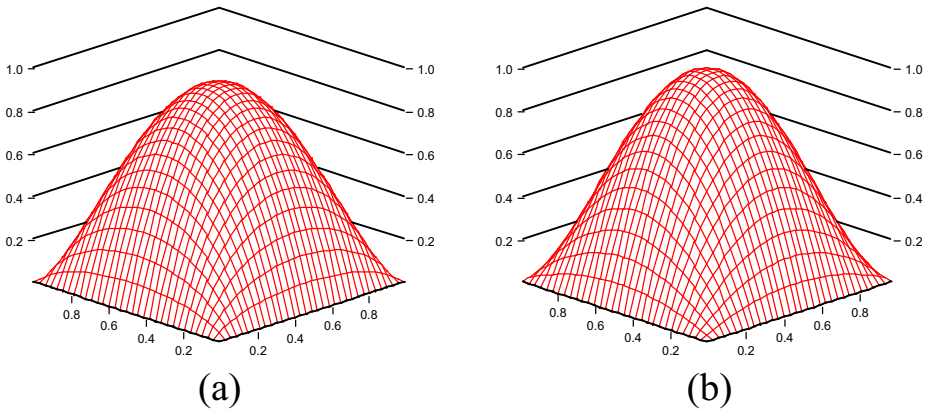

Figure 1.12: Problem 1.21. (a) Exact and (b) variational potential configuration. Pretty isn't it?

Problem 1.22

Two-dimensional relaxation calculations commonly use sites on a square lattice with spacing $\Delta x = \Delta y = h$, and label the sites by (i, j), where i, j are integers and $x_i = ih + x_0, y_j = jh + y_0$. The value of the potential at (i, j) can be approximated by the average of the values at neighboring sites. [Recall the relevant theorem about harmonic functions.] But what average?

(a) If $F(x, y)$ is a well-behaved function in the neighborhood of the origin, but not necessarily harmonic, by explicit Taylor series expansions, show that the "cross" sum

$$S_c = F(h, 0) + F(0, h) + F(-h, 0) + F(0, -h)$$

can be expressed as

$$S_c = 4F(0, 0) + h^2\nabla^2 F + \frac{h^4}{12}(F_{xxxx} + F_{yyyy}) + O(h^6)$$

(b) Similarly, show that the "square" sum,

$$S_s = F(h, h) + F(-h, h) + F(-h, -h) + F(h, -h)$$

can be expressed as

$$S_s = 4F(0, 0) + 2h^2\nabla^2 F - \frac{h^4}{3}(F_{xxxx} + F_{yyyy}) + \frac{h^4}{2}\nabla^2(\nabla^2 F) + O(h^6)$$

Here F_{xxxx} is the fourth partial derivative of F with respect to x, evaluated at $x = 0, y = 0$, etc. If $\nabla^2 F = 0$, the averages $S_c/4$ and $S_s/4$ each give the value of $F(0, 0)$, correct to order h^3 inclusive. Note that an improvement can be obtained by forming the "improved" average,

$$\tilde{S} = \frac{1}{5}\left[S_c + \frac{1}{4}S_s\right]$$

where

$$\tilde{S} = F(0, 0) + \frac{3}{10}h^2\nabla^2 F + \frac{h^4}{40}\nabla^2(\nabla^2 F) + O(h^6)$$

If $\nabla^2 F = 0$, then $\tilde{S}$ gives $F(0, 0)$, correct to order h^5 inclusive. For Poisson's equation, the charge density and its lowest order Laplacian can be inserted for the same accuracy.

Solution

Since we are expanding around the origin, we need a two-dimensional Maclaurin series, which can be written like this:

$$F(x, y) = \sum_{n=0}^{\infty} \frac{1}{n!}\left\{xF_x(0, 0) + yF_y(0, 0)\right\}^n$$

where $F_x(0, 0),\ F_y(0, 0)$ denote partial derivatives with respect to $x,\ y$ evaluated at $x = 0,\ y = 0$.

