UNIVERSITY OF DUBLIN

XSCH3042

TRINITY COLLEGE

FACULTY OF ENGINEERING, MATHEMATICS
AND SCIENCE

SCHOOL OF MATHEMATICS

Mathematics
TSM Mathematics
Foundation Scholarship

Hilary Term 2011

MATHEMATICS PAPER 3

Friday, January 14 UPPER LUCE HALL 9:30—12:30

Dr. D. Kitson, Dr. V. Dotsenko, Dr. R. Levene

Attempt SIX questions.

Please begin each question in a fresh answer book.

Mathematics students should answer questions from Parts A and B, **and not** *C.*

TSM students should answer questions from Parts A and C, **and not** *B.*

Non-programmable calculators are permitted for this examination, — please indicate the make and model of your calculator on each answer book used.

PART A — MA1111, MA1214, MA2215 and MA2223

All students answer from this part

1. (MA1111, Linear Algebra I)

   (a) Given that for a $n \times n$-matrix A both matrices A and A^{-1} have integer entries, show that $\det(A) = \pm 1$.

   (b) For how many $n \times n$-matrices A both matrices A and A^{-1} have *nonnegative* integer entries?

2. (MA1214, Introduction to group theory)

   Let G be a group, and let us write $\mathrm{Aut}(G)$ for the set of all group isomorphisms from G to G. That is,

   $$\mathrm{Aut}(G) = \{\theta \colon G \to G \ : \ \theta \text{ is an isomorphism}\}.$$

   (a) Prove that $\mathrm{Aut}(G)$ is a group under composition. You may use standard properties of isomorphisms without proof.

   (b) Show that the map $\theta \colon G \to G$, $\theta(a) = a^{-1}$ is in $\mathrm{Aut}(G)$ if and only if G is abelian.

   (c) Prove that $\mathrm{Aut}(S_3)$ is isomorphic to S_3.

3. (MA1214, Introduction to group theory)

   (a) Find a group G and elements $x, y, z \in G$ so that $o(x) = 5$, $o(y) = o(z) = 7$, and $o(xy) = 35$ but $o(xz) \neq 35$.

   (b) Show that if G is a group and $o(x) \leq 2$ for all $x \in G$, then G is abelian but not necessarily cyclic.

   (c) Let G be a finite group. By considering the size of the set $\{x \in G \colon o(x) \geq 3\}$, prove that there is an element $x \in G$ with $o(x) = 2$ if and only if $|G|$ is even.

4. (MA2215, Fields, rings and modules)

   (a) Explain briefly why $R = \{a + bi\sqrt{3} \colon a, b \in \mathbb{Z}\}$ is an integral domain. Then compute $\mathrm{Units}(R)$ and show that R is neither a principal ideal domain nor a unique factorisation domain.

   (b) Show that there is an ideal $I \lhd R$ so that $R/I \approx \mathbb{Z}_7$, but that there is no ideal $J \lhd R$ so that $R/J \approx \mathbb{Z}_5$.

5. (a) (MA2223, Metric spaces)

   Show that if A and B are subsets of a metric space (X, d) and have nonempty intersection then

   $$diam(A \cup B) \leq diam(A) + diam(B)$$

   Give an example of a metric space and two sets A and B where the above inequality does not hold.

   (b) Prove that if A and B are complete subspaces of a metric space (X, d) then $A \cap B$ and $A \cup B$ are complete subspaces.

   (c) Show that if A and B are compact subsets of a normed vector space $(X, \|.\|)$ then $A + B = \{a + b : a \in A,\, b \in B\}$ is compact.

6. (MA2223, Metric spaces)

(a) The spectral norm (2-norm) of the following matrix is a non-negative integer. Compute the spectral norm.

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

(b) Let $C[a,\, b]$ be the set of continuous real-valued functions $f : [a,\, b] \to \mathbb{R}$ with the usual pointwise vector space operations and supremum norm

$$\|f\|_\infty = \sup_{x \in [a,\, b]} |f(x)|$$

Give an example of a linear operator $T : C[0,\, 1] \to C[2,\, 3]$ which is an isometry.

(c) Prove that if $T : X \to Y$ is a continuous mapping between two metric spaces and A is a dense subset of X then $T(A)$ is a dense subset of $T(X)$.

PART B — MA1212

Mathematics students only may answer from this section, not TSM

7. (MA1212, Linear Algebra II)

   Given that for a square matrix A we have $\mathrm{rk}(A) = 1$, show that $\det(I + A) = 1 + \mathrm{tr}(A)$.

8. (MA1212, Linear Algebra II)

   Show that the quadratic form

   $$Q(x_1, \ldots, x_n) = x_1^2 + x_2^2 + \ldots + x_n^2 - x_1x_2 - x_2x_3 - \ldots - x_{n-1}x_n$$

   is positive definite.

PART C — MA2215, MA2223

TSM students only may answer from this section, not Mathematics

9. (MA2215, Fields, rings and modules)

   (a) Find a real number α so that $\mathbb{Q}(\sqrt{3}, \sqrt{7}) = \mathbb{Q}(\alpha)$, compute $[\mathbb{Q}(\sqrt{3}, \sqrt{7}) \colon \mathbb{Q}]$ and determine the minimum polynomial of α over $\mathbb{Q}$.

   (b) Let $p \in \mathbb{N}$ be a prime integer, and define an equivalence relation $\sim$ on $\mathbb{Z}_p[x]$ by

   $$f \sim g \iff f(\alpha) = g(\alpha) \text{ for all } \alpha \in \mathbb{Z}_p.$$

   A theorem from number theory asserts that $\alpha^p = \alpha$ for all $\alpha \in \mathbb{Z}_p$. Use this to show that for $f, g \in \mathbb{Z}_p[x]$ we have $f \sim g \iff x^p - x | f - g$.

10. (MA2223, Metric spaces)

    Suppose (X, d) is a metric space.

    (a) Show that if every subset of X is an open set then there are no limit points in X.

    (b) Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be continuous mappings. Show that

    $$A = \{x \in X : f(x) = g(x)\}$$

    is a closed subset of X.

    (c) Prove that if $a \in X$ then the mapping $T : X \to \mathbb{R}$, $x \mapsto d(a, x)$ is uniformly continuous.