

11. Find all functions f, g such that

$$f, g : \mathbb{R} \setminus \{-1, 0, 1\} \longrightarrow \mathbb{R}$$

$$x f(x) = 1 + \frac{1}{x} g\left(\frac{1}{x}\right) \text{ and } \frac{1}{x^2} f\left(\frac{1}{x}\right) = x^2 g(x)$$

Solution. We have equations

$$x f(x) = 1 + \frac{1}{x} g\left(\frac{1}{x}\right) \quad (1)$$

$$\frac{1}{x^2} f\left(\frac{1}{x}\right) = x^2 g(x) \quad (2)$$

Since $x \neq 0$, On replacing x with $\frac{1}{x}$ in equation (2) we get

$$x^2 f(x) = \frac{1}{x^2} g\left(\frac{1}{x}\right) \quad (3)$$

But from (1) we have

$$g\left(\frac{1}{x}\right) = x^2 f(x) - x \quad (4)$$

From replacing (4) in (3), we have..

$$x^2 f(x) = \frac{1}{x^2} (x^2 f(x) - x)$$

$$x^2 f(x) = f(x) - \frac{1}{x}$$

Rearranging the above equation we have

$$f(x) (1 - x^2) = \frac{1}{x} \quad (5)$$

Replacing equation (5) in equation (4), we get..

$$g\left(\frac{1}{x}\right) = x^2 \left(\frac{1}{x(1-x^2)} \right) - x$$

$$g\left(\frac{1}{x}\right) = \frac{x}{1-x^2} - x$$

$$g\left(\frac{1}{x}\right) = \frac{x^3}{1-x^2}$$

Replacing x with $\frac{1}{x}$ in the above equation gives..

$$g(x) = \frac{1}{x^3 \left(1 - \frac{1}{x^2}\right)}$$

$$g(x) = \frac{1}{x^3 - x}$$