Part a

For the cross sum, we need e.g. $F(\pm h, 0)$

$$\begin{aligned} F(\pm h,0) &= \sum_{n=0}^{\infty} \frac{1}{n!} \{xF_x(0,0)\}^n \\ &= F(0,0) \pm hF_x + \frac{h^2}{2!}F_{xx} \pm \frac{h^3}{3!}F_{xxx} + \frac{h^4}{4!}F_{xxxx} \pm \frac{h^5}{5!}F_{xxxxx} + O(h^6) \end{aligned}$$

We see at once that odd terms cancel so the cross sum is

$$\begin{aligned} S_c &= 4F(0,0) + h^2(F_{xx} + F_{yy}) + \frac{h^4}{24}(2F_{xxxx} + 2F_{yyyy}) + O(h^6) \\ &= 4F(0,0) + h^2\nabla^2 F + \frac{h^4}{12}(F_{xxxx} + F_{yyyy}) + O(h^6) \end{aligned}$$

Part b

For the square sum, odd terms cancel as well so I shall only consider even ones. Furthermore, since for even terms the sign under the n exponent in

$$F(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!} \{xF_x(0,0) + yF_y(0,0)\}^n$$

is of no consequence, we shall only consider the first two terms of the square sum. The desired result is just twice the sum of these two terms. Here goes:

$$\begin{aligned} F(h,h) &= F(0,0) + \frac{h^2}{2!}(F_{xx} + 2F_{xy} + F_{yy}) + \frac{h^4}{4!}(F_{xxxx} + 4F_{xxxy} + 6F_{xxyy} + 4F_{xyyy} + F_{yyyy}) + O(h^6) \\ F(h,-h) &= F(0,0) + \frac{h^2}{2!}(F_{xx} - 2F_{xy} + F_{yy}) + \frac{h^4}{4!}(F_{xxxx} - 4F_{xxxy} + 6F_{xxyy} - 4F_{xyyy} + F_{yyyy}) + O(h^6) \end{aligned}$$

The desired sum is then

$$\begin{aligned} S_s &= 4F(0,0) + 2h^2(F_{xx} + F_{yy}) + \frac{h^4}{6}(F_{xxxx} + 6F_{xxyy} + F_{yyyy}) + O(h^6) \\ &= 4F(0,0) + 2h^2\nabla^2 F + \frac{h^4}{2}(F_{xxxx} + 2F_{xxyy} + F_{yyyy}) - \frac{h^4}{6}(2F_{xxxx} + 2F_{yyyy}) + O(h^6) \\ &= 4F(0,0) + 2h^2\nabla^2 F - \frac{h^4}{3}(F_{xxxx} + F_{yyyy}) + \frac{h^4}{2}\nabla^2(\nabla^2 F) + O(h^6) \end{aligned}$$

The improved average is then formed by adding $\frac{1}{4}S_s$ to S_c, thus eliminating the horrible fourth partial derivatives, and we get the expression $\tilde{S}$ given in (1.81) and the problem text.

Problem 1.23

A transmission line consists of a long straight conductor with a hollow square region in its interior, with a square conductor of one-quarter the area of the hollow region centered in the empty space, with edges parallel to the inner sides of outer conductor. If the conductors are raised to different potentials, the potential and electric field in the space between them exhibit an eightfold symmetry; the basic unit is sketched in the accompanying figure. The efficacy of the relaxation method in determining the properties of the transmission line can be illustrated by a simple calculation.

(a) Using only the four interior points indicated in the figure, write down the relaxation equation for each point for the "cross" and the "improved" averaging schemes (defined in Problem 1.22) if the inner conductor has Φ=100 V and the outer conductor has Φ=0. By performing either the relaxation iteration process or solving the set of algebraic equations for each scheme, find estimates for the potential at each of the four points for the two schemes.

(b) From the results of part a make the best estimate (or estimates) you can for the capacitance per unit length of the transmission line.

(c) (Optional) Using your favorite computational tools, repeat the relaxation calculation with half the lattice spacing (21 interior points) and compare.

Answer: Φ_1=48.872 V, Φ_2=47.171 V, Φ_3=38.297 V, Φ_4=19.786 V and C= 10.257 ϵ_0 F/m [from an accurate numerical calculation].[2]

Solution

The conductor configuration is shown in Fig. 1.13a. The dashed lines outline the basic unit in this eight-fold symmetric setup though I haven't tried to exploit this symmetry in the numerical calculations. With the limited number of points involved in parts a and b, the symmetry is easily exploited.

Part a

Cross averaging is simply the average of the potential of the four points closest to the points of interest. i.e. $F = \frac{1}{4}S_c$ with the definition of the cross sum from Problem 1.22. The algebraic equations look like this with Φ=100 V and $\Phi_0 = 0$

$$\begin{aligned}
\Phi_1 &= \frac{1}{4}\left[\Phi_2 + \Phi_2 + \Phi + \Phi_0\right] = \frac{1}{4}\left[2\Phi_2 + \Phi\right] \\
\Phi_2 &= \frac{1}{4}\left[\Phi_1 + \Phi_3 + \Phi + \Phi_0\right] = \frac{1}{4}\left[\Phi_1 + \Phi_3 + \Phi\right] \\
\Phi_3 &= \frac{1}{4}\left[\Phi_2 + \Phi_4 + \Phi + \Phi_0\right] = \frac{1}{4}\left[\Phi_2 + \Phi_4 + \Phi\right] \\
\Phi_3 &= \frac{1}{4}\left[\Phi_3 + \Phi_3 + \Phi_0 + \Phi_0\right] = \frac{1}{2}\Phi_3
\end{aligned}$$

[2] I give values slightly different from Jackson's; I have performed about 5000 iterations on a 256×256 grid to achieve excellent accuracy. If you disagree, let me know.

The corresponding augmented matrix is

$$\left[\begin{array}{cccc|c} 4 & -2 & 0 & 0 & 1 \\ -1 & 4 & -1 & 0 & 1 \\ 0 & -1 & 4 & -1 & 1 \\ 0 & 0 & -1 & 2 & 0 \end{array}\right]$$

where I solve for Φ_i in units of Φ, the potential on the inner conductor. The solution is Φ_1= 44/90 Φ=48.89 V, Φ_2= 43/90 Φ=47.78 V, Φ_3= 38/90 Φ=42.22 V, and Φ_4= 19/90 Φ=21.11 V.

For the improved averaging scheme we need $\frac{1}{5}\left[S_c + \frac{1}{4}S_s\right]$ and the corresponding augmented matrix is, again in units of Φ

$$\left[\begin{array}{cccc|c} 5 & -2 & 0 & 0 & 3/2 \\ -1 & 5 & -1 & 0 & 3/2 \\ 0 & -1 & 19/4 & -1 & 5/4 \\ 0 & 0 & -2 & 5 & 1/4 \end{array}\right]$$

Here we get Φ_1=49.21 V, Φ_2=48.03 V, Φ_3=40.93 V, and Φ_4=21.37 V. The two methods are both quite close to the correct values but fail badly when the potential changes rapidly, i.e. close to the corners. The improved averaging scheme doesn't seem to perform better than the cross average in terms of accuracy.

Part b

To find the capacitance (per unit length) C of the transmission line, we need the charge per unit length on the inner conductor. For convenience, we can find the charge per unit length Q on one side and multiply by four to get the total charge. Since the electric field is perpendicular to the surface of the inner conductor, we can find the surface charge density σ as the normal derivative at the conductor surface

$$\sigma = \epsilon_0 \frac{\partial \Phi}{\partial n}$$

For the present discrete lattice, the lattice spacing h is the same in both directions so the contribution from a horisontal segment becomes

$$\Delta Q = \epsilon_0 \frac{\partial \Phi}{\partial n} d\ell = \epsilon_0 \frac{\Phi - \Phi'}{\Delta y} \Delta x = \Phi - \Phi'$$

where Φ' is the potential at the point next to the inner conductor. For instance, charge and capacitance per unit length for the cross and improved averaging schemes are

$$\begin{aligned} Q_c &= \epsilon_0(5\Phi - \Phi_1 - 2\Phi_2 - 2\Phi_3) = 271.1\epsilon_0\,\mathrm{C/m} \\ Q_i &= \epsilon_0(5\Phi - \Phi_1 - 2\Phi_2 - 2\Phi_3) = 272.9\epsilon_0\,\mathrm{C/m} \\ C_c &= 4\frac{Q_c}{\Phi} = 10.84\epsilon_0\,\mathrm{F/m} \\ C_i &= 4\frac{Q_i}{\Phi} = 10.91\epsilon_0\,\mathrm{F/m} \end{aligned}$$

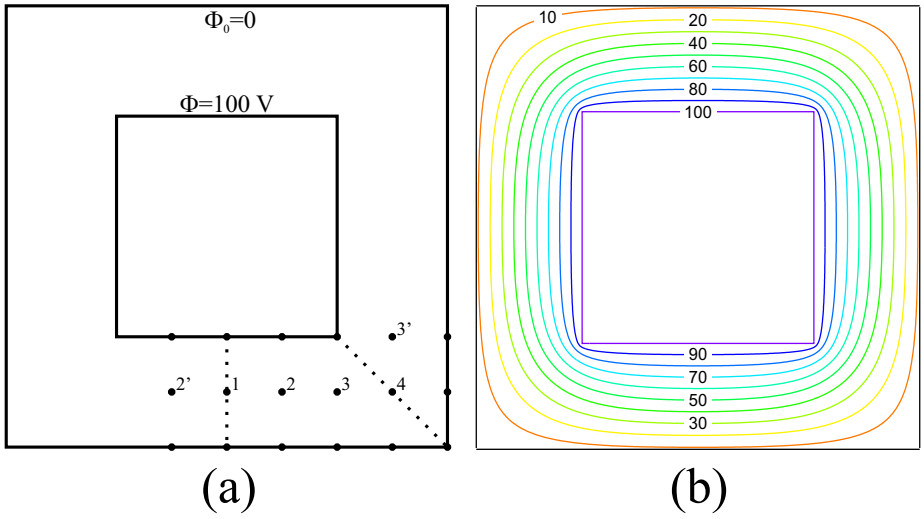

Figure 1.13: Problem 1.23. (a) The conductor configuration and (b) contour plot for an accurate calculation with h=1/256.

Part c

With half the lattice spacing (21 interior points), the algebraic method becomes impractical and the relaxation method has to be employed. I used the improved averaging Gauss-Seidel iteration scheme to achieve faster convergence. I get Φ_1=49.00 V, Φ_2=47.50 V, Φ_3=39.29 V, and Φ_4=20.40 V and C=10.54ϵ_0 F/m, clearly close to the correct values. For h =1/256 I get the accurate values cited in the problem answer. Figure 1.13b shows a contour plot for this latter calculation. Figure 1.14 shows a 3D plot.

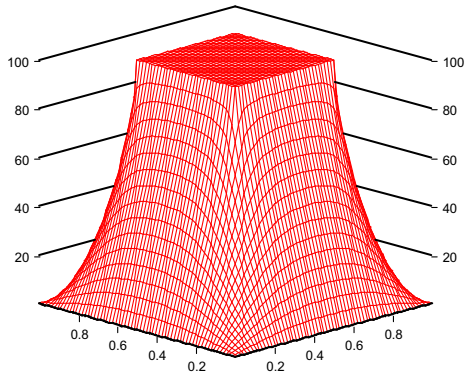

Figure 1.14: Problem 1.23. 3D plot of the potential.

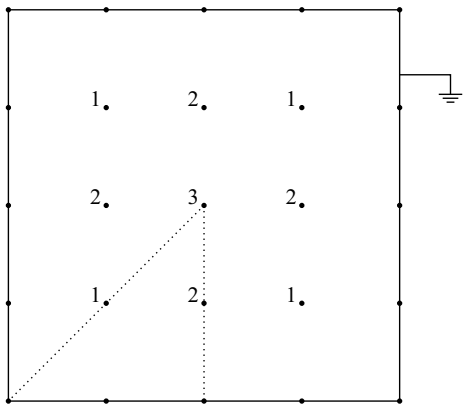

Figure 1.15: Problem 1.24. Sketch of the 2D situation.

Problem 1.24

Consider solution of the two-dimensional Poisson equation problem of Problem 1.21, a unit square with zero potential on the boundary and a constant unit charge density in the interior, by the technique of relaxation. Choose h=0.25 so that there are nine interior sites. Use symmetry to reduce the number of needed sites to three, at (0.25, 0.25), (0.5, 0.25), and (0.5, 0.5). With so few sites, it is easy to do the iterations with a block of paper and a pocket calculator, but suit yourself.

(a) Use the "improved grid" averaging of Problem 1.22 and the simple (Jacobian) iteration scheme, starting with $4\pi\epsilon_0$=1.0 at all three interior sites. Do at least six iterations, preferably eight or ten.

(b) Repeat the iteration procedure with the same starting values, but using Gauss-Seidel iteration.

(c) Graph the two sets of results of each iteration versus iteration number and compare with the exact values, $4\pi\epsilon_0\Phi(0.25, 0.25) = 0.5691$, $4\pi\epsilon_0\Phi(0.5, 0.25) = 0.7205$, $4\pi\epsilon_0\Phi(0.5, 0.5) = 0.9258$. Comment on rate of convergence and final accuracy.

Solution

By symmetry it is easy to find the equivalent points shown in Fig. 1.15.

Parts a, b and c

Defining $\Psi = 4\pi\epsilon_0\Phi$ and $\langle\langle\Psi(x,y)\rangle\rangle = \frac{1}{5}\left[S_c + \frac{1}{4}S_s\right]$, we can use the expression from the end of Problem 1.22, which is the same as (1.81), or (1.82). One gets

$$\Psi(x,y) = \langle\langle\Psi(x,y)\rangle\rangle + \frac{6}{5}\pi h^2 = \langle\langle\Psi(x,y)\rangle\rangle + \frac{3}{40}\pi$$

i.e. the surface charge density has the effect of adding a constant during each iteration. Fig. 1.16 shows the evolution of the potential for both the Jacobian

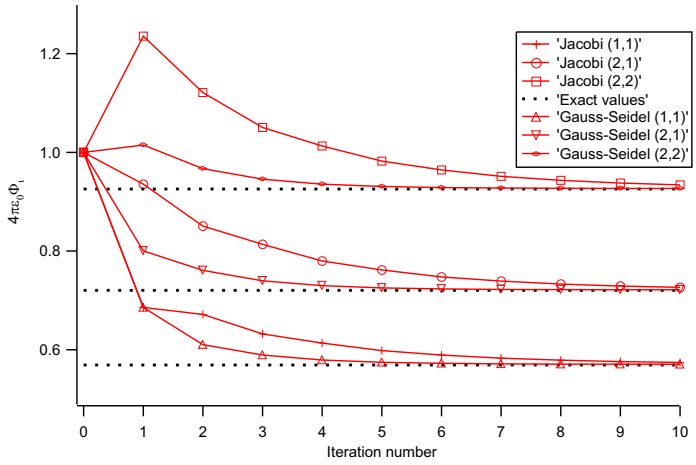

Figure 1.16: Problem 1.24. Evolution of the grid values in the Jacobian and Gauss-Seidel iteration schemes.

and Gauss-Seidel iteration scheme. Both methods converge towards the same value, but the Gauss-Seidel converges much more rapidly. As Jackson remarks bottom p. 49, the Gauss-Seidel method benefits immediately from the improved values so that $\langle\langle\Psi(x,y)\rangle\rangle$ is typically made up half of old values and half of new ones. The accuracy is the same for both methods and actually surprisingly close to the exact values given in the problem text